JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 21, No. 4, December 2008

A WEIERSTRASS POINT OF $\Gamma_1(4p)$

SoYoung Choi*

Abstract. We show that the cusp 1/2 is a Weierstrass point of $\Gamma_1(4p)$ if p is a prime greater than 7.

1. Introduction

At an arbitrary point P of Riemann surface S, genus $g \geq 2$, there is in general no element of K that has a pole of order $\leq g$ at P and is regular elsewhere. Those points P for which a function exists are the Weierstrass points, where K is the algebraic function field of S.

For a congruence subgroup Γ of $SL_2(\mathbb{Z})$ we have a Riemann surface $S = \Gamma\backslash\mathfrak{H}^*$. We may then speak of the Weierstrass points of Γ. J. Lehner and M. Newman studied the Wierestrass points of $\Gamma_0(n)$ in [2] as follows : they showed that cusps 0 and ∞ are Wierestrass points of $\Gamma_0(n)$ if $n \equiv 0 \pmod 4$ or $\pmod 9$ and that some interior points of of $\Gamma_0(n)$ are Wierestrass points of $\Gamma_0(n)$ for all positive integer n but finitely many integer n. The Wierestrass points of $\Gamma(n)$ were studied by Schoeneberg, Bruno in [3]. In this paper we show that the cusp 1/2 is a weierstrass point of $\Gamma_1(4p)$ for a prime integer $p > 7$. Historically, Weierstrass points were used by Schwartz and Hurwitz to determine an automorphism group of a compact Riemann surface of genus g greater than 1. Meanwhile, recently, those points play an important role in the study of algebraic geometric coding theory, that is, if we know a Weierstrass nongap sequence of a Weierstrass point then we are able to estimate parameters of codes in a concrete way. Therefore, the result in this paper is worth of further research.

Received August 28, 2008; Accepted November 20, 2008.
2000 Mathematics Subject Classification: Primary 11F, 14H55.
Key words and phrases: Weierstrass points, congruence subgroups.

2. Premilaries

Let $\Gamma_1(n)$ be the congruence subgroup of $SL_2(\mathbb{Z})$ whose elements are congruent to $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$ mod n ($n = 1, 2, 3, \cdots$). Let $\mathfrak{N}(\Gamma_1(n))$ be the normalizer in $SL_2(\mathbb{R})/\{\pm 1\}$ of $\Gamma_1(n)$. For any $W \in \mathfrak{N}(\Gamma_1(n))$ of order 2 in the factor group $\mathfrak{N}(\Gamma_1(n))/\Gamma_1(n)$, let $< W, \Gamma_1(n) >$ be the group generated by W and $\Gamma_1(n)$. Denote the genus of this group by g_W and g is the genus of $\Gamma_1(n)$. Then Schoeneberg gave the following

Proposition 2.1. *Let $P_{\Gamma_1(n)}$ be the set of all cups of $\Gamma_1(n)$ and α be the fixed point of W in $\mathfrak{H} \cup P_{\Gamma_1(n)}$. Then α is a weierstrass points of $\Gamma_1(n)$ if $g_W \neq [g/2]$ and $g \geq 2$.*

Proof. See [3] or [2, Schoeneberg)]. □

We need some facts related to $\Gamma_1(n)$.

Proposition 2.2. *let $a, c, a^{'}, c^{'} \in \mathbb{Z}$ with $gcd(a, c) = gcd(a^{'}, c^{'}) = 1$.*
(1) a/c and $a^{'}/c^{'}$ are $\Gamma_1(n)$-equivalent $\Leftrightarrow \pm \begin{pmatrix} a^{'} \\ c^{'} \end{pmatrix} \equiv \left(\begin{smallmatrix} a+mc \\ c \end{smallmatrix}\right)$ (mod n) for some $m \in \mathbb{Z}$.
(2) If p is an odd prime number, then $1/2, 1/(2p), 3/(2p), \cdots, (p-2)/(2p)$ are $\Gamma_1(4p)$-inequivalent cusps.

Proof. For (1) see [1, Lemma 3]. By the assertion (1) we know that $1/2$ and $i/(2p)$ are $\Gamma_1(4p)$-inequivalent cusps. If $i/(2p)$ and $j/(2p)$ are $\Gamma_1(4p)$-equivalent cusps for some $i, j \in \{1, 3, \cdots, p-2\}$, then by (1) we have that $j \equiv \pm i$ (mod $2p$). But $|j \pm i| < 2p$. Hence $i = j$. This implies the assertion (2) holds.

□

3. Schoenberg's theorem applied to cups of $\Gamma_1(4p)$

In this section, we assume that p is a prime number greater than 7. Let
$\mathfrak{H}$: the complex upper half plane
$\mathfrak{H}^*$: the extended complex upper half plane
$W = \begin{pmatrix} 2p+1 & -p \\ 4p & -2p+1 \end{pmatrix} \in \Gamma_1(2p)$.
g_W : the genus of $< \Gamma_1(4p), W >$
g : the genus of $\Gamma_1(4p)$
Then easily we see that W is contained in $\mathfrak{N}(\Gamma_1(4p))$ and has order 2 in the factor group $\mathfrak{N}(\Gamma_1(4p))/\Gamma_1(4p)$.

THEOREM 3.1. *The cusp* $1/2$ *is a Weierstrass point of* $\Gamma_1(4p)$.

Proof. Consider the natural commutative diagram:

$$\begin{array}{ccc} \mathfrak{H}^* & \xrightarrow{id} & \mathfrak{H}^* \\ \pi \downarrow & & \downarrow \pi_W \\ \Gamma_1(4p)\backslash\mathfrak{H}^* & \xrightarrow{\varphi} & <\Gamma_1(4p), W> \backslash\mathfrak{H}^* \end{array}$$

By the Hurwitz formula applied to the natural projection φ, we obtain

$$2g-2 = 2(2g_W - 2) + \sum_{q \in \Gamma_1(4p)\backslash\mathfrak{H}^*} (e_q - 1).$$

We notice that if $z \in \mathfrak{H}^*$, then we have $e_{\pi(z)} = [\overline{<\Gamma_1(4p), W>_z} : \overline{\Gamma_1(4p)_z}]$ ([4, Proposition 1.37]), where $\overline{\Gamma}$ means $\Gamma/\{\pm 1\}$ for a subgroup Γ of $SL_2(\mathbb{R})$.

Since the degree of φ is two, $e_{\pi(z)} = 1$ or 2 for $z \in \mathfrak{H}^*$. Where π is the canonical map from $\mathfrak{H}^*$ into $\Gamma_1(4p)\backslash\mathfrak{H}^*$. We know that by [4, Proposition 1.37] if a cusp z of $\Gamma_1(4p)$ and Wz are $\Gamma_1(4p)$-equivalent, then $e_{\pi(z)} = 2$. We will find such cusps of $\Gamma_1(4p)$. Let $i \in \{1, 3, \cdots, p-2\}$. Then

$$W\frac{i}{2p} = \frac{i + 2p(i-p)}{2p(2i - 2p + 1)}.$$

Suppose that q is any prime divisor of $2i-2p+1$ and let $2i-2p+1 = q\alpha$ for some integer α. Then we have that $i + 2p(i-p) = i + p(q\alpha - 1) \equiv i - p \pmod q$. If q divides $i+2p(i-p)$, then q divides $i-p$ and so does 1. This is impossible. Thus, we obtain $\gcd(i+2p(i-p), 2p(2i-2p+1)) = 1$. Now applying Proposition 2.2(1) to cusps $i/(2p)$ and $W\frac{i}{2p}$, we have that

$$W\frac{i}{2p} \text{ and } \frac{i}{2p} \text{ are } \Gamma_1(4p)\text{-equivalent.}$$

Therefore for a cusp $i/(2p)$, we have $e_{\pi(i/(2p))} = 2$ ($i \in \{1, 3, \cdots, p-2\}$). Moreover $e_{\pi(1/2)} = 2$ because W fixes $1/2$. We now obtain

$$\sum_{q \in \Gamma_1(4p)\backslash\mathfrak{H}^*} (e_q - 1) \geq \frac{p-1}{2} + 1$$

and so

$$g \geq 2g_W + \frac{p-3}{4} \geq 2g_W + 2$$

which implies $[g/2] \neq g_w$. Consequently, Proposition 2.1 implies that the fixed point $1/2$ of W is a Weierstrass point of $\Gamma_1(4p)$. $\square$

References

[1] C. H. Kim and J. K. Koo, Self-recursion formulas satisfied by Fourier coefficients of some modular functions. J. Pure Appl. Algebra **160** (2001), no. 1, 53–65.

[2] J. Lehner and M. Newman. Weierstrass points of $\Gamma_0(n)$, Ann. of Math. (2) **79** (1964) 360–368.

[3] B. Schoeneberg, Uber die Weierstrass-Punkte in den Korpern der elliptischen Modulfunktionen, (German) Abh. Math. Sem. Univ. Hamburg **17** (1951). 104–111.

[4] G. Shimura, Introduction to the Arithmetic Theory of Automorphic Functions, Publ. Math. Soc. Japan, No11. Tokyo Prineton, 1971.

*

Department of Mathematics Education
Dongguk University
Gyeongju 780-714, Republic of Korea.
E-mail: young@dongguk.ac.kr