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FUNCTIONAL EQUATIONS ASSOCIATED WITH INNER PRODUCT SPACES

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Abstract. In [7], Th.M. Rassias proved that the norm defined over a real vector space V is induced by an inner product if and only if for a fixed integer $n \geq 2$

$$n\left\|\frac{1}{n}\sum_{i=1}^{n} x_i\right\|^2 + \sum_{i=1}^{n}\left\|x_i - \frac{1}{n}\sum_{j=1}^{n} x_j\right\|^2 = \sum_{i=1}^{n}\|x_i\|^2$$

holds for all $x_1, \cdots, x_n \in V$.

Let V, W be real vector spaces. It is shown that if a mapping $f : V \to W$ satisfies

$$(0.1) \qquad nf\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) + \sum_{i=1}^{n} f\left(x_i - \frac{1}{n}\sum_{j=1}^{n} x_j\right) = \sum_{i=1}^{n} f(x_i)$$

for all $x_1, \cdots, x_n \in V$, then the mapping $f : V \to W$ satisfies

$$(0.2) \qquad 2f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) + f\left(\frac{y-x}{2}\right) = f(x) + f(y)$$

for all $x, y \in V$.

Furthermore, we prove the generalized Hyers-Ulam stability of the functional equation (0.2) in real Banach spaces.

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1. Introduction

The stability problem of functional equations originated from a question of Ulam [15] concerning the stability of group homomorphisms. Hyers [5] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [6] for linear mappings by considering an unbounded Cauchy difference. The paper of Th.M. Rassias [6] has provided a lot of influence in the development of what we call *generalized Hyers-Ulam stability* of functional equations. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [4] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach.

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. A generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [14] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [2] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. In [3], Czerwik proved the generalized Hyers-Ulam stability of the quadratic functional equation. Several functional equations have been investigated in [8]–[13].

Throughout this paper, assume that n is a fixed integer greater than 1. Let X be a real normed vector space with norm $||\cdot||$, and Y a real Banach space with norm $\|\cdot\|$.

In this paper, we investigate the functional equation (0.2), and prove the generalized Hyers-Ulam stability of the functional equation (0.2) in real Banach spaces.

2. Jensen quadratic mappings associated with inner product spaces

We investigate the functional equations (0.1) and (0.2).

LEMMA 2.1. *Let V and W be real vector spaces. If a mapping $f : V \to W$ satisfies*

$$(2.1) \quad nf\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) + \sum_{i=1}^{n} f\left(x_i - \frac{1}{n}\sum_{j=1}^{n} x_j\right) = \sum_{i=1}^{n} f(x_i)$$

for all $x_1, \cdots, x_n \in V$, then the mapping $f : V \to W$ satisfies

$$(2.2) \quad 2f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) + f\left(\frac{y-x}{2}\right) = f(x) + f(y)$$

for all $x, y \in V$.

Proof. Assume that $f : V \to W$ satisfies (2.1).

Letting $x_n = \frac{\sum_{i=1}^{n-1} x_i}{n-1}$, we get

$$(n-1)f\left(\frac{1}{n-1}\sum_{i=1}^{n-1} x_i\right) + \sum_{i=1}^{n-1} f\left(x_i - \frac{1}{n-1}\sum_{j=1}^{n-1} x_j\right) = \sum_{i=1}^{n-1} f(x_i)$$

for all $x_1, \cdots, x_{n-1} \in V$. Applying continuously this method $n-3$ times, we get

$$2f\left(\frac{x_1+x_2}{2}\right) + f\left(\frac{x_1-x_2}{2}\right) + f\left(\frac{x_2-x_1}{2}\right) = f(x_1) + f(x_2)$$

for all $x_1, x_2 \in V$, as desired. □

One can easily show that an even mapping $f : V \to W$ satisfies (2.2) if and only if the even mapping $f : V \to W$ is a Jensen quadratic mapping, i.e.,

$$2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) = f(x) + f(y),$$

and that an odd mapping $f : V \to W$ satisfies (2.2) if and only if the odd mapping mapping $f : V \to W$ is a Jensen additive mapping, i.e.,

$$2f\left(\frac{x+y}{2}\right) = f(x) + f(y).$$

For a given mapping $f : X \to Y$, we define

$$Df(x,y) := 2f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) + f\left(\frac{y-x}{2}\right) - f(x) - f(y)$$

for all $x, y \in X$.

We prove the generalized Hyers-Ulam stability of the functional equation $Df(x,y) = 0$ in real Banach spaces: an even case.

THEOREM 2.2. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\varphi : X^2 \to [0,\infty)$ such that*

$$\widetilde{\varphi}(x,y) : = \sum_{j=0}^{\infty} 4^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty, \tag{2.3}$$

$$\|Df(x,y)\| \leq \varphi(x,y) \tag{2.4}$$

for all $x, y \in X$. Then there exists a unique Jensen quadratic mapping $Q : X \to Y$ such that

$$\|f(x) + f(-x) - Q(x)\| \leq \widetilde{\varphi}(x,0) + \widetilde{\varphi}(-x,0) \tag{2.5}$$

for all $x \in X$.

Proof. Letting $y = 0$ in (2.4), we get

$$\left\|3f\left(\frac{x}{2}\right) + f\left(\frac{-x}{2}\right) - f(x)\right\| \leq \varphi(x,0) \tag{2.6}$$

for all $x \in X$. Replacing x by $-x$ in (2.6), we get

$$\left\|3f\left(\frac{-x}{2}\right) + f\left(\frac{x}{2}\right) - f(-x)\right\| \leq \varphi(-x,0) \tag{2.7}$$

for all $x \in X$. Let $g(x) := f(x) + f(-x)$ for all $x \in X$. It follows from (2.6) and (2.7) that

$$\left\|g(x) - 4g\left(\frac{x}{2}\right)\right\| \leq \varphi(x,0) + \varphi(-x,0) \tag{2.8}$$

for all $x \in X$. Hence

$$\|4^l g(\frac{x}{2^l}) - 4^m g(\frac{x}{2^m})\| \leq \sum_{j=l}^{m-1} 4^j \varphi\left(\frac{x}{2^j}, 0\right) + \sum_{j=l}^{m-1} 4^j \varphi\left(-\frac{x}{2^j}, 0\right) \tag{2.9}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.3) and (2.9) that the sequence $\{4^k g(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{4^k g(\frac{x}{2^k})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{k\to\infty} 4^k g\left(\frac{x}{2^k}\right)$$

for all $x \in X$.

By (2.3) and (2.4),

$$\begin{aligned}\|DQ(x,y)\| &= \lim_{k\to\infty} 4^k \left\|Dg\left(\frac{x}{2^k}, \frac{y}{2^k}\right)\right\| \\ &\leq \lim_{k\to\infty} 4^k \left(\varphi\left(\frac{x}{2^k}, \frac{y}{2^k}\right) + \varphi\left(-\frac{x}{2^k}, -\frac{y}{2^k}\right)\right) = 0\end{aligned}$$

for all $x, y \in X$. So $DQ(x, y) = 0$. Since $g : X \to Y$ is even, $Q : X \to Y$ is even. So the mapping $Q : X \to Y$ is Jensen quadratic. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.9), we get (2.5). So there exists a Jensen quadratic mapping $Q : X \to Y$ satisfying (2.5).

Now, let $Q' : X \to Y$ be another Jensen quadratic mapping satisfying (2.5). Then we have

$$\begin{aligned} \|Q(x) - Q'(x)\| &= 4^q \left\| Q\left(\frac{x}{2^q}\right) - Q'\left(\frac{x}{2^q}\right) \right\| \\ &\le 4^q \left\| Q\left(\frac{x}{2^q}\right) - f\left(\frac{x}{2^q}\right) - f\left(\frac{-x}{2^q}\right) \right\| \\ &+ 4^q \left\| Q'\left(\frac{x}{2^q}\right) - f\left(\frac{x}{2^q}\right) - f\left(\frac{-x}{2^q}\right) \right\| \\ &\le 2 \cdot 4^q \widetilde{\varphi}\left(\frac{x}{2^q}, 0\right) + 2 \cdot 4^q \widetilde{\varphi}\left(\frac{-x}{2^q}, 0\right), \end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $Q(x) = Q'(x)$ for all $x \in X$. This proves the uniqueness of Q. □

Corollary 2.3. *Let $p > 2$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping such that*

$$\|Df(x, y)\| \le \theta(||x||^p + ||y||^p) \tag{2.10}$$

for all $x, y \in X$. Then there exists a unique Jensen quadratic mapping $Q : X \to Y$ such that

$$\|f(x) + f(-x) - Q(x)\| \le \frac{2^{p+1}\theta}{2^p - 4} ||x||^p$$

for all $x \in X$.

Proof. Define $\varphi(x, y) = \theta(||x||^p + ||y||^p)$, and apply Theorem 2.2 to get the desired result. □

Theorem 2.4. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\varphi : X^2 \to [0, \infty)$ satisfying (2.4) such that*

$$\widetilde{\varphi}(x, y) := \sum_{j=1}^{\infty} 4^{-j} \varphi(2^j x, 2^j y) < \infty \tag{2.11}$$

for all $x, y \in X$. Then there exists a unique Jensen quadratic mapping $Q : X \to Y$ such that

$$\|f(x) + f(-x) - Q(x)\| \le \widetilde{\varphi}(x, 0) + \widetilde{\varphi}(-x, 0) \tag{2.12}$$

for all $x \in X$.

Proof. It follows from (2.8) that

$$\left\|g(x) - \frac{1}{4}g(2x)\right\| \leq \frac{1}{4}\varphi(2x,0) + \frac{1}{4}\varphi(-2x,0)$$

for all $x \in X$. So

$$\left\|\frac{1}{4^l}g(2^l x) - \frac{1}{4^m}g(2^m x)\right\| \leq \sum_{j=l+1}^{m} \frac{1}{4^j}\varphi(2^j x,0) + \sum_{j=l+1}^{m} \frac{1}{4^j}\varphi(-2^j x,0) \tag{2.13}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.11) and (2.13) that the sequence $\{\frac{1}{4^k}g(2^k x)\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^k}g(2^k x)\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{k\to\infty} \frac{1}{4^k}g(2^k x)$$

for all $x \in X$.

By (2.4) and (2.11),

$$\begin{aligned}\|DQ(x,y)\| &= \lim_{k\to\infty} \frac{1}{4^k}\|Dg(2^k x, 2^k y)\| \\ &\leq \lim_{k\to\infty} \frac{1}{4^k}(\varphi(2^k x, 2^k y) + \varphi(-2^k x, -2^k y)) = 0\end{aligned}$$

for all $x, y \in X$. So $DQ(x,y) = 0$. Since $g : X \to Y$ is even, $Q : X \to Y$ is even. So the mapping $Q : X \to Y$ is Jensen quadratic. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.13), we get (2.12). So there exists a Jensen quadratic mapping $Q : X \to Y$ satisfying (2.12).

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let $p < 2$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying* (2.10). *Then there exists a unique Jensen quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) + f(-x) - Q(x)\| \leq \frac{2^{p+1}\theta}{4 - 2^p}||x||^p$$

for all $x \in X$.

Proof. Define $\varphi(x,y) = \theta(||x||^p + ||y||^p)$, and apply Theorem 2.4 to get the desired result. □

3. Jensen additive mappings associated with inner product spaces

In this section, we prove the generalized Hyers-Ulam stability of the functional equation $Df(x,y) = 0$ in real Banach spaces: an odd case.

THEOREM 3.1. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\varphi : X^2 \to [0,\infty)$ such that*

$$\Phi(x,y) := \sum_{j=0}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty, \tag{3.1}$$

$$\|Df(x,y)\| \leq \varphi(x,y) \tag{3.2}$$

for all $x, y \in X$. Then there exists a unique Jensen additive mapping $A : X \to Y$ such that

$$\|f(x) - f(-x) - A(x)\| \leq \Phi(x,0) + \Phi(-x,0) \tag{3.3}$$

for all $x \in X$.

Proof. Letting $y = 0$ in (3.2), we get

$$\left\|3f\left(\frac{x}{2}\right) + f\left(\frac{-x}{2}\right) - f(x)\right\| \leq \varphi(x,0) \tag{3.4}$$

for all $x \in X$. Replacing x by $-x$ in (3.4), we get

$$\left\|3f\left(\frac{-x}{2}\right) + f\left(\frac{x}{2}\right) - f(-x)\right\| \leq \varphi(-x,0) \tag{3.5}$$

for all $x \in X$. Let $h(x) := f(x) - f(-x)$ for all $x \in X$. It follows from (3.4) and (3.5) that

$$\left\|h(x) - 2h\left(\frac{x}{2}\right)\right\| \leq \varphi(x,0) + \varphi(-x,0) \tag{3.6}$$

for all $x \in X$. Hence

$$\begin{aligned} \|2^l h(\frac{x}{2^l}) - 2^m h(\frac{x}{2^m})\| &\leq \sum_{j=l}^{m-1} 2^j \varphi\left(\frac{x}{2^j}, 0\right) \\ &+ \sum_{j=l}^{m-1} 2^j \varphi\left(-\frac{x}{2^j}, 0\right) \end{aligned} \tag{3.7}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.1) and (3.7) that the sequence $\{2^k h(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{2^k h(\frac{x}{2^k})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty} 2^k h\left(\frac{x}{2^k}\right)$$

for all $x \in X$.

By (3.1) and (3.2),

$$\begin{aligned} \|DA(x,y)\| &= \lim_{k\to\infty} 2^k \left\| Dh\left(\frac{x}{2^k}, \frac{y}{2^k}\right)\right\| \\ &\leq \lim_{k\to\infty} 2^k \left(\varphi\left(\frac{x}{2^k}, \frac{y}{2^k}\right) + \varphi\left(-\frac{x}{2^k}, -\frac{y}{2^k}\right)\right) = 0 \end{aligned}$$

for all $x, y \in X$. So $DA(x,y) = 0$. Since $h : X \to Y$ is odd, $A : X \to Y$ is odd. So the mapping $A : X \to Y$ is Jensen additive. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.7), we get (3.3). So there exists a Jensen additive mapping $A : X \to Y$ satisfying (3.3).

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 3.2. *Let $p > 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping such that*

$$\|Df(x,y)\| \leq \theta(||x||^p + ||y||^p) \tag{3.8}$$

for all $x, y \in X$. Then there exists a unique Jensen additive mapping $A : X \to Y$ such that

$$\|f(x) - f(-x) - A(x)\| \leq \frac{2^{p+1}\theta}{2^p - 2}||x||^p$$

for all $x \in X$.

Proof. Define $\varphi(x,y) = \theta(||x||^p + ||y||^p)$, and apply Theorem 3.1 to get the desired result. □

Theorem 3.3. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\varphi : X^2 \to [0,\infty)$ satisfying (3.2) such that*

$$\Phi(x,y) := \sum_{j=1}^{\infty} 2^{-j}\varphi(2^j x, 2^j y) < \infty \tag{3.9}$$

for all $x, y \in X$. Then there exists a unique Jensen additive mapping $A : X \to Y$ such that

$$\|f(x) - f(-x) - A(x)\| \leq \Phi(x,0) + \Phi(-x,0) \tag{3.10}$$

for all $x \in X$.

Proof. It follows from (3.6) that

$$\left\| h(x) - \frac{1}{2}h(2x) \right\| \leq \frac{1}{2}\varphi(2x,0) + \frac{1}{2}\varphi(-2x,0)$$

for all $x \in X$. So

$$\left\| \frac{1}{2^l}h(2^l x) - \frac{1}{2^m}h(2^m x) \right\| \leq \sum_{j=l+1}^{m} \frac{1}{2^j}\varphi(2^j x, 0) + \sum_{j=l+1}^{m} \frac{1}{2^j}\varphi(-2^j x, 0) \tag{3.11}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.9) and (3.11) that the sequence $\{\frac{1}{2^k}h(2^k x)\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^k}h(2^k x)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty} \frac{1}{2^k}h(2^k x)$$

for all $x \in X$.

By (3.2) and (3.9),

$$\begin{aligned} \|DA(x,y)\| &= \lim_{k\to\infty} \frac{1}{2^k}\|Dh(2^k x, 2^k y)\| \\ &\leq \lim_{k\to\infty} \frac{1}{2^k}(\varphi(2^k x, 2^k y) + \varphi(-2^k x, -2^k y)) = 0 \end{aligned}$$

for all $x, y \in X$. So $DA(x,y) = 0$. Since $h : X \to Y$ is odd, $A : X \to Y$ is odd. So the mapping $A : X \to Y$ is Jensen additive. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.11), we get (3.10). So there exists a Jensen additive mapping $A : X \to Y$ satisfying (3.10).

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 3.4. *Let $p < 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying* (3.8). *Then there exists a unique Jensen additive mapping $A : X \to Y$ such that*

$$\|f(x) - f(-x) - A(x)\| \leq \frac{2^{p+1}\theta}{2 - 2^p}||x||^p$$

for all $x \in X$.

Proof. Define $\varphi(x,y) = \theta(||x||^p + ||y||^p)$, and apply Theorem 3.3 to get the desired result. □

Note that

$$\sum_{j=0}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) \le \sum_{j=0}^{\infty} 4^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right).$$

Combining Theorem 2.2 and Theorem 3.1, we obtain the following result.

THEOREM 3.5. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\varphi : X^2 \to [0, \infty)$ satisfying* (2.3) *and* (2.4)*. Then there exist a unique Jensen additive mapping $A : X \to Y$ and a unique Jensen quadratic mapping $Q : X \to Y$ such that*

$$\|2f(x) - A(x) - Q(x)\| \le \widetilde{\varphi}(x, 0) + \widetilde{\varphi}(-x, 0) + \Phi(x, 0) + \Phi(-x, 0)$$

for all $x \in X$, where $\widetilde{\varphi}$ and Φ are defined in (2.3) *and* (3.1)*, respectively.*

COROLLARY 3.6. *Let $p > 2$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying* (2.10)*. Then there exist a unique Jensen additive mapping $A : X \to Y$ and a unique Jensen quadratic mapping $Q : X \to Y$ such that*

$$\|2f(x) - A(x) - Q(x)\| \le \left(\frac{2^{p+1}}{2^p - 2} + \frac{2^{p+1}}{2^p - 4}\right)\theta||x||^p$$

for all $x \in X$.

Proof. Define $\varphi(x, y) = \theta(||x||^p + ||x||^p)$, and apply Theorem 3.5 to get the desired result. □

Note that

$$\sum_{j=1}^{\infty} 4^{-j}\varphi(2^j x, 2^j y) \le \sum_{j=1}^{\infty} 2^{-j}\varphi(2^j x, 2^j y).$$

Combining Theorem 2.4 and Theorem 3.3, we obtain the following result.

THEOREM 3.7. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\varphi : X^2 \to [0, \infty)$ satisfying* (2.4) *and* (3.9)*. Then there exist a unique Jensen additive mapping $A : X \to Y$ and a unique Jensen quadratic mapping $Q : X \to Y$ such that*

$$\|2f(x) - A(x) - Q(x)\| \le \widetilde{\varphi}(x, 0) + \widetilde{\varphi}(-x, 0) + \Phi(x, 0) + \Phi(-x, 0)$$

for all $x \in X$, where $\widetilde{\varphi}$ and Φ are defined in (2.11) *and* (3.9)*, respectively.*

Corollary 3.8. *Let $p < 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying* (2.10)*. Then there exist a unique Jensen additive mapping $A : X \to Y$ and a unique Jensen quadratic mapping $Q : X \to Y$ such that*

$$\|2f(x) - A(x) - Q(x)\| \leq \left(\frac{2^{p+1}}{2 - 2^p} + \frac{2^{p+1}}{4 - 2^p} \right) \theta ||x||^p$$

for all $x \in X$.

Proof. Define $\varphi(x, y) = \theta(||x||^p + ||y||^p)$, and apply Theorem 3.7 to get the desired result. □

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