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CRITICAL POINTS RESULT FOR THE $C^{1,1}$ FUNCTIONAL AND THE RELATIVE CATEGORY THEORY

Tacksun Jung* and Q-Heung Choi**

Abstract. We show the existence of at least four nontrivial critical points of the $C^{1,1}$ functional f on the Hilbert space $H = X_0 \oplus X_1 \oplus X_2 \oplus X_3 \oplus X_4$, X_i, $i = 0, 1, 2, 3$ are finite dimensional, with $f(0) = 0$ when two sublevel subsets, torus with three holes and sphere, of f link, the functional f satisfies sup-inf variatinal linking inequality on the linking subspaces, the functional f satisfies $(P.S.)_c$ condition, and $f|_{X_0 \oplus X_4}$ has no critical point with level c. We use the deformation lemma, the relative category theory and the critical point theory for the proof of main result.

1. Introduction and statement of main result

Let f be a $C^{1,1}$ functional defined on a Hilbert space H with $f(0) = 0$. Here H is a Hilbert space which is a direct sum of five closed subspaces X_0, X_1, X_2, X_3 and X_4 with X_0, X_1, X_2, X_3 of finite dimension. In this paper we investigate the number of nontrivial critical points of the $C^{1,1}$ functional f on H under some conditions on the sublevels sets, torus with three holes and sphere, of f and the shape of f. Micheletti and Saccon prove in [4] that the functional $f \in C^{1,1}(H, R)$ has at least two nontrivial critical points under the conditions that $H = X_0 \oplus X_1 \oplus X_2$, X_1 is finite dimensional, the sublevel sets are the Torue with one hole and the sphere, f satisfies the Torus-Sphere variational linking inequality, f satisfies the $(P.S.)_c$ condition, and $f|_{X_0 \oplus X_2}$ has no critical point with level c. In this paper we improve this result to the case that the sublevel sets are the torus with three holes and sphere. We show the existence of at least four nontrivial critical points of f on H when the sublevels

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sets, the torus with three holes and sphere, of f link, the functional f satisfies the Sphere-Torus with three holes variational linking inequality, the functional f satisfies $(P.S.)_c$ condition, and $f|_{X_0 \oplus X_4}$ has no critical point with level c.

Now, we state the main result:

THEOREM 1.1. *Let $f : H \to R$ be a $C^{1,1}$ functional defined on a Hilbert space $H = X_0 \oplus X_1 \oplus X_2 \oplus X_3 \oplus X_4$ with $f(0) = 0$, where X_0, X_1, X_2, X_3 and X_4 are five closed subspaces of H and X_0, X_1, X_2, X_3 are finite dimensional subspaces. Let $\rho_1 > 0$, $\rho_2 > 0$, $\rho_3 > 0$, $r > 0$, $R > 0$ with $r < R$ and $R_1 > 0$; we define*

$$S_i(\rho_i) = \{z \in X_i |\ \|z\| = \rho_i\},\ i = 1, 2, 3.$$

$$S_i(\rho_i) - w_i = \{z - w_i |\ z \in S_i(\rho_i), w_i \in X_i\},\ i = 1, 2, 3.$$

$$\begin{aligned}
&\Delta^3_R(S_1(\rho_1) - w_1, S_2(\rho_2) - w_2, S_3(\rho_3) - w_3, X_4)\\
&= \{z = (z_1 - w_1) + (z_2 - w_2) + (z_3 - w_3) + z_4 |\ z_i \in X_i,\ i = 1, 2, 3, 4,\\
&\ \rho_1 \le \|z_1 - w_1\| \le R,\ \rho_2 \le \|z_2 - w_2\| \le R,\\
&\rho_3 \le \|z_3 - w_3\| \le R,\ \|z_4\| \le R_1,\ \|z\| \le R\},
\end{aligned}$$

$$\begin{aligned}
&\Sigma^3_R(S_1(\rho_1) - w_1, S_2(\rho_2) - w_2, S_3(\rho_3) - w_3, X_4)\\
&= \{z = (z_1 - w_1) + (z_2 - w2) + (z_3 - w_3) + z_4 |\ z_i \in X_i,\ i = 1, 2, 3, 4,\\
&\|z_4\| \le R_1,\ \|z_1 - w_1\| = \rho_1,\ \|z_2 - w_2\| = \rho_2,\\
&\ \|z_3 - w_3\| = \rho_3,\ \|z\| = R\}\\
&\cap\{z = z_1 + z_2 + z_3 + z_4 |\ z_i \in X_i,\ i = 1, 2, 3, 4,\ \|z_4\| = R_1,\\
&\qquad \rho_1 \le \|z_1 - w_1\| \le R,\ \|z\| = R\ , w_1 \in X_1\}\\
&\cap\{z = z_1 + z_2 + z_3 + z_4 |\ \ z_i \in X_i,\ \ , i = 1, 2, 3, 4,\\
&\qquad \|z_4\| = R_1,\ \rho_2 \le \|z_2 - w_2\| \le R,\ \|z\| = R\ , w_2 \in X_2\}\\
&\cap\{z = z_1 + z_2 + z_3 + z_4 |\ z_i \in X_i,\ i = 1, 2, 3, 4,\ \ \|z_4\| = R_1,\\
&\qquad \rho_3 \le \|z_3 - w_3\| \le R,\ \|z\| = R,\ w_3 \in X_3\}.
\end{aligned}$$

Let

$$\alpha = \inf_{\Delta^3_R(S_1(\rho_1) - w_1, S_2(\rho_2) - w_2, S_3(\rho_3) - w_3, X_4)} f(z),$$

$$\beta = \sup_{S_r(X_0 \oplus X_1 \oplus X_2 \oplus X_3)} f(z).$$

(i) Assume that

$$\sup_{z \in S_r(X_0 \oplus X_1 \oplus X_2 \oplus X_3)} f(z) < \inf_{z \in \Sigma^3_R(S_1(\rho_1) - w_1, S_2(\rho_2) - w_2, S_3(\rho_3) - w_3, X_4)} f(z).$$

(ii) Assume that the $(P.S.)_c$ condition holds for f, $\forall c \in [\alpha, \beta]$.
(iii) Assume that $f|_{X_0 \oplus X_4}$ has no critical points with $\alpha \leq f(z) \leq \beta$.
(iv) Moreover we assume $\beta < +\infty$.
Then there exist at least four nontrivial critical points z_1, z_2, z_3 and z_4 for f in $X_1 \oplus X_2 \oplus X_3$ such that

$$\inf_{z \in \Delta^3_R(S_1(\rho_1)-w_1, S_2(\rho_2)-w_2, S_3(\rho_3)-w_3, X_4)} f(z)$$
$$\leq f(z_i) \leq \sup_{z \in S_r(X_0 \oplus X_1 \oplus X_2 \oplus X_3)} f(z), \quad i = 1, 2, 3, 4.$$

In section 2, we recall the notion of the relative category, the deformation lemma and the multiplicity theorem in [4]. In section 3, by the deformation lemma, the relative category theory and the multiplicity theorem in [4] we prove the main theorem.

2. Critical point theory on the manifold

Now, we will consider the critical point theory on the manifold with boundary. Let H be a Hilbert space and M be the closure of an open subset of H such that M can be endowed with the structure of C^2 manifold with boundary. Let $f : W \to R$ be a $C^{1,1}$ functional, where W is an open set containing M. For applying the usual topological methods of critical points theory we need a suitable notion of critical point for f on M. We recall the following notions: lower gradient of f on M, $(P.S.)_c$ condition and the relative category (see [3]).

DEFINITION 2.1. If $u \in M$, the lower gradient of f on M at u is defined by

$$grad^-_M f(u) = \begin{cases} \nabla f(u) & \text{if } u \in int(M), \\ \nabla f(u) + [< \nabla f(u), \nu(u) >]^- \nu(u) & \text{if } u \in \partial M, \end{cases}$$

where we denote by $\nu(u)$ the unit normal vector to ∂M at the point u, pointing outwards.

We say that u is a lower critical for f on M, if $grad^-_M f(u) = 0$.

DEFINITION 2.2. Let $c \in R$. We say that f satisfies the $(P.S.)_c$ condition for $c \in R$, on the manifold with boundary M, if for any sequence $(u_n)_n$ in M such that $f(u_n) \to c$, $grad^-_M f(u_n) \to 0$, there exists a subsequence of $(u_n)_n$ which converges to a point $u \in M$ such that $grad^-_M f(u) = 0$.

Let Y be a closed subspace of M.

Definition 2.3. Let B be a closed subset of M with $Y \subset B$. We define the relative category $cat_{M,Y}(B)$ of B in (M,Y), as the least integer h such that there exist $h+1$ closed subsets U_0, $U_1, \ldots, U_h$ with the following properties:
$B \subset U_0 \cup U_1 \cup \ldots \cup U_h$;
$U_1, \ldots, U_h$ are contractible in M;
$Y \subset U_0$ and there exists a continuous map $F : U_0 \times [0,1] \to M$ such that

$$\begin{aligned} F(x,0) &= x \qquad \forall x \in U_0, \\ F(x,t) &\in Y \qquad \forall x \in Y,\ \forall t \in [0,1], \\ F(x,1) &\in Y \qquad \forall x \in U_0. \end{aligned}$$

If such an h does not exist, we say that $cat_{M,Y}(B) = +\infty$.

Let Y be a fixed subset of M. We set

$$\mathcal{B}_{\rm i} = \{{\rm B} \subset {\rm M}|\ {\rm cat}_{({\rm M,Y})}({\rm B}) \geq {\rm i}\},$$
$$c_i = \inf_{B \in \mathcal{B}_{\rm i}} \sup_{x \in B} f(x).$$

We have the following multiplicity theorems, which was proved in [4].

Theorem 2.4. *Let $i \in N$ and assume that*
(1) $c_i < +\infty$,
(2) $\sup_{x \in Y} f(x) < c_i$,
(3) *the $(P.S.)_c$ condition holds for $c \in R$.*
Then there exists a lower critical point x such that $f(x) = c_i$. If

$$c_i = c_{i+1} = \ldots = c_{i+k-1} = c,$$

then

$$cat_M(\{x \in M | f(x) = c,\ grad_M^- f(x) = 0\}) \geq k.$$

We recall the "nonsmooth" version of the classical Deformation Lemma in [1].

Lemma 2.5. *(Deformation Lemma) Let $h : H \to R \cup \{+\infty\}$ be a lower semi-continuous function and assume h to be φ-convex of order 2. Let $c \in R$, $\delta > 0$ and D be a closed set in H such that*

$$\inf\{\|grad_M^- h(x)\| |\ c-\delta \leq h(x) \leq c+\delta, \qquad dist(x,D) < \delta\} > 0.$$

Then there exists $\epsilon > 0$ and a continuous deformation $\eta : h^{c+\epsilon} \cap D \times [0,1] \to h^{c+\epsilon} \cap D_\delta$ (D_δ is the δ-neighborhood of D and $h^c = \{x | h(x) \leq c\}$) such that
(i) $\eta(x,0) = x \qquad \forall x \in h^{c+\epsilon} \cap D$,

(ii) $\eta(x,t) = x \qquad \forall x \in h^{c-\epsilon} \cap D, \forall t \in [0,1],$
(iii) $\eta(x,1) \in h^{c-\epsilon} \qquad \forall x \in h^{c+\epsilon} \cap D, \forall t \in [0,1].$

3. Proof of Theorem 1.1

Let H be a Hilbert space with $H = X_0 \oplus X_1 \oplus X_2 \oplus X_3 \oplus X_4$. Here X_0, X_1, X_2, X_3 and X_4 are five closed subspaces of H and X_0, X_1, X_2, X_3 are finite dimensional subspaces. Let $f : H \to R$ be a $C^{1,1}$ functional defined on a Hilbert space H with $f(0) = 0$. Let $P_{X_1 \oplus X_2 \oplus X_3}$ be the orthogonal projection from H onto $X_1 \oplus X_2 \oplus X_3$ and

$$C = \{z \in H| \ \|P_{X_1 \oplus X_2 \oplus X_3} z\| \geq 1\}. \tag{3.1}$$

Then C is the smooth manifold with boundary. Let us define a functional $\Psi : H \setminus \{X_0 \oplus X_4\} \to H$ by

$$\Psi(z) = z - \frac{P_{X_1 \oplus X_2 \oplus X_3} z}{\|P_{X_1 \oplus X_2 \oplus X_3} z\|} = P_{X_0 \oplus X_4} z$$
$$+(1 - \frac{1}{\|P_{X_1 \oplus X_2 \oplus X_3} z\|}) P_{X_1 \oplus X_2 \oplus X_3} z. \tag{3.2}$$

We have

$$\nabla \Psi(z)(w) = w - \frac{1}{\|P_{X_1 \oplus X_2 \oplus X_3} z\|} (P_{X_1 \oplus X_2 \oplus X_3} w$$
$$- \langle \frac{P_{X_1 \oplus X_2 \oplus X_3} z}{\|P_{X_1 \oplus X_2 \oplus X_3} z\|}, w \rangle \frac{P_{X_1 \oplus X_2 \oplus X_3} z}{\|P_{X_1 \oplus X_2 \oplus X_3} z\|}). \tag{3.3}$$

Let us define the functional $\tilde{f} : C \to R$ by

$$\tilde{f} = f \circ \Psi. \tag{3.4}$$

Then $\tilde{f} \in C^{1,1}_{loc}$. We note that if $\tilde{z}$ is the critical point of $\tilde{f}$ and lies in the interior of C, then $z = \Psi(\tilde{z})$ is the critical point of f. So it suffices to find the nontrivial critical points of $\tilde{f}$. We note that

$$\|grad_C^- \tilde{f}(\tilde{z})\| \geq \|P_{X_0 \oplus X_4} \nabla f(\Psi(\tilde{z}))\| \qquad \forall \tilde{z} \in \partial C. \tag{3.5}$$

Let us set

$$\tilde{S_r} = \Psi^{-1}(S_r(X_0 \oplus X_1 \oplus X_2 \oplus X_3)),$$
$$\tilde{B_r} = \Psi^{-1}(B_r(X_0 \oplus X_1 \oplus X_2 \oplus X_3)),$$
$$\tilde{\Sigma^3_R} = \Psi^{-1}(\Sigma^3_R(S_1(\rho_1) - w_1, S_2(\rho_2) - w_2, S_3(\rho_3) - w_3, X_4)),$$
$$\tilde{\Delta^3_R} = \Psi^{-1}(\Delta^3_R(S_1(\rho_1) - w_1, S_2(\rho_2) - w_2, S_3(\rho_3) - w_3, X_4)).$$

We note that $\tilde{S_r}$, $\tilde{B_r}$, $\tilde{\Sigma^3_R}$ and $\tilde{\Delta^3_R}$ have the same topological structure as S_r, B_r, Σ^3_R and Δ^3_R respectively.

By condition (i) of Theorem 1.1, there exist $\rho_1 > 0$, $\rho_2 > 0$, $\rho_3 > 0$, $r > 0$, $R > 0$ with $R > r$ and $R_1 > 0$ such that

$$\sup_{\tilde{z}\in\tilde{\Sigma}^3_R}(-\tilde{f})(\tilde{z}) = \sup_{z\in\Sigma^3_R(S_1(\rho_1)-w_1,S_2(\rho_2)-w_2,S_3(\rho_3)-w_3,X_4)}(-f)(z) <$$

$$\inf_{z\in S_r(X_0\oplus X_1\oplus X_2\oplus X_3)}(-f)(z) = \inf_{\tilde{z}\in\tilde{S}_r}(-\tilde{f})(\tilde{z}),$$

$$\sup_{\tilde{z}\in\tilde{\Delta}^3_R}(-\tilde{f})(\tilde{z}) = \sup_{z\in\Delta^3_R(S_1(\rho_1)-w_1,S_2(\rho_2)-w_2,S_3(\rho_3)-w_3,X_4)}(-f)(z) < \infty$$

and

$$\inf_{\tilde{z}\in\tilde{B}_r}(-\tilde{f})(\tilde{z}) = \inf_{z\in B_r(X_0\oplus X_1\oplus X_2\oplus X_3)}(-f)(z) > -\infty.$$

By condition (ii) of Theorem 1.1,, $-\tilde{f}$ satisfies the $(P.S.)_{\tilde{c}}$ condition for every real number $\tilde{c}$ such that

$$\inf_{\tilde{z}\in\tilde{S}_r}(-\tilde{f})(\tilde{z}) \le \tilde{c} \le \sup_{\tilde{z}\in\tilde{\Delta}^3_R}(-\tilde{f})(\tilde{z}). \tag{3.6}$$

We claim that

$$cat_{(C,\tilde{\Sigma^3})}(\tilde{\Delta^3}) = 4. \tag{3.7}$$

In fact, we consider a continuous deformation $r : \tilde{S}_r\backslash X_0 \times [0,1] \to \tilde{S}_r\backslash X_0$ such that

- $r(x,0) = x, \ \forall x \in \tilde{S}_r\backslash X_0,$
- $r(x,t) = x, \ \forall x \in \tilde{S}_r \cap (X_1 \oplus X_2 \oplus X_3), \ \forall t \in [0,1],$
- $r(x,1) \in \tilde{S}_r \cap (X_1 \oplus X_2 \oplus X_3), \ \forall x \in \tilde{S}_r\backslash X_0.$

Now we can define, if $x = x_0 + x_{123} + x_4 \in X_0 \oplus (X_1 \oplus X_2 \oplus X_3) \oplus X_4$, $t \in [0,1]$,

$$r_1(x,t) = x_0 + \|x_{123} + x_4\| r(\frac{x_{123} + x_4}{\|x_{123} + x_4\|}, t). \tag{3.8}$$

Using r_1, it is easy to construct a continuous deformation $\eta : C \times [0,1] \to C$ such that

- $\eta(x,0) = x, \quad \forall x \in C$
- $\eta(x,t) = x, \quad \forall x \in \tilde{\Delta}^3, \forall t \in [0,1],$
- $\eta(x,1) \in \tilde{\Delta^3}, \quad \forall x \in C,$
- $\eta(x,t) \in C\backslash\tilde{S}_r, \quad \forall x \in C\backslash\tilde{S}_r, \quad \forall t \in [0,1].$

The existence of η implies that

$$cat_{(C,\tilde{\Sigma^3})}(\tilde{\Delta^3}) = cat_{(\tilde{\Delta^3},\tilde{\Sigma^3})}(\tilde{\Delta^3}). \tag{3.9}$$

We note that the pair $(\tilde{\Delta^3}, \tilde{\Sigma^3})$ is homeomorphic to the pair (Δ^3, Σ^3) and the pair (Δ^3, Σ^3) is homeomorphic to the pair $(\mathcal{B}^{p+1} \times \{(\mathcal{S}^{q_1-1} - w_1) \cup (\mathcal{S}^{q_2-1} - w_2) \cup (\mathcal{S}^{q_3-1} - w_3)\}, \mathcal{S}^p \times \{(\mathcal{S}^{q_1-1} - w_1) \cup (\mathcal{S}^{q_2-1} - w_2) \cup$

$(\mathcal{S}^{q_3-1}-w_3)\})$, where $p=\dim X_1$, $q_1=\dim X_1$, $q_2=\dim X_2$, $q_3=\dim X_3$ and $\mathcal{B}^r$, $\mathcal{S}^r$ denote the r-dimensional ball, the r-dimensional sphere, respectively. Thus the pair $(\tilde{\Delta^3},\tilde{\Sigma^3})$ is homeomorphic to the pair $(\mathcal{B}^{p+1}\times\{(\mathcal{S}^{q_1-1}-w_1)\cup(\mathcal{S}^{q_2-1}-w_2)\cup(\mathcal{S}^{q_3-1}-w_3)\},\mathcal{S}^p\times\{(\mathcal{S}^{q_1-1}-w_1)\cup(\mathcal{S}^{q_2-1}-w_2)\cup(\mathcal{S}^{q_3-1}-w_3)\})$. This fact imply that

$$cat_{(C,\tilde{\Sigma^3_R})}(\tilde{\Delta^3_R})=4. \tag{3.10}$$

Let us set

$$\begin{aligned}\mathcal{A}_1&=\{A\subset C|\ cat_{(C,\tilde{\Sigma^3_R})}(A)\geq 1\},\\ \mathcal{A}_2&=\{A\subset C|\ cat_{(C,\tilde{\Sigma^3_R})}(A)\geq 2\},\\ \mathcal{A}_3&=\{A\subset C|\ cat_{(C,\tilde{\Sigma^3_R})}(A)\geq 3\},\\ \mathcal{A}_4&=\{A\subset C|\ cat_{(C,\tilde{\Sigma^3_R})}(A)\geq 4\}.\end{aligned} \tag{3.11}$$

Since $cat_{(C,\tilde{\Sigma^3_R})}(\tilde{\Delta^3_R})=4$, $\tilde{\Delta^3_R}\in\mathcal{A}_i$, $i=1,2,3$. Let us set

$$\begin{aligned}\tilde{c_1}&=\inf_{A\in\mathcal{A}_1}\sup_{\tilde{z}\in A}(-\tilde{f})(\tilde{z}),\quad \tilde{c_2}=\inf_{A\in\mathcal{A}_2}\sup_{\tilde{z}\in A}(-\tilde{f})(\tilde{z}),\\ \tilde{c_3}&=\inf_{A\in\mathcal{A}_3}\sup_{\tilde{z}\in A}(-\tilde{f})(\tilde{z}),\quad \tilde{c_4}=\inf_{A\in\mathcal{A}_4}\sup_{\tilde{z}\in A}(-\tilde{f})(\tilde{z}).\end{aligned} \tag{3.12}$$

We first claim that $\tilde{c_i}<\infty$, $i=1,2,3,4$. In fact, from the facts that

$$\sup_{z\in\Delta^3_R(S_1(\rho_1)-w_1,S_2(\rho_2)-w_2,S_3(\rho_3)-w_3,X_4)}(-f)(z)<\infty$$

and $\tilde{\Delta^3_R}\in\mathcal{A}_i$, $i=1,2,3,4$, we have that

$$\begin{aligned}\tilde{c_i}&=\inf_{A\in\mathcal{A}_i}\sup_{\tilde{z}\in A}(-\tilde{f})(\tilde{z})\leq\sup_{\tilde{z}\in\tilde{\Delta^3_R}}(-\tilde{f})(\tilde{z})\\ &=\sup_{z\in\Delta^3_R(S_1(\rho_1)-w_1,S_2(\rho_2)-w_2,S_3(\rho_3)-w_3,X_4)}(-f)(z)<\infty.\end{aligned}$$

We also claim that $\sup_{\tilde{z}\in\tilde{\Sigma^3_R}}(-\tilde{f})(\tilde{z})\leq\tilde{c_i}$, $i=1,2,3,4$. In fact, for any $A\in\mathcal{A}_i$ with $\tilde{\Sigma^3_R}\subset A$, $i=1,2,3,4$,

$$\sup_{\tilde{z}\in\tilde{\Sigma^3_R}}(-\tilde{f})(\tilde{z})\leq\sup_{\tilde{z}\in A}(-\tilde{f})(\tilde{z}), \tag{3.13}$$

and hence

$$\sup_{\tilde{z}\in\tilde{\Sigma^3_R}}(-\tilde{f})(\tilde{z})\leq\inf_{A\in\mathcal{A}_i}\sup_{\tilde{z}\in A}(-\tilde{f})(\tilde{z})=\tilde{c_i},\ i=1,2,3,4. \tag{3.14}$$

By condition (ii) of Theorem 1.1, $-\tilde{f}$ satisfies the $(P.S.)_{\tilde{c}}$ condition for any real number $\tilde{c}$ with $\inf_{\tilde{z}\in\tilde{S_r}}(-\tilde{f})(\tilde{z}) \le \tilde{c} \le \sup_{\tilde{z}\in\tilde{\Delta^3_R}}(-\tilde{f})(\tilde{z})$. Thus, by Theorem 2.1, there exist four nontrivial critical points $\tilde{z_1}$, $\tilde{z_2}$, $\tilde{z_3}$, $\tilde{z_4}$ of the functional $-\tilde{f}$ such that

$$\tilde{c_1} = (-\tilde{f})(\tilde{z_1}), \quad \tilde{c_2} = (-\tilde{f})(\tilde{z_2}), \quad \tilde{c_3} = (-\tilde{f})(\tilde{z_3}), \quad \tilde{c_4} = (-\tilde{f})(\tilde{z_4}). \tag{3.15}$$

We claim that

$$\inf_{\tilde{z}\in\tilde{S_r}}(-\tilde{f})(\tilde{z}) \le \tilde{c_1} \le \tilde{c_2} \le \tilde{c_3} \le \tilde{c_4} \le \sup_{\tilde{z}\in\tilde{\Delta^3_R}}(-\tilde{f})(\tilde{z}). \tag{3.16}$$

Since $cat_{(C,\tilde{\Sigma^3_R})}(\tilde{\Delta^3_R}) = 4$, $\tilde{\Delta^3_R} \in \mathcal{A}_4$ and hence

$$\tilde{c_4} = \inf_{A\in\mathcal{A}_4}\sup_{\tilde{z}\in A}(-\tilde{f})(\tilde{z}) \le \sup_{\tilde{z}\in\tilde{\Delta^3_R}}(-\tilde{f})(\tilde{z}), \ \forall A \in \mathcal{A}_4. \tag{3.17}$$

For the proof of $\tilde{c_1} \ge \inf_{\tilde{z}\in\tilde{S_r}}(-\tilde{f})(\tilde{z})$, we construct a deformation $\eta' : C\backslash\tilde{S_r} \times [0,1] \to C\backslash\tilde{S_r}$ such that

- $\eta'(x,0) = x, \quad \forall x \in C\backslash\tilde{S_r}$,
- $\eta'(x,t) = x, \quad \forall x \in \tilde{\Sigma^3}, \forall t \in [0,1]$,
- $\eta'(x,1) \in \tilde{\Sigma^3}, \quad \forall x \in C$.

Actually η' can be defined by taking the retraction of η on $C\backslash\tilde{S_r}$ followed by a retraction of $\tilde{\Delta^3}\backslash\tilde{S_r}$ to $\tilde{\Sigma^3}$. The existence of η' implies that any $A \in \mathcal{A}_1$ must intersect $\tilde{S_r}$. So $\sup(-\tilde{f})(A) \ge \inf_{\tilde{z}\in\tilde{S_r}}(-\tilde{f})(\tilde{z}), \ \forall A \in \mathcal{A}_1$. So we have $\tilde{c_1} = \inf_{A\in\mathcal{A}_1}\sup_{\tilde{z}\in A}(-\tilde{f})(\tilde{z}) \ge \inf_{\tilde{z}\in\tilde{S_r}}(-\tilde{f})(\tilde{z})$. Therefore there exist at least four nontrivial critical points $\tilde{z_1}$, $\tilde{z_2}$, $\tilde{z_3}$, $\tilde{z_4}$ for the functional $-\tilde{f}$ such that

$$\inf_{\tilde{z}\in\tilde{S_r}}(-\tilde{f})(\tilde{z}) \le (-\tilde{f})(\tilde{z_1}) \le (-\tilde{f})(\tilde{z_2}) \le (-\tilde{f})(\tilde{z_3})$$
$$\le (-\tilde{f})(\tilde{z}) \le \sup_{\tilde{z}\in\tilde{\Delta^3_R}}(-\tilde{f})(\tilde{z}).$$

Setting $z_i = \Psi(\tilde{z_i})$, $i = 1,2,3,4$, we have

$$\inf_{z\in S_r}(-f)(z) = \inf_{\tilde{z}\in\tilde{S_r}}(-\tilde{f})(\tilde{z}) \le (-f)(z_1) \le (-f)(z_2) \le (-f)(z_3)$$
$$\le (-f)(z_4) \le \sup_{\tilde{z}\in\tilde{\Delta^3_R}}(-\tilde{f})(\tilde{z}) = \sup_{z\in\Delta^3_R}(-f)(z). \tag{3.18}$$

Thus we have

$$\inf_{z\in\Sigma^3_R(S_1(\rho_1)-w_1,S_2(\rho_2)-w_2,S_3(\rho_3)-w_3,X_4)} f(z) \le f(z_4) \le f(z_3)$$

$$\leq f(z_2) \leq f(z_1) \leq \sup_{z \in S_r(X_0 \oplus X_1 \oplus X_2 \oplus X_3)} f(z). \tag{3.19}$$

We claim that $\tilde{z}_i \notin \partial C$, that is $z_i \notin X_0 \oplus X_4$, which implies that z_i are the critical points for f in $X_1 \oplus X_2 \oplus X_3$. For this we assume by contradiction that $z_i \in X_0 \oplus X_4$. From (3.5), $P_{X_0 \oplus X_4} \nabla f(z_i) = 0$, namely, z_i, $i = 1, 2, 3, 4$, are the critical points for $f|_{X_0 \oplus X_4}$. By condition (iii) of Theorem 1.1, the critical points z_i in $X_0 \oplus X_4$ has no critical values in $[\inf_{z \in \Delta^3_R(S_1(\rho_1)-w_1, S_2(\rho_2)-w_2, S_3(\rho_3)-w_3, X_4)} f(z), \sup_{z \in S_r(X_0 \oplus X_1 \oplus X_2 \oplus X_3)} f(z)]$, which contradicts to (3.19). Thus $z_i \notin X_0 \oplus X_4$. This proves Theorem 1.1.

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Department of Mathematics
Kunsan National University
Kunsan 573-701, Republic of Korea
E-mail: tsjung@kunsan.ac.kr

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Department of Mathematics Education
Inha University
Incheon 402-751, Republic of Korea
E-mail: qheung@inha.ac.kr