JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **21**, No. 4, December 2008

ASYMPTOTIC ERROR ANALYSIS OF k-FOLD PSEUDO-NEWTON'S METHOD LOCATING A SIMPLE ZERO

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Abstract. The k-fold pseudo-Newton's method is proposed and its convergence behavior is investigated near a simple zero. The order of convergence is proven to be at least $k+2$. The asymptotic error constant is explicitly given in terms of k and the corresponding simple zero. High-precison numerical results are successfully implemented via Mathematica and illustrated for various examples.

1. A high-order Newton-type iterative method

The classical *Newton's method* of order 2 is often used to obtain a good approximated root to a given nonlinear algebraic equation. It is relatively simple to implement only with evaluations of the given function and its exact derivative symbolically computed by software such as *Mathematica* or *Maple*. In this paper, we propose a high order iterative method from the classical Newton's method and develop some theoretical results on the order of convergence as well as the asymptotic error constant of the method.

Suppose that a function $f : \mathbb{R} \to \mathbb{R}$ has a simple real zero α, and f is sufficiently smooth in a neighborhood of α. We seek an approximated α by a method

$$x_{n+1} = g(x_n),\ n = 0,\ 1,\ 2, \cdots, \tag{1.1}$$

where $g : \mathbb{R} \to \mathbb{R}$ is an iteration function and x_0 is given. Suppose further that α is a fixed point of g, and g is sufficiently smooth in a

Received September 06, 2008; Accepted November 20, 2008.
2000 Mathematics Subject Classification: Primary 65H05, 65H99.
Key words and phrases: pseudo-Newton's method, order of convergence, asymptotic error constant.
* The author was supported by the research fund of Dankook University in 2007.

neighborhood of α. Given $p \in \mathbb{N}$, we assume

$$\begin{cases} \left|\frac{d^p}{dx^p} g(x)\right|_{x=\alpha} = |g^{(p)}(\alpha)| < 1, & \text{if } p = 1. \\ g^{(i)}(\alpha) = 0 \text{ for } 1 \le i \le p-1 \text{ and } g^{(p)}(\alpha) \neq 0, & \text{if } p \ge 2. \end{cases} \quad (1.2)$$

Let x_n belong to a sufficiently small neighborhood of α for $n \in \mathbb{N} \cup \{0\}$. Then Taylor series[1,8] expansion about α immediately gives

$$x_{n+1} = g(x_n) = g(\alpha) + g^{(p)}(\xi)\ (x_n - \alpha)^p/p!, \quad (1.3)$$

where $\xi \in (a, b)$ with $a = \min(\alpha, x_n)$ and $b = \max(\alpha, x_n)$. In view of the continuity of g at α, for all given $\epsilon > 0$, there exists a number $\delta > 0$ such that

$$|x_{n+1} - \alpha| = |g(x_n) - g(\alpha)| = |g^{(p)}(\xi)|\ \frac{|(x_n - \alpha)^{p-1}|}{p!}\ |x_n - \alpha| < \epsilon, \quad (1.4)$$

whenever $|x_n - \alpha| < \delta$. Let $\boldsymbol{J} = \{x : |x - \alpha| \le \delta\}$. Then the continuity of $g^{(p)}$ on $\boldsymbol{J}$ ensures the existence of a number $M > 0$ satisfying $|g^{(p)}(x)| \le M$ for all $x \in \boldsymbol{J}$. Choose

$$\delta = \begin{cases} \min(\epsilon,\ 1/M), & \text{if } p = 1. \\ \{\min(\epsilon^{p-1},\ p!/M)\}^{1/(p-1)}, & \text{if } p \ge 2. \end{cases}$$

Then $|x_{n+1} - \alpha| = |g(x_n) - g(\alpha)| \le |x_n - \alpha|$. Hence $g : \boldsymbol{J} \to \boldsymbol{J}$. Since $|x_n - \alpha| < \delta$, it follows from (1.4) that

$$|x_{n+1} - \alpha| = |g(x_n) - g(\alpha)| \le K|x_n - \alpha|, \quad (1.5)$$

where $0 < K = \sup\{M\ |(x_n - \alpha)|^{p-1}/p!\ :\ n \in \mathbb{N} \cup \{0\}\} < M\ \delta^{p-1}/p!\ \le 1$ for $p \ge 2$. If $p = 1$, then $K = M < 1$ can be chosen according to (1.2). Hence g is contractive on $\boldsymbol{J}$ for any $p \in \mathbb{N}$ and the sequence $\{x_n\}_{n=0}^{\infty}$ with $x_0 \in \boldsymbol{J}$ defined by (1.1) converges to a fixed point $\alpha \in \boldsymbol{J}$[2,8,10]. Now introducing $e_n = x_n - \alpha$ with the fact that $\lim_{n \to \infty} \xi = \alpha$, for the iterative method (1.1) we obtain the *asymptotic error constant* η and *order of convergence* p[2,10] as follows:

$$\eta = \lim_{n \to \infty} \left|\frac{e_{n+1}}{{e_n}^p}\right| = \frac{|g^{(p)}(\alpha)|}{p!}. \quad (1.6)$$

Given an arbitrary $x \in \mathbb{R}$, we now define a function $F : \mathbb{R} \to \mathbb{R}$ by

$$F(w) = w - f(w)/f'(x) \quad (1.7)$$

with $f'(x) \neq 0$. Let $w_0 = w_0(x) = F(x) = x - f(x)/f'(x)$, and let for $k \in \mathbb{N}$

$$w_k(x) = w_k = F(w_{k-1}(x)) = w_{k-1}(x) - f(w_{k-1}(x))/f'(x). \quad (1.8)$$

Hence $w_k(x) = F^k(w_0) = F^{k+1}(x)$ for $k \in \mathbb{N}$, where $F^k(w_0) = F(F(\cdots$

$F(w_0)\cdots))$. Then the iterative method with $x_0 \in \mathbb{R}$

$$x_{n+1} = F^{k+1}(x_n) = g(x_n) \tag{1.9}$$

is called the *k-fold pseudo-Newton's method*. If $k = 0$, it becomes the classical *Newton's method*. If $k = 1$, it is simply called the *pseudo-Newton's method* which has the cubic convergence as shown in Halley's method[9,11], leap-frogging Newton's method[4,6] as well as other methods of Laguerre's type[5,7]

2. Convergence analysis

In this section, we wish to show that the order of convergence of k-fold pseudo-Newton's method is to be at least $k+2$ and express the theoretical asymptotic error constant of the method in terms of k and α. Since $f'(\alpha) \neq 0$, it can be easily shown from (1.8) that

$$w_k(\alpha) = \alpha, \text{ for all } k \in \mathbb{N} \cup \{0\}, \tag{2.1}$$

$$w_0{}'(\alpha) = \frac{d}{dx} w_0(x)\Big|_{x=\alpha} = 0, \; w_0{}''(\alpha) = f''(\alpha)/f'(\alpha). \tag{2.2}$$

We further wish to establish the following Lemma 2.1.

LEMMA 2.1. *Let $w_k^{(m)}(\alpha) = \frac{d^m}{dx^m} w_k(x)|_{x=\alpha}$ for any $k, m \in \mathbb{N} \cup \{0\}$. Then the following holds.*

$$w_k^{(m)}(\alpha) = \begin{cases} \alpha, \; if \; m = 0. \\ 0, \; if \; 1 \le m \le k+1. \\ \frac{(k+2)!}{2} c^{k+1} \; with \; c = f''(\alpha)/f'(\alpha), \; if \; m = k+2. \end{cases} \tag{2.3}$$

Proof. If $k = 0$, then the assertion holds from (2.1) and (2.2). It suffices to consider $k \in \mathbb{N}$. If $m = 0$, then the assertion immediately holds from (2.1). The remaining proof will be given based on induction on $m \ge 1$. We rewrite (1.8) to obtain for $k \in \mathbb{N}$

$$f'(x)(w_k - w_{k-1}) = -f(w_{k-1}). \tag{2.4}$$

Differentiating both sides of (2.4) with respect to x and evaluating at $x = \alpha$ yields

$$f''(x)(w_k - w_{k-1})|_{x=\alpha} + f'(x)(w_k{}' - w_{k-1}{}')|_{x=\alpha} = -f'(w_{k-1}) \cdot w_{k-1}{}'|_{x=\alpha}. \tag{2.5}$$

Hence we get $f'(\alpha) \cdot \Big(w_k{}'(\alpha) - w_{k-1}{}'(\alpha)\Big) = -f'(\alpha) \cdot w_{k-1}{}'(\alpha)$, which states

$$w_k{}'(\alpha) = 0 \text{ for all } k \; \in \mathbb{N}. \tag{2.6}$$

Suppose now (2.3) holds for $m \geq 1$. By differentiating $(m+1)$ times both sides of (2.4) with respect to x via Leibnitz Rule[8] and evaluating at $x = \alpha$ we obtain

$$\sum_{r=0}^{m+1} {}_{m+1}C_r f^{(m-r+2)}(x) \cdot \Big(w_k^{(r)}(x) - w_{k-1}^{(r)}(x) \Big)\Big|_{x=\alpha}$$
$$= -\frac{d^m}{dx^m}\Big(f'(w_{k-1}(x)) \cdot {w_{k-1}}'(x) \Big)\Big|_{x=\alpha}$$
$$= -\sum_{r=0}^{m} {}_mC_r \Big[f'(w_{k-1}) \Big]^{(m-r)}(x) \cdot w_{k-1}^{(r+1)}(x)\Big|_{x=\alpha}, \qquad (2.7)$$

where ${}_mC_r = \frac{m!}{(m-r)!r!}$. Since $w_k^{(r)}(\alpha) - w_{k-1}^{(r)}(\alpha) = 0$ for $0 \leq r \leq m-1 \leq k$, the leftmost side of (2.7) has possible nonvanishing terms for $r = m$ and $r = m+1$ as follows.

$$(m+1)f''(\alpha) \cdot \Big(w_k^{(m)}(\alpha) - w_{k-1}^{(m)}(\alpha) \Big) + f'(\alpha) \cdot \Big(w_k^{(m+1)}(\alpha) - w_{k-1}^{(m+1)}(\alpha) \Big)$$
$$= -(m+1)f''(\alpha) \cdot w_{k-1}^{(m)}(\alpha) + f'(\alpha) \cdot \Big(w_k^{(m+1)}(\alpha) - w_{k-1}^{(m+1)}(\alpha) \Big) \qquad (2.8)$$

in view of the induction hypothesis that $w_k^{(m)}(\alpha) = 0$ for $1 \leq m \leq k+1$. Similarly, owing to the induction hypothesis that $w_{k-1}^{(r+1)}(\alpha) = 0$ for $1 \leq r+1 \leq m \leq k$, the rightmost side of (2.7) has a possible nonvanishing term for $r = m$ as follows.

$$-f'(\alpha) \cdot w_{k-1}^{(m+1)}(\alpha). \qquad (2.9)$$

Hence it follows from the right side of (2.8) and (2.9) that

$$w_k^{(m+1)}(\alpha) = \begin{cases} 0, \text{ if } 2 \leq m+1 \leq k+1. \\ c\ (m+1)\ w_{k-1}^{(m)}(\alpha), \text{ if } m+1 = k+2. \end{cases} \qquad (2.10)$$

We also find for $m+1 = k+2$ that

$$w_k^{(m+1)}(\alpha) = w_k^{(k+2)}(\alpha) = c(k+2)\ w_{k-1}^{(k+1)}(\alpha) = c^2\ (k+2)(k+1)\ w_{k-2}^{(k)}(\alpha)$$
$$= (k+2)(k+1)(k)\cdots 4 \cdot 3 \cdots c^k \cdot {w_0}''(\alpha) = \frac{(k+2)!}{2} c^{k+1}. \qquad (2.11)$$

Hence (2.3) also holds for $m+1$, completing the induction proof. □

As a result of the preceding analysis, we obtain the following theorem.

THEOREM 2.2. *Let $k \in \mathbb{N} \cup \{0\}$ be given and α be a simple zero of the smooth function f described in Section 1. Then the k-fold pseudo-Newton's method defined by (1.9) is at least of order $k+2$ and its asymptotic error constant η is given by $|c^{k+1}|/2$, where $c = f''(\alpha)/f'(\alpha)$.*

TABLE 1. Convergence of k-fold pseudo-Newton's method for $f(x) = e^{x/2} + x^3 - x - \sqrt{e}$

k	n	x_n	$f(x_n)$	$\mid x_n - \alpha \mid$	$\lvert e_{n+1}/{e_n}^{k+2} \rvert$	η
	0	1.50000000000000	2.3432787	0.500000		
	1	1.15583039684897	0.52189168	0.15583	0.6233215874	
	2	1.02197739041691	0.063631617	0.0219774	0.9050483270	
	3	1.00052936890028	0.0014960273	0.000529369	1.095988954	
	4	1.00000031783299	8.97675×10^{-7}	3.17833×10^{-7}	1.134180376	
	5	1.00000000000011	3.23872×10^{-13}	1.14671×10^{-13}	1.135155498	1.135156084
0	6	1.00000000000000	4.21582×10^{-26}	1.49266×10^{-26}	1.135156084	
	7	1.00000000000000	7.14332×10^{-52}	2.52918×10^{-52}	1.135156084	
	8	1.00000000000000	2.05086×10^{-103}	7.26132×10^{-104}	1.135156084	
	9	1.00000000000000	1.69047×10^{-206}	5.98531×10^{-207}	1.135156084	
	10	1.00000000000000	5.87545×10^{-255}	2.08025×10^{-255}		
	11	1.00000000000000	5.85767×10^{-255}	2.43896×10^{-260}		
	0	1.50000000000000	2.3432787	0.500000		
	1	1.07917743644445	0.24423842	0.0791774	0.6334194916	
	2	1.00095340131643	0.0026956643	0.000953401	1.920752638	
	3	1.00000000222496	6.28409×10^{-9}	2.22496×10^{-9}	2.567409009	
1	4	1.00000000000000	8.0173×10^{-26}	2.83863×10^{-26}	2.577158648	2.57715867
	5	1.00000000000000	1.66489×10^{-76}	2.89475×10^{-77}	2.577158670	
	6	1.00000000000000	1.49093×10^{-228}	5.27882×10^{-229}	2.577158670	
	7	1.00000000000000	5.85767×10^{-255}	2.43896×10^{-260}		
	8	1.00000000000000	5.85767×10^{-255}	2.43896×10^{-260}		
	0	1.50000000000000	2.3432787	0.500000		
	1	1.04330486322393	0.12840498	0.0433049	0.6928778116	
	2	1.00001614296551	0.0000455944392	0.000016143	4.590253115	
2	3	1.00000000000000	1.12212×10^{-18}	3.97301×10^{-19}	5.850410166	
	4	1.00000000000000	4.11742×10^{-73}	1.45782×10^{-73}	5.850954689	5.850954689
	5	1.00000000000000	5.53909×10^{-255}	1.96115×10^{-255}		
	6	1.00000000000000	5.85767×10^{-255}	2.43896×10^{-260}		
	0	1.50000000000000	2.3432787	0.5		
	1	1.02444535198647	0.070973481	0.0244454	0.7822512636	
	2	1.00000009669892	2.73113×10^{-7}	9.66989×10^{-8}	11.07745851	
3	3	1.00000000000000	3.17205×10^{-34}	1.1231×10^{-34}	13.28348392	13.28349363
	4	1.00000000000000	6.70398×10^{-169}	2.37363×10^{-169}	13.28349363	
	5	1.00000000000000	5.85769×10^{-255}	3.25195×10^{-260}		
	6	1.00000000000000	5.85767×10^{-255}	2.43896×10^{-260}		
	0	1.50000000000000	2.3432787	0.500000		
	1	1.01402110566350	0.040233800	0.0140211	0.8973507625	
4	2	1.00000000000000	5.68454×10^{-10}	2.01268×10^{-10}	26.48996622	30.15767721
	3	1.00000000000000	5.66197×10^{-57}	2.00469×10^{-57}	30.15767716	
	4	1.00000000000000	5.85767×10^{-255}	2.43896×10^{-260}		
	5	1.00000000000000	5.85766×10^{-255}	2.07395×10^{-255}		
	0	1.50000000000000	2.34328	0.500000		
	1	1.00811175707122	0.023122	0.00811176	1.038304905	
5	2	1.00000000000014	4.08545×10^{-13}	1.4465×10^{-13}	62.59149507	68.46734154
	3	1.00000000000000	2.56238×10^{-88}	9.07241×10^{-89}	68.46734154	
	4	1.00000000000000	5.85766×10^{-255}	2.07395×10^{-255}		
	5	1.00000000000000	5.85766×10^{-255}	2.07395×10^{-255}		
	0	1.50000000000000	2.34328	0.500000		
	1	1.00471570183098	0.0133902	0.00471570	1.207219669	
6	2	1.00000000000000	1.01043×10^{-16}	3.57756×10^{-17}	146.2904295	155.4422386
	3	1.00000000000000	1.1781×10^{-129}	4.17121×10^{-130}	155.4422386	
	4	1.00000000000000	5.85766×10^{-255}	2.07395×10^{-255}		
	5	1.00000000000000	5.85767×10^{-255}	2.43896×10^{-260}		
	0	1.50000000000000	2.34328	0.500000		
	1	1.00274900614084	0.00778843	0.00274901	1.407491144	
7	2	1.00000000000000	8.5831×10^{-21}	3.03895×10^{-21}	338.9552803	352.9024058
	3	1.00000000000000	2.20338×10^{-182}	7.80133×10^{-183}	352.9024058	
	4	1.00000000000000	5.85767×10^{-255}	2.43896×10^{-260}		
	5	1.00000000000000	5.85767×10^{-255}	2.43896×10^{-260}		

TABLE 2. Convergence of k-fold pseudo-Newton's method for $f(x) = 2x(1 + x - x^2)\ln x - x^2 + 1$

k	n	x_n	$f(x_n)$	$\mid x_n - \alpha \mid$	$\lvert e_{n+1}/e_n{}^{k+2}\rvert$	η
	0	0.400000000000000	-0.0689604	0.0720322		
	1	0.306685110747460	0.0268329	0.0212827	4.101782893	
	2	0.326786157628102	0.0014083	0.00118163	2.608724731	
	3	0.327963839686230	4.68685×10^{-6}	3.94565×10^{-6}	2.825899702	
	4	0.327967785287608	5.25159×10^{-11}	4.42112×10^{-11}	2.839857327	
0	5	0.327967785331819	6.59366×10^{-21}	5.55097×10^{-21}	2.839904339	2.83990434
	6	0.327967785331819	1.03944×10^{-40}	8.75068×10^{-41}	2.839904340	
	7	0.327967785331819	2.58312×10^{-80}	2.17464×10^{-80}	2.839904340	
	8	0.327967785331819	1.59528×10^{-159}	1.34301×10^{-159}	2.839904340	
	9	0.327967785331819	$-1.14317 \times 10^{-260}$	1.47326×10^{-251}		
	10	0.327967785331819	1.92706×10^{-260}	1.47326×10^{-251}		
	0	0.400000000000000	-0.0689604	0.0720322		
	1	0.342994437219077	-0.0170960	0.0150267	40.20517185	
	2	0.328031970218841	-0.0000762275	0.0000641849	18.91673172	
	3	0.327967785336087	-5.06964×10^{-12}	4.26795×10^{-12}	16.14066044	16.13011332
1	4	0.327967785331819	-1.48955×10^{-33}	1.254×10^{-33}	16.13011332	
	5	0.327967785331819	-3.77819×10^{-98}	3.18073×10^{-98}	16.13011332	
	6	0.327967785331819	$-1.42559 \times 10^{-262}$	1.47326×10^{-251}		
	7	0.327967785331819	$-2.85855 \times 10^{-263}$	1.47326×10^{-251}		
	0	0.400000000000000	-0.0689604	0.0720322		
	1	0.319860760331448	0.0098529	0.00810703	301.1302897	
2	2	0.327967433519558	4.17897×10^{-7}	3.51812×10^{-7}	81.44509260	
	3	0.327967785331819	1.66713×10^{-24}	1.4035×10^{-24}	91.61548223	91.61595761
	4	0.327967785331819	4.2226×10^{-94}	3.55486×10^{-94}	91.61595761	
	5	0.327967785331819	$-4.91463 \times 10^{-261}$	1.47326×10^{-251}		
	6	0.327967785331819	1.38082×10^{-260}	1.47326×10^{-251}		
	0	0.400000000000000	-0.0689604	0.0720322		
	1	0.333193365416221	-0.00611539	0.00522558	2694.639489	
	2	0.327967787577164	-2.66711×10^{-9}	2.24535×10^{-9}	576.2496212	
3	3	0.327967785331819	-3.52759×10^{-41}	2.96976×10^{-41}	520.3611338	520.3611112
	4	0.327967785331819	$-1.42780 \times 10^{-200}$	1.20201×10^{-200}	520.3611112	
	5	0.327967785331819	2.28231×10^{-263}	1.47326×10^{-251}		
	6	0.327967785331819	9.17649×10^{-265}	1.47326×10^{-251}		
	0	0.400000000000000	-0.0689604	0.0720322		
	1	0.324918224317845	0.00365383	0.00304956	21831.15541	
	2	0.327967785329608	2.62656×10^{-12}	2.21121×10^{-12}	2749.198517	2955.551556
4	3	0.327967785331819	4.10371×10^{-67}	3.45477×10^{-67}	2955.551556	
	4	0.327967785331819	$-4.23772 \times 10^{-263}$	1.47326×10^{-251}		
	5	0.327967785331819	$-7.62972 \times 10^{-265}$	1.47326×10^{-251}		
	0	0.400000000000000	-0.0689604	0.0720322		
	1	0.329862466012106	-0.00223848	0.00189468	188299.2775	
5	2	0.327967785331820	-1.84495×10^{-15}	1.5532×10^{-15}	17720.45312	16786.96738
	3	0.327967785331819	$-4.34830 \times 10^{-100}$	3.66068×10^{-100}	16786.96738	
	4	0.327967785331819	1.92012×10^{-260}	1.47326×10^{-251}		
	5	0.327967785331819	1.92139×10^{-260}	1.47326×10^{-251}		
	0	0.400000000000000	-0.0689604	0.0720322		
	1	0.326833425262387	0.00135178	0.00113436	$1.565080894 \times 10^{6}$	
	2	0.327967785331819	2.9912×10^{-19}	2.51818×10^{-19}	91850.08854	95346.76301
6	3	0.327967785331819	1.8313×10^{-144}	1.54171×10^{-144}	95346.76301	
	4	0.327967785331819	$-1.16589 \times 10^{-260}$	1.47326×10^{-251}		
	5	0.327967785331819	1.92136×10^{-260}	1.47326×10^{-251}		
	0	0.400000000000000	-0.0689604	0.0720322		
	1	0.328662611133129	-0.000823714	0.000694826	$1.330868140 \times 10^{7}$	
7	2	0.327967785331819	-2.49252×10^{-23}	2.09836×10^{-23}	555900.8411	541551.3721
	3	0.327967785331819	$-5.07369 \times 10^{-199}$	4.27136×10^{-199}	541551.3721	
	4	0.327967785331819	1.92027×10^{-260}	1.47326×10^{-251}		
	5	0.327967785331819	$-1.15581 \times 10^{-260}$	1.47326×10^{-251}		

TABLE 3. Convergence behavior for various functions

$f(x)$	α	x_0	k	p	ν	η
$x^8 - 14x^4 \sin\frac{\pi x}{4} - 32$	2	1.87	0	2	10	2.647720887
			1	3	8	14.02085179
			2	4	6	74.24660427
			3	5	6	393.1685698
			4	6	5	2082.001268
			5	7	6	11025.11649
$3x^7 - 37x^4 + 208$	2	1.958	0	2	10	7.05
			1	3	8	99.405
			2	4	6	1401.6105
			3	5	7	19762.70805
			4	6	5	278654.1835
			5	7	6	3929023.987
$e^{-x}\sin x + \ln[1 + (x-\pi)^2]$	π	2.8	0	2	13	24.14069263
			1	3	9	1165.546082
			2	4	7	56274.17941
			3	5	6	2716995.337
			4	6	6	131180298.6
			5	7	5	6333566536
$\cos x - x$	0.739085133215161	0.6	0	2	8	0.2208053959
			1	3	5	0.0975100456
			2	4	4	0.0430614884
			3	5	4	0.0190164180
			4	6	3	0.0083978500
			5	7	3	0.0037085837
$x^2\sin(\pi x/8) + e^{(x-2)^2} - 1 - 2\sqrt{2}$	21.7		0	2	9	0.659974721
			1	3	6	0.871133264
			2	4	5	1.149851867
			3	5	5	1.517746330
			4	6	4	2.003348418
			5	7	4	2.644318631
$e^{(x^2+7x-30)} - 1$	3	2.94	0	2	10	6.576923077
			1	3	8	86.51183432
			2	4	6	1137.963359
			3	5	6	14968.59495
			4	6	5	196894.5952
			5	7	6	2589921.213
$\sin(\pi x/(2\sqrt{2})) - x^4 + 3$	$\sqrt{2}$	1.6	0	2	9	1.115182548
			1	3	6	2.487264232
			2	4	5	5.547507330
			3	5	4	12.37296672
			4	6	4	27.59623312
			5	7	4	61.54967515
$\sin^2 x - x^2 + 1$	1.40449164821534	1.27	0	2	8	0.783570950
			1	3	6	1.227966868
			2	4	5	1.924398331
			3	5	4	3.015805257
			4	6	4	4.726194782
			5	7	4	7.406617873

Proof. Let $g(x) = w_k(x) = F^{k+1}(x)$ as described in (1.8) and (1.9). Define the iteration $x_{n+1} = g(x_n)$ with $x_0 \in \boldsymbol{J}$ and the error $e_n = x_n - \alpha$ for $n \in \mathbb{N} \cup \{0\}$. Then Lemma 2.1 yields the asymptotic error constant η and the order of convergence $p = k + 2$ in view of (1.6)

$$\eta = \lim_{n\to\infty} \left| \frac{e_{n+1}}{{e_n}^{k+2}} \right| = \frac{1}{(k+2)!} \, |g^{(k+2)}(\alpha)| = \frac{1}{(k+2)!} \, |w_k^{(k+2)}(\alpha)| = \frac{1}{2} \, |c^{k+1}|,$$

completing the proof. □

3. Algorithm and numerical results

Based on the discussion in Sections 1 and 2, we construct a zero-finding algorithm with the aid of symbolic and computational ability of *Mathematica*[11] as follows.

Algorithm 3.1 (Zero-Finding Algorithm)
Step 1. For $k \in \mathbb{N} \cup \{0\}$, construct the iteration function $g = F^{k+1}$ with the given function f having a simple real zero α, according to the description in Section 1.
Step 2. Set the minimum number of precision digits. With exact zero α or most accurate zero, supply the theoretical asymptotic error constant η. Set the error range ϵ, the maximum iteration number n_{max} and choose the initial value x_0 sufficiently close to α. Compute $f(x_0)$ and $|x_0 - \alpha\ |$.
Step 3. Compute $x_{n+1} = g(x_n)$ for $0 \le n \le n_{max}$ and display the computed values of $n,\ x_n, f(x_n), |x_n - \alpha|,\ |e_{n+1}/{e_n}^{k+2}|$ and η.

To achieve sufficient accuracy, the minimum number of precision digits was chosen as 250 by assigning *$MinPrecision=250* in Mathematica. The error bound ϵ for $|\ x_n - \alpha\ | < \epsilon$ was chosen as 0.5×10^{-235}. The symbolic computation of $f'(x)$ in (1.8) has been easily done with the aid of Mathematica. In Table 1, we illustrate the order of convergence and asymptotic error constant for a nonlinear function

$$f(x) = e^{x/2} + x^3 - x - \sqrt{e}$$

having a simple zero $\alpha = 1$. The number of computation gets smaller as k increases due to high-order convergence. For each $0 \le k \le 7$, the order of convergence has been confirmed to be of at least $k + 2$. As the second example, we take $f(x) = 2x(1+x-x^2)\ln x - x^2 + 1$ with a simple zero

$$\alpha = 0.327967785331818805262244062619867169281398606960022721174026491663921570\\ 332495459610381664214643576518207702120569678002225882133859815729417121 80$$

1580835920390080126334828769626066822822507332954433877923534794244336508 6
753831312157004303583377836781852 6,

which is accurate up to 250 significant decimal digits. Table 2 also shows a good agreement with the theory presented in this paper. The computed asymptotic error constants were found in good agreement with the theoretical values η up to 10 significant digits. The computed root was rounded to be accurate up to the 235 significant digits. Nevertheless, the limited space allows us to list it only up to 15 significant digits. Our analysis has been confirmed through more test functions that are listed below:

(1) $f(x) = x^8 - 14x^4 \sin \frac{\pi x}{4} - 32$, $\alpha = 2$
(2) $f(x) = 3x^7 - 37x^4 + 208$, $\alpha = 2$
(3) $f(x) = e^{-x} \sin x + \ln[1 + (x - \pi)^2]$, $\alpha = \pi$
(4) $f(x) = \cos x - x$, $\alpha = 0.739085133215161$
(5) $f(x) = x^2 \sin(\pi x/8) + e^{(x-2)^2} - 1 - 2\sqrt{2}$, $\alpha = 2$
(6) $f(x) = e^{(x^2+7x-30)} - 1$, $\alpha = 3$
(7) $f(x) = \sin(\pi x/(2\sqrt{2}) - x^4 + 3$, $\alpha = \sqrt{2}$
(8) $f(x) = \sin^2 x - x^2 + 1$, $\alpha = 1.40449164821534$

Table 3 shows convergence behavior for the above test functions as k varies by displaying the initial guess x_0, the order of convergence p, the least iteration number ν for convergence and the asymptotic error constant η. The high-order convergence[5] established in Theorem 1 will play a role in the highly accurate computation of zeros for the nonlinear equation. The current study will be extended to the case for functions with zeros of multiplicity higher than 1.

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