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A NEW FUNCTION SPACE $L_\alpha(X)$ VERSION 1.1

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Abstract. We develop a new function space $L_\alpha(X)$ that generalizes the classical Lebesgue space $L^p(X)$. The generalization is focused on a better explanation of the flux terms arising from many dynamics.

1. Introduction

The function space of p^{th}-power summable functions $L^p(X)$ has been commonly used in analysis since the beginning of the 20th century. The spaces $L^p(X)$(named the classical Lebesgue space) are of such extreme importance since it has fundamental roles in modern analysis and it has not been replaced by significantly different, better function spaces. In fact, $L^p(X)$ formulates basic structures on another function spaces such as (fractional) Sobolev spaces $W^{s,p}$, Besov spaces $B^s_{p,q}$, Triebel-Lizorkin spaces $F^s_{p,q}$. Even more, by virtue of Littlewood-Paley decomposition, it has been known that the spaces $L^p(X)$ have a connection with Hölder spaces C^s, Zygmund spaces Λ^s and BMO. In this sense one may say that all of those spaces are under the influence of $L^p(X)$.

In this paper we attempt to develop a new function space that generalizes the classical Lebesgue space. The motivation of this research starts from a close look at the L^p-norm: $\|f\|_{L^p} = \left(\int_X |f(x)|^p \, d\mu\right)^{1/p}$ of the classical Lebesgue spaces $L^p(X)$, $1 \le p < \infty$. It can be rewritten as

$$\|f\|_{L^p} := \alpha^{-1}\left(\int_X \alpha(|f(x)|)\, d\mu\right),$$

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with

$$\alpha(x) := x^p.$$

Even though the positive-real-variable function $\alpha(x) := x^p$ has very beautiful and convenient algebraic and geometric properties(Section 2.1), it also has some practical limitations. One of such practical examples which cannot be overlooked arises from some problems of partial differential equations which contain the *flux* term $\mathbf{J}$: for example, the elliptic equations:

$$-\nabla \cdot \mathbf{J} = f$$

the parabolic equations:

$$\frac{\partial}{\partial t} c(u) - \nabla \cdot \mathbf{J} = f$$

(and also hyperbolic problems, of course). We explain the flux terms of the above equations in detail because not only is it related with our forthcoming study [4] but also it justifies this report.

For an irrotational flux($\nabla \times \mathbf{J} = 0$), $\mathbf{J}$ can be written as $\mathbf{J} = \nabla\phi(u)$ with a prescribed physical content u (in a simply connected domain). In general, physical observations tell us that $\mathbf{J}$ depends on u and its gradient at each point x, that is, $\mathbf{J} = \mathbf{J}(x, \nabla, u(x))$. Therefore for the linear case, one can simply represent $\mathbf{J}$ as $\mathbf{J} = c\nabla u$ (on isotropic medium) or $\mathbf{J} = A\nabla u$ with a square matrix A (on an-isotropic medium). But for the nonlinear case, the situation can be much more complicated. One of the most common assumptions is that $\mathbf{J} = |\nabla u|^{p-2}\nabla u$ is to produce the p-Laplacian

$$\Delta_p u := \nabla \cdot |\nabla u|^{p-2}\nabla u.$$

Putting so, one of the main benefits of what we may get in working with this new function spaces is that solutions can be discussed inside the underlying set L^p. Even though, slightly more generally, the Leray-Lions type conditions[1] might be given on $\mathbf{J}$, it is still dominated by L^p-theory[3]. Hence we carefully point out that the growth condition of the flux is *artificial* enough to handle those solutions inside L^p hierarchy. But in practical physical problems, the flux may not be so nice as the above. Our new function space is designed to handle solutions of nonlinear equations having more general flux term than the equations having either p-Laplacian or flux term of the Leray-Lions type conditions[4].

[1] there are positive constants c_1, c_2 such that for all $\xi \in \mathbb{R}^n$ and almost every x

$$c_1|\xi|^p \le \mathbf{J}(x,\xi)\cdot\xi, \quad |\mathbf{J}(x,\xi)| \le c_2|\xi|^{p-1}.$$

2. The space $L_\alpha(X)$

We introduce some terminologies to define Lebesgue type function spaces $L_\alpha(X)$, which generalize the classical Lebesgue spaces $L^p(X)$. In this paper, $(X, \mathfrak{M}, \mu)$ always represents a given measurable space.

A *pre-Hölder's function* $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$[2] is an *absolutely continuous bijective* function satisfying

$$[H1] \qquad \alpha(0) = 0, \quad \alpha(1) = 1.$$

Suppose there exists a pre-Hölder's function β satisfying

$$[H2] \qquad \alpha^{-1}(x)\beta^{-1}(x) = x \qquad \text{for all } x \geq 0,$$

then β is called the *conjugate (pre-Hölder's) function* of α. A typical example of pre-Hölder's functions is $\alpha(x) = x^p$, $p > 1$ with the conjugate function $\beta(x) = x^q$, $\frac{1}{p} + \frac{1}{q} = 1$. In the relation [H2], the notation α^{-1}, β^{-1} are meant to be the inverse functions of α, β, respectively. In the following, a function g represents the two-variable function g on $\mathbb{R}_+ \times \mathbb{R}_+$ defined by:

$$g(x, y) := \alpha^{-1}(x)\beta^{-1}(y)$$

provided that those pre-Hölder's pair (α, β) exists.

Definition 2.1. *A pre-Hölder's function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ together with the conjugate function β is said to be a Hölder's function if for any positive constants $a, b > 0$, there exist positive constants θ_1, θ_2 (depending on a, b) such that*

$$\theta_1 + \theta_2 \leq 1$$

and that a comparable condition

$$[H3] \qquad g(x, y) \leq \theta_1 \frac{ab}{\alpha(b)}\, x + \theta_2 \frac{ab}{\beta(a)}\, y$$

holds for all $(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+$.

The above definition means that the linear transformation

$$\begin{aligned} T(x, y) &:= \theta_1 \frac{ab}{\alpha(b)}\, x + \theta_2 \frac{ab}{\beta(a)}\, y \\ &= (\theta_1 \ \ \theta_2) \begin{pmatrix} \frac{b}{\alpha(b)} & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & \frac{a}{\beta(a)} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \end{aligned}$$

is greater than or equal to g; that is to say,

$$g(x, y) \leq T(x, y)$$

[2] $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$

for all $(x, y) \in \mathbb{R}^2_+$. In this sense we call it a *dominating plane* of g.

First, we introduce some basic useful identities for a pre-Hölder's pair (α, β). We solve for $\beta^{-1}(x)$ in the conjugate identity $\alpha^{-1}(x)\beta^{-1}(x) = x$ to get $\beta^{-1}(x) = \dfrac{x}{\alpha^{-1}(x)}$, which in turn yields

$$x = \beta\left(\frac{x}{\alpha^{-1}(x)}\right) \quad \text{or} \quad \alpha(x) = \beta\left(\frac{\alpha(x)}{x}\right). \tag{1}$$

Similarly, from a variance of the conjugate identity: $\alpha^{-1}(x) = \dfrac{x}{\beta^{-1}(x)}$, we have

$$x = \frac{\alpha(x)}{\beta^{-1}(\alpha(x))}. \tag{2}$$

Also, differentiating on both sides of $\alpha^{-1}(x)\beta^{-1}(x) = x$ (if they are differentiable), we can obtain

$$\frac{\beta^{-1}(x)}{\alpha'(\alpha^{-1}(x))} + \frac{\alpha^{-1}(x)}{\beta'(\beta^{-1}(x))} = 1. \tag{3}$$

Then we have for $\alpha(x) = \beta(y)$, that is, for $y := \frac{\alpha(x)}{x}$,

$$\frac{y}{\alpha'(x)} + \frac{x}{\beta'(y)} = 1. \tag{4}$$

Therefore we get

$$\alpha'(x) = \frac{\alpha(x)}{x} + \frac{\alpha(x)}{\beta'\left(\frac{\alpha(x)}{x}\right) - x}. \tag{5}$$

We say that α obeys a *commutative condition* if

$$[\beta, \alpha](x) := \beta \circ \alpha \circ \beta^{-1} \circ \alpha^{-1}(x) = x \tag{C}$$

for all $x \geq 0$. Then it is easy to check that

$$\beta \circ \alpha \circ \beta^{-1} = \alpha, \quad \beta \circ \alpha^{-1} \circ \beta^{-1} = \alpha^{-1} \quad \text{and} \quad \alpha^{-1} \circ \beta \circ \alpha = \beta.$$

Identities above together with the conjugate identity (1) lead to get

$$\begin{aligned}\beta \circ \alpha^{-1}\left(\frac{\alpha(a)}{a}\right) &= \beta \circ \alpha^{-1} \circ \beta^{-1} \circ \beta\left(\frac{\alpha(a)}{a}\right) \\ &= (\beta \circ \alpha^{-1} \circ \beta^{-1}) \circ \alpha(a) = \alpha^{-1} \circ \alpha(a) = a,\end{aligned}$$

and

$$\alpha \circ \beta^{-1}\left(\frac{\beta(a)}{a}\right) = a,$$

for any $a > 0$. Therefore we can notice that

$$\frac{\alpha(x)}{x} = \left(\frac{\beta(x)}{x}\right)^{-1} = \alpha \circ \beta^{-1}(x) = \beta^{-1} \circ \alpha(x), \tag{6}$$

$x > 0$, provided that the pre-Hölder's function α is commutative with its conjugate function β.

2.1. Classical Lebesgue spaces

The L^p-norm $\|f\|_{L^p}$ of the classical Lebesgue spaces $L^p(X)$, $1 \le p < \infty$ can be rewritten as

$$\|f\|_{L^p} := \alpha^{-1}\left(\int_X \alpha(|f(x)|)\, d\mu\right),$$

where we let

$$\alpha(x) := x^p.$$

Besides the fact that $\alpha(x) = x^p$ is a pre-Hölder's function, the main ingredients of $\alpha(x) = x^p$ in this case are;

(A) α satisfies a *sub-homomorphic condition*:

$$\alpha(xy) \ge \alpha(x)\alpha(y) \tag{$[H3']$}$$

for all $x, y > 0$, and

(B) the α is *convex* on $\mathbb{R}_+$.

In fact, these conditions have a close connection with the comparable condition $[H3]$. To illustrate this we first observe that the condition $[H3']$ leads to

$$\alpha^{-1}(xy) \le \alpha^{-1}(x)\alpha^{-1}(y) \quad \text{and} \quad \beta^{-1}(xy) \ge \beta^{-1}(x)\beta^{-1}(y), \tag{7}$$

and the convexity of α implies the concaveness of α^{-1}.

Lemma 2.1. *Let α be a convex pre-Hölder's function satisfying the sub-homomorphic condition $[H3']$ and posses the corresponding conjugate function β. Then for every point $t \in \mathbb{R}_+$ at which α is differentiable, we have*

$$g(x,y) := \alpha^{-1}(x)\beta^{-1}(y) \le \frac{\beta^{-1}(t)}{\alpha'(\alpha^{-1}(t))}\, x + \frac{\alpha^{-1}(t)}{\beta'(\beta^{-1}(t))}\, y$$

for all $x, y \ge 0$.

Proof. We compute

$$\nabla g(t,t) = \left(\frac{\beta^{-1}(t)}{\alpha'(\alpha^{-1}(t))}, \ \frac{\alpha^{-1}(t)}{\beta'(\beta^{-1}(t))} \right)$$

to find the tangent plane to the graph of the function g at (t,t). We obtain the equation of the tangent plane:

$$\begin{aligned} z &= g_x(t,t)(x-t) + g_y(t,t)(y-t) + g(t,t) \\ &= \frac{\beta^{-1}(t)}{\alpha'(\alpha^{-1}(t))}(x-t) + \frac{\alpha^{-1}(t)}{\beta'(\beta^{-1}(t))}(y-t) + \alpha^{-1}(t)\beta^{-1}(t) \\ &= \frac{\beta^{-1}(t)}{\alpha'(\alpha^{-1}(t))}\, x + \frac{\alpha^{-1}(t)}{\beta'(\beta^{-1}(t))}\, y \equiv T(x,y) \end{aligned}$$

by virtue of the identity (3) and the conjugate identity $\alpha^{-1}(t)\beta^{-1}(t) = t$. Since the restriction $z = T(x,1)$ of the tangent plane $z = T(x,y)$ is the tangent line to the graph $g(x,1) = \alpha^{-1}(x)$ located inside x-z plane and α^{-1} is concave up on $\mathbb{R}_+$, we obtain inequality

$$\alpha^{-1}(x) = g(x,1) \le T(x,1) = \frac{\beta^{-1}(t)}{\alpha'(\alpha^{-1}(t))}\, x + \frac{\alpha^{-1}(t)}{\beta'(\beta^{-1}(t))}.$$

For any $x, y > 0$, we write $x := \lambda y$ for some $\lambda > 0$ to have

$$\alpha^{-1}(x) \le \alpha^{-1}(\lambda)\alpha^{-1}(y).$$

Therefore we conclude

$$\begin{aligned} g(x,y) = \alpha^{-1}(x)\beta^{-1}(y) &\le \alpha^{-1}(\lambda)\, \alpha^{-1}(y)\beta^{-1}(y) \\ &= \alpha^{-1}(\lambda) y \\ &\le \left(\frac{\beta^{-1}(t)}{\alpha'(\alpha^{-1}(t))}\, \lambda + \ \frac{\alpha^{-1}(t)}{\beta'(\beta^{-1}(t))} \right) y \\ &= \frac{\beta^{-1}(t)}{\alpha'(\alpha^{-1}(t))}\, x + \ \frac{\alpha^{-1}(t)}{\beta'(\beta^{-1}(t))}\, y, \end{aligned}$$

where we used the fact that $\alpha^{-1}(y)\beta^{-1}(y) = y$. □

Suppose that α is a function obeying the conditions in Lemma 2.1, and that the conjugate function β of α also satisfies the sub-homomorphic condition $[H3']$. Then we can easily see that both α and β are (multiplicative) homomorphisms:

(8) $$\alpha(xy) = \alpha(x)\alpha(y) \quad \text{and} \quad \beta(xy) = \beta(x)\beta(y).$$

COROLLARY 2.2. *Any convex pre-Hölder's function α with the homomorphic condition satisfies the comparable condition $[H3]$ if there exists the conjugate function β of α. In fact, for any point $t \in \mathbb{R}_+$ at which α is differentiable, we have*

$$g(x,y) \le \frac{\beta^{-1}(t)}{\alpha'(\alpha^{-1}(t))} \frac{ab}{\alpha(b)}\, x + \frac{\alpha^{-1}(t)}{\beta'(\beta^{-1}(t))} \frac{ab}{\beta(a)}\, y$$

for all $x, y \ge 0$. Therefore α is a Hölder's function.

Proof. For any $b > 0$, one of the conjugate identities (1) leads to have

$$\beta \circ \alpha^{-1}\left(\frac{\alpha(b)}{b}\right) = \beta\left(\frac{b}{\alpha^{-1}(b)}\right) = b.$$

Similarly, we have for $a > 0$,

$$\alpha \circ \beta^{-1}\left(\frac{\beta(a)}{a}\right) = a.$$

Hence by virtue of the homomorphic properties, we obtain

$$\begin{aligned}
\alpha^{-1}(x)\beta^{-1}(y) &= \alpha^{-1}\left(\frac{b}{\alpha(b)} \frac{\alpha(b)}{b}\, x\right)\, \beta^{-1}(y)\\
&= \alpha^{-1}\left(\frac{b}{\alpha(b)} x\right) \left(\alpha^{-1}\left(\frac{\alpha(b)}{b}\right)\, \beta^{-1}(y)\right)\\
&= \alpha^{-1}\left(\frac{b}{\alpha(b)} x\right)\, \beta^{-1}\left(\beta \circ \alpha^{-1}\left(\frac{\alpha(b)}{b}\right)\, y\right)\\
&= \alpha^{-1}\left(\frac{b}{\alpha(b)} x\right)\, \beta^{-1}\left(b\, y\right)\\
&\le \frac{\beta^{-1}(t)}{\alpha'(\alpha^{-1}(t))} \frac{b}{\alpha(b)} x + \frac{\alpha^{-1}(t)}{\beta'(\beta^{-1}(t))}\, b\, y.
\end{aligned}$$

More generally, one delivers that for $a, b > 0$,

$$\begin{aligned}
\alpha^{-1}(x)\beta^{-1}(y) &= \alpha^{-1}\left(\frac{b}{\alpha(b)} \frac{\alpha(b)}{b}\, x\right)\, \beta^{-1}\left(\frac{\beta(a)}{a} \frac{a}{\beta(a)}\, y\right)\\
&\le \frac{\beta^{-1}(t)}{\alpha'(\alpha^{-1}(t))} \frac{ab}{\alpha(b)}\, x + \frac{\alpha^{-1}(t)}{\beta'(\beta^{-1}(t))} \frac{ab}{\beta(a)}\, y.
\end{aligned}$$

This completes the proof. □

A typical example of Hölder's functions is the pre-Hölder's function $\alpha(x) = x^p$, $p > 1$. Indeed, we can notice that for any $a, b > 0$,

$$x^{1/p} y^{1/q} \le \frac{ab^{1-p}}{p}\, x + \frac{a^{1-q} b}{q}\, y,$$

with $\frac{1}{p}+\frac{1}{q}=1$. In this case, $\theta_1=\frac{1}{p}$, $\theta_2=\frac{1}{q}$ are *independent* of a and b.

Now, we define the Lebesgue type function spaces $L_\alpha(X)$:

$$L_\alpha(X) := \{f : f \text{ is a measurable function on } X \text{ satisfying } \|f\|_{L_\alpha} < \infty\},$$

where we set

$$\|f\|_{L_\alpha} := \alpha^{-1}\left(\int_X \alpha(|f(x)|)\, d\mu\right).$$

We point out that the Layer cake representation can be read as

$$\|f\|_{L_\alpha} = \alpha^{-1}\left(\int_0^\infty \alpha'(t)\, \lambda_f(t)\, dt\right),$$

where $\lambda_f(t) := \mu(\{x : |f(x)| > t\})$.

2.2. Hölder's and Minkowski's inequalities on $L_\alpha(X)$

First, we present a Hölder type inequality on the new space $L_\alpha(X)$.

PROPOSITION 2.1 (Hölder's inequality). *Let α be a Hölder's function and β be the corresponding Hölder's conjugate function. Then for any $f \in L_\alpha(X)$ and any $g \in L_\beta(X)$, we have*

$$\left|\int_X f(x)g(x)\, d\mu\right| \le \|f\|_{L_\alpha}\|g\|_{L_\beta}.$$

Proof. The result is obvious if $\|f\|_{L_\alpha}=0$ or if $\|g\|_{L_\beta}=0$. Otherwise, substituting $a := \|g\|_{L_\beta}$ and $b := \|f\|_{L_\alpha}$, there exist θ_1, θ_2 such that

$$\begin{aligned}|f(x)g(x)| &= \alpha^{-1}(\alpha(|f(x)|))\beta^{-1}(\beta(|g(x)|))\\ &\le \theta_1\frac{ab}{\alpha(b)}\,\alpha(|f(x)|) + \theta_2\frac{ab}{\beta(a)}\,\beta(|g(x)|).\end{aligned}$$

Integration of both sides yields

$$\begin{aligned}\int_X |f(x)g(x)|\, d\mu &\le \theta_1\frac{ab}{\alpha(b)}\int_X \alpha(|f(x)|)\, d\mu + \theta_2\frac{ab}{\beta(a)}\int_X \beta(|g(x)|)\, d\mu\\ &= (\theta_1+\theta_2)\|f\|_{L_\alpha}\|g\|_{L_\beta}\\ &\le \|f\|_{L_\alpha}\|g\|_{L_\beta}.\end{aligned}$$

□

As a typical application of Hölder's inequality, we also have Minkowski's inequality on $L_\alpha(X)$.

Proposition 2.2 (Minkowski's inequality). *Let α be a Hölder's function. Then for any $f, g \in L_\alpha(X)$, we have*

$$\|f+g\|_{L_\alpha} \le \|f\|_{L_\alpha} + \|g\|_{L_\alpha}.$$

Proof. Without loss of generality, we may assume that $f(x)+g(x) \neq 0$ almost every $x \in X$ by restricting the domain X if necessary. Hence we have

$$\begin{aligned}\int_X \alpha(|f(x)+g(x)|)\,d\mu &= \int_X \frac{\alpha(|f(x)+g(x)|)}{|f(x)+g(x)|}\,|f(x)+g(x)|\,d\mu \\ &\le \int_X \frac{\alpha(|f(x)+g(x)|)}{|f(x)+g(x)|}\,(|f(x)|+|g(x)|)\,d\mu.\end{aligned}$$

Applying Hölder's inequality, we obtain

$$\begin{aligned}&\int_X \alpha(|f(x)+g(x)|)\,d\mu \\ &\le\ \alpha^{-1}\left(\int_X \alpha(|f(x)|)\,d\mu\right)\beta^{-1}\left(\int_X \beta\left(\frac{\alpha(|f(x)+g(x)|)}{|f(x)+g(x)|}\right)d\mu\right) \\ &\quad +\ \alpha^{-1}\left(\int_X \alpha(|g(x)|)\,d\mu\right)\beta^{-1}\left(\int_X \beta\left(\frac{\alpha(|f(x)+g(x)|)}{|f(x)+g(x)|}\right)d\mu\right) \\ &= (\|f\|_{L_\alpha}+\|g\|_{L_\alpha})\beta^{-1}\left(\int_X \beta\left(\frac{\alpha(|f(x)+g(x)|)}{|f(x)+g(x)|}\right)d\mu\right) \\ &= (\|f\|_{L_\alpha}+\|g\|_{L_\alpha})\beta^{-1}\left(\int_X \alpha(|f(x)+g(x)|)\,d\mu\right).\end{aligned}$$

The last equality follows from the identity (1). Therefore we have

$$\frac{\alpha(\|f+g\|_{L_\alpha})}{\beta^{-1}(\alpha(\|f+g\|_{L_\alpha}))} \le\ \|f\|_{L_\alpha} + \|g\|_{L_\alpha}.$$

By the identity (2), we conclude

$$\|f+g\|_{L_\alpha} \le \|f\|_{L_\alpha} + \|g\|_{L_\alpha}.$$

□

2.3. Completeness of $L_\alpha(X)$

The functional $\|\cdot\|_{L_\alpha}$ on $L_\alpha(X)$ may not produce a norm, since it does not always satisfy the homogeneity required for norms. Instead, by virtue of Minkowski's inequality (Proposition 2.2), we define a metric on $L_\alpha(X)$:

$$d(f,g) := \|f-g\|_{L_\alpha}, \qquad \text{for } f, g \in L_\alpha(X).$$

It produces a complete metric space on $L_\alpha(X)$:

THEOREM 2.3. *The metric space $L_\alpha(X)$ is complete.*

Proof. The proof is very similar to the classical Riesz-Fischer Theorem. □

The following proposition says that $L_\alpha(X)$ is a *topological vector space* with *inhomogeneous* norm $\|\cdot\|_{L_\alpha}$.

PROPOSITION 2.3. *The scalar multiplication* $\cdot : \mathbb{R} \times L_\alpha(X) \to \mathbb{R}$ *is continuous. Furthermore, for* $k \geq 0$

$$\lceil k^{-1} \rceil^{-1} \|f\|_{L_\alpha} \leq \|k\, f\|_{L_\alpha} \leq \lceil k \rceil \, \|f\|_{L_\alpha},$$

where $\lceil k \rceil$ *is the ceiling of* k, *the smallest integer that is not less than* k.

Proof. The monotonicity of α, α^{-1} and Minkowski's inequality deliver

$$\|k\, f\|_{L_\alpha} \leq \|\lceil k \rceil\, f\|_{L_\alpha} \leq \lceil k \rceil \, \|f\|_{L_\alpha}. \tag{9}$$

Hence $f_n \to f$ in $L_\alpha(X)$ implies $kf_n \to kf$ in $L_\alpha(X)$. In turn, it leads $\|f\|_{L_\alpha} = \|\frac{1}{k}\, k\, f\|_{L_\alpha} \leq \lceil \frac{1}{k} \rceil \|k\, f\|_{L_\alpha}$, and so $\lceil \frac{1}{k} \rceil^{-1} \|f\|_{L_\alpha} \leq \|k\, f\|_{L_\alpha}$. Now, it remains to check that $k_n \to k$ and $f \in L_\alpha(X)$ imply that $k_n f \to kf$ in $L_\alpha(X)$. Indeed, the inequality (9) implicitly shows the sequence $\{\alpha(|(k_n - k)f|)\}$ is dominated by a L_α-function $\alpha\,(2M|f|)$ with $M := \left\lceil \max_{n\in\mathbb{N}} |k_n| \right\rceil$, hence by Lebesgue dominated convergence theorem, we have

$$\lim_{n\to\infty} \int_X \alpha(|(k_n - k)f|)d\mu = \int_X \lim_{n\to\infty} \alpha(|(k_n - k)f(x)|)d\mu = 0.$$

This gives the desired convergence. □

Appendix: Proof of Theorem 2.3

Suppose that $\{f_n\}$ is a Cauchy sequence in $L_\alpha(X)$. Then there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $d(f_{n_{k+1}}, f_{n_k}) \leq 2^{-k}$, $k = 1, 2, \cdots$. Setting F with

$$F(x) = |f_{n_1}(x)| + \sum_{k=1}^{\infty} |f_{n_{k+1}}(x) - f_{n_k}(x)|,$$

we can notice that $|F(x)| < \infty$ *almost everywhere* $x \in X$. *In fact, from the fact that*

$$\|F\|_{L_\alpha} \leq \|f_{n_1}\|_{L_\alpha} + \sum_{k=1}^{\infty} \|f_{n_{k+1}} - f_{n_k}\|_{L_\alpha} = \|f_{n_1}\|_{L_\alpha} + 1 < \infty,$$

there exists a null set $N \subset X$ *such that* $F(x) < \infty$ *for all* $x \in N^{\mathsf{C}}$. Therefore for any $x \in N^{\mathsf{C}}$, the absolute convergence of the series $f_{n_1}(x) + \sum_{k=1}^{\infty}[f_{n_{k+1}}(x) - f_{n_k}(x)]$ makes it possible to define $f(x) := \lim_{k\to\infty} f_{n_k}(x)$ on N^{C}. Since $\|f\|_{L_\alpha} \le \|f - f_{n_k}\|_{L_\alpha} + \|f_{n_k}\|_{L_\alpha}$, we have $f \in L_\alpha$. Also, the following fact

$$\|f - f_{n_k}\|_{L_\alpha} = \left\| \sum_{j=k+1}^{\infty} f_{n_{j+1}} - f_{n_j} \right\|_{L_\alpha} \le \sum_{j=k+1}^{\infty} \|f_{n_{j+1}} - f_{n_j}\|_{L_\alpha} = 2^{-k}$$

yields the convergence of $\{f_{n_k}\}$ to f in $L_\alpha(X)$, which, in turn, implies the convergence of the original *Cauchy* sequence $\{f_n\}$ in $L_\alpha(X)$.

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