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C-DUNFORD AND C-PETTIS INTEGRALS

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Abstract. In this paper, we give some extensions of Dunford integral and Pettis integral, C-Dunford integral and C-Pettis integral. We also discuss the relation among the C-Dunford integral, C-Pettis integral and C-integral.

1. Introduction

In 1986 A. M. Bruckner, R. J. Fleissner and J. Fordan [1] researched the following function

$$F(x) = \begin{cases} x^2 \sin \frac{1}{x^2} & if \quad 0 < x \leq 1 \\ 0 & if \quad x = 0 \end{cases} .$$

It is a primitive for the Riemann improper integral and therefore for the Henstock integral, but it is neither a Lebesgue primitive, neither a differential function, nor a sum of Lebesgue primitive and a differentiable function. It is natural to ask whether there is a minimal integral including the Lebesgue integral and the derivative.

In 1996 B. Bongiorno [2] provided a new solution to the problem of recovering a function from its derivative by integration by introducing a constructive minimal integration process of Riemann type, called C-integral, which includes the Lebesgue integral and also integrates the derivatives of differentiable function. B. Bongiorno and L. Di Piazza in [3]-[4] discussed some properties of the C-integral of real-valued functions. The C-integral is a natural extension of the Lebesgue integral. In [9]-[11], Dafang Zhao and Guoju Ye studied the Banach-valued C-integral.

The authors of [5]-[7] studied the Denjoy extension of the McShane integral and others of functions mapping an interval [a, b] into a Banach

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space X. In this paper, we will study the C-extension of the Dunford and Pettis integrals to C-Dunford integral and C-Pettis integral. We prove that a function f is C-Dunford integrable if and only if x^*f is C-integrable for all $x^* \in X^*$. Further we discuss the relation among the C-Dunford integral , C-Pettis integral and C-integral.

2. Definition and basic properties

Throughout this paper, X will denote a real Banach space with norm $||.||$ and X^* its dual. $[a, b]$ is a compact interval in R. A partition D is a finite collection of interval-point pairs $\{[u_i, v_i], \xi_i\}$, where $[u_i, v_i]$ are non-overlapping subintervals of $[a, b]$. $\delta(\xi)$ is a positive function on $[a, b]$, i.e, $\delta(\xi) : [a, b] \to R^+$ and we call it a gauge. We say D=$\{[u_i, v_i], \xi_i\}_{i=1}^n$ is

(1) a partial partition of $[a, b]$ if $\bigcup_{i=1}^n [u_i, v_i] \subset [a, b]$.

(2) δ-fine Mcshane partition of $[a, b]$ if $[u_i, v_i] \subset B(\xi_i, \delta(\xi_i)) = (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ and $\xi_i \in [a, b]$ for all $i = 1, 2, ..., n$.

(3) δ-fine C-partition of $[a, b]$ if it is a δ-fine Mcshane partition of $[a, b]$ and satisfying the condition

$$\sum_{i=1}^n dist(\xi_i, [u_i, v_i]) < \frac{1}{\varepsilon}.$$

Definition 2.1. A function $f : [a, b] \to X$ is C-integrable if there exists a vector $A \in X$ such that for each $\varepsilon > 0$, there is a gauge δ such that

$$\|(D) \sum_{i=1}^n f(\xi_i)(v_i - u_i) - A\| < \epsilon$$

for each $\delta-$fine C-partition $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$ of $[a, b]$. A is called the C-integral of f on $[a, b]$, and we write $A = \int_a^b f$ or $A =$ (C) $\int_a^b f$. The function f is C-integrable on the set $E \subset [a, b]$ if the function $f\chi_E$ is C-integrable on $[a, b]$. We write $\int_E f = \int_a^b f\chi_E$.

From the Definition of C-integral, we can easily obtain the following Theorem 2.1 and Theorem 2.2.

Theorem 2.1. *A function $f : [a, b] \to X$ is C-integrable if and only if for each $\varepsilon > 0$ there is a gauge δ such that*

$$\|(D_1) \sum f(\xi_i)(v_i - u_i) - (D_2) \sum f(\eta_j)(t_j - s_j)\| < \epsilon$$

for arbitrary δ - fine C-partition $D_1 = \{([u_i, v_i], \xi_i)\}_{i=1}^n$ and $D_2 = \{([s_j, t_j], \eta_j)\}_{j=1}^p$ of $[a, b]$.

Theorem 2.2. *Let $f : [a, b] \to X$ and $g : [a, b] \to X$.*

(1) If f is $C-$integrable on $[a, b]$, then f is $C-$integrable on every subinterval of $[a, b]$.

(2) If f is $C-$integrable on each of the intervals I_1 and I_2, where I_i are nonoverlapping and $I_1 \bigcup I_2 = [a, b]$, then f is $C-$integrable on $[a, b]$ and $\int_{I_1} f + \int_{I_2} f = \int_a^b f$.

(3) If f and g are $C-$integrable on $[a, b]$ and if α and β are real numbers, then $\alpha f + \beta g$ is $C-$integrable on $[a, b]$ and $\int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g$.

Theorem 2.3. *Let $f : [a, b] \to X$. If $f = \theta$ almost everywhere on $[a, b]$, then f is $C-$integrable and $\int_a^b f = \theta$.*

Proof. Assume $E = \{\xi \in [a, b] : f(\xi) \neq \theta\}$ then $E = \bigcup_n E_n \subset [a, b]$, where $E_n = \{\xi \in E : n - 1 \leq \|f(\xi)\| < n\}$. Obviously, $\mu(E) = 0$ and $\mu(E_n) = 0$ for every $n \in N$. Then there are open sets $G_n \subset [a, b]$ such that $E_n \subset G_n$ and $\mu(G_n) < \frac{\varepsilon}{n \cdot 2^n}$. We define a gauge δ in such a way that $\delta(\xi) = 1$ if $\xi \in [a, b] \backslash E$ and $B(\xi, \delta(\xi)) \subset G_n$ if $\xi \in E_n$.

Suppose that $D = \{([u_i, v_i], \xi) i = 1, 2, ..., m\}$ is a $\delta-$ fine $C-$partition of $[a, b]$. Then

$$\| \sum_{i=1}^m f(\xi)(v_i - u_i) \| \leq \sum_n n \frac{\varepsilon}{n \cdot 2^n} < \varepsilon.$$

Hence, f is $C-$integrable on $[a, b]$ and $\int_a^b f = \theta$.

□

The following Lemma has been proved in [9]. For convenience to use it we present it here.

Lemma 2.4 (Saks-Henstock). *Let $f : [a, b] \to X$ be C-integrable on $[a, b]$. Then for $\varepsilon > 0$ there is a gauge δ such that*

$$\|(D) \sum_{i=1}^n f(\xi_i)(v_i - u_i) - \int_a^b f \| < \epsilon$$

for each δ - fine C-partition $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$ of $[a, b]$. Particularly, if $D' = \{([u_i, v_i], \xi_i)\}_{i=1}^m$ is an arbitrary δ - fine partial C-partition of

$[a, b]$, *we have*

$$\|(D') \sum_{i=1}^{m} f(\xi_i)(v_i - u_i) - \sum_{i=1}^{m} \int_{u_i}^{v_i} f(\xi_i)\| \leq \epsilon.$$

3. C-Dunford integral and C-Pettis integral

Definition 3.1. A function $f : [a, b] \to X$ is C-Dunford integrable on $[a, b]$ if $x^* f$ is C-integrable on $[a, b]$ for each $x^* \in X^*$ and if for every subinterval $[c, d] \subset [a, b]$ there exists an element $x^{**}_{[c,d]} \in X^{**}$ such that $\int_c^d x^* f = x^{**}_{[c,d]}(x^*)$ for each $x^* \in X^*$. We write

$$(CD) \int_c^d f = x^{**}_{[c,d]} \in X^{**}.$$

A function $f : [a, b] \to X$ is C-Pettis integrable on $[a, b]$ if f is C-Dunford on $[a, b]$ and $(CD) \int_c^d f \in X$ for every interval $[c, d] \subset [a, b]$. We write

$$(CP) \int_c^d f = (CD) \int_c^d f \in X.$$

The function f is C-Dunford (C-Pettis) integrable on the set $E \subset [a, b]$ if the function $f\chi_E$ is C-Dunford (C-Pettis) integrable on $[a, b]$. We write$(CD) \int_E f = (CD) \int_a^b f\chi_E$ $((CP) \int_E f = (CP) \int_a^b f\chi_E)$.

Theorem 3.1. *Let $f : [a, b] \to X$ be C-Dunford integrable on $[a, b]$ if and only if $x^* f$ is C-integrable on $[a, b]$ for each $x^* \in X^*$.*

Proof. If f is C-Dunford integrable on $[a, b]$, then $x^* f$ is C-integrable on $[a, b]$ for each $x^* \in X^*$.

Now we prove the "only if " part.

If $x^* f$ is C-integrable on $[a, b]$ for each $x^* \in X^*$, then $x^* f$ is Denjoy integrable on $[a, b]$ and the Denjoy integral $(D) \int_a^b x^* f = (C) \int_a^b x^* f$. From Theorem 3 in [12], f is Denjoy-Dunford integrable on $[a, b]$ and for every subinterval $[c, d] \subset [a, b]$ there exists an element $x^{**}_{[c,d]} \in X^{**}$ such that $(D) \int_c^d x^* f = x^{**}_{[c,d]}(x^*)$ for each $x^* \in X^*$.

Since $x^* f$ is C-integrable on $[c, d]$ and

$$(D) \int_c^d x^* f = (C) \int_c^d x^* f = x^{**}_{[c,d]}(x^*)$$

for each $x^* \in X^*$. Hence f is C-Dunford integrable on $[a, b]$.

□

The following theorem can be proved easily by [14].

THEOREM 3.2. *If $f : [a, b] \to R$ is C-integrable on $[a, b]$, then there exists a nondegenerate subset $J \subset [a, b]$ such that f is Lebesgue integrable on J.*

THEOREM 3.3. *If X contains no copy of c_0 and $f : [a, b] \to X$ is C-Dunford integrable, then there exists a subinterval $E_N \subset [a, b]$ such that f is Dunford integrable on E_N.*

Proof. Let $\{E_n\}$ be the sequence of all open intervals in $[a, b]$ that have rational endpoints. For each pair of positive integers m and n let $E_m^n = \{x^* \in X^* : \int_{E_n} |x^*f| \leq m\}$. Then $X^* = \bigcup_m^\infty \bigcup_n^\infty E_m^n$.

For each m and n we have $E_m^n \subset X^*$, so $\bigcup_m^\infty \bigcup_n^\infty E_m^n \subset X^*$. On the other hand, for every $x^* \in X^*$, by Theorem 3.1, x^*f is C-integrable on $[a, b]$. It follows from Theorem 3.2 that there exists a nondegenerate interval $J \subset [a, b]$ such that x^*f is Lebesgue integrable on J. So there is a $n_0 \in N$ such that $E_{n_0} \in \{E_n\}$, $E_{n_0} \subset J$ and therefore x^*f is Lebesgue integrable on E_{n_0}. Hence there is a m_0 such that $\int_{E_{n_0}} |x^*f| \leq m_0$. This means $x^* \in E_{m_0}^{n_0}$. So $X^* = \bigcup_m^\infty \bigcup_n^\infty E_m^n$.

Now we prove each of the sets E_m^n is closed.

Let x^* be a limit point of E_m^n and $\{x_k^*\}$ a sequence in E_m^n that converges to x^*. Then the sequence $\{|x_k^*f|\}$ converges pointwise on $[a, b]$ to the function $|x^*f|$ and by Fatou's Lemma we have

$$\int_{E_n} |x^*f| \leq \liminf_{k \to \infty} \{\int_{E_n} |x_k^*f|\} \leq m.$$

This shows $x^* \in E_m^n$ and concludes that the set E_m^n is closed.

By the Baire Category Theorem there exists M ,N, x_0^*, and $r > 0$ such that $\{x^* : \|x^* - x_0^*\| \leq r\} \subset E_M^N$. For each x^* in X^* with $\|x^*\| \neq 0$ we find that

$$\int_{E_N} |x^*f| \leq \frac{\|x^*\|}{r} \{\int_{E_N} |\frac{r}{\|x^*\|} x^*f + x_0^*f| + \int_{E_N} |x_0^*f|\} \leq \frac{2M}{r} \|x^*\|.$$

Hence, for each x^* in X^* the function x^*f is Lebesgue integrable on E_N. So f is Dunford integrable on E_N.

□

THEOREM 3.4. *If the function $f : [a, b] \to X$ is C-Dunford integrable on $[a, b]$, then there is a sequence $\{E_n\}$ of closed subsets such that $E_n \subset E_{n+1}$ for all n, $\bigcup_{n=1}^\infty E_n = [a, b]$, f is Dunford integrable on each E_n and*

$$\lim_{n \to \infty} (Dunford) \int_{E_n \bigcap [a,x]} f(t)dt = (CD) \int_a^x f(t)dt$$

weakly uniformly on $[a, b]$.

Proof. Since $f : [a, b] \to X$ is C-Dunford integrable on $[a, b]$, $x^* f$ is C-integrable on $[a, b]$ for each $x^* \in X^*$. Then there is a sequence $\{E_n\}$ of closed subsets such that $E_n \subset E_{n+1}$ for all n, $\bigcup_{n=1}^{\infty} E_n = [a, b]$, $x^* f$ is Lebesgue integrable on each E_n and

$$\lim_{n\to\infty} (L) \int_{E_n \bigcap [a,x]} x^* f(t)dt = (C) \int_a^x x^* f(t)dt$$

uniformly on $[a, b]$ for each $x^* \in X^*$. Hence f is Dunford integrable on each E_n and

$$\lim_{n\to\infty} (Dunford) \int_{E_n \bigcap [a,x]} f(t)dt = (CD) \int_a^x f(t)dt$$

weakly uniformly on $[a, b]$.

□

According to [8], we can easily obtain the following two theorems:

Theorem 3.5. *If the function* $f : [a, b] \to X$ *is C-Dunford integrable on* $[a, b]$*, then each perfect set in* $[a, b]$ *contains a portion on which* f *is Dunford integrable.*

Theorem 3.6. *Suppose that* X *contains no copy of* c_0*. If the function* $f : [a, b] \to X$ *is C-Pettis integrable on* $[a, b]$*, then each perfect set in* $[a, b]$ *contains a portion on which* f *is Pettis integrable.*

Let $F : [a, b] \to X$ be a function and let E be a subset of $[a, b]$.

Definition 3.2. (a) A function F is AC_c on E if for each $\varepsilon > 0$ there is a constant $\eta > 0$ and a gauge δ such that $\sum_i \|F[u_i, v_i]\| < \varepsilon$ for each δ-fine partial C-partition $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$ of $[a, b]$ satisfying the endpoints of I_i belonging to E and $\sum_{i=1}^n (v_i - u_i) < \eta$, where $F[u_i, v_i] = F(v_i) - F(u_i)$.
(b) The function F is ACG_c on E if F is continuous on E and E can be expressed as a union of sets on each of which F is AC_c.

From Definition 3.2 above, we can see if a function F is AC_c on E then F is AC on E and if a function F is ACG_c on E, then F is ACG on E.

Theorem 3.7. *If a function* $f : [a, b] \to X$ *is C-Dunford integrable on* $[a, b]$*, then there exists a sequence* $\{X_k\}$ *of closed sets,* $\bigcup_{k=1}^{\infty} X_k = [a, b]$*,* f *is Dunford integrable on each* X_k.

Proof. Since f is C-Dunford on $[a,b]$, for each $x^* \in X^*$, x^*f is C-integrable on $[a,b]$. Let F be the primitive of f. Then for every interval $[u_i,v_i] \subset [a,b], i = 1,2,...n$, $\int_{u_i}^{v_i} x^*f = x^* \int_{u_i}^{v_i} f = x^*F[u_i,v_i]$ and x^*F is ACG_c on $[a,b]$ for each $x^* \in X^*$. So there is a sequence $\{X_k\}$ of closed subsets such that $\bigcup_{k=1}^{\infty} X_k = [a,b]$ and x^*F is VB^* on each X_k for each $x^* \in X^*$. From [15], x^*f is Lebesgue integrable on each X_k for each $x^* \in X^*$. So f is Dunford integrable on each X_k and $\bigcup_{k=1}^{\infty} X_k = [a,b]$. □

THEOREM 3.8. *Suppose that X contains no copy of c_0 and $f : [a,b] \to X$ is measurable. If the function $f : [a,b] \to X$ is C-Pettis integrable on $[a,b]$, then there exists a sequence $\{X_k\}$ of closed sets with $X_k \uparrow [a,b]$ such that f is Pettis integrable on each X_k, and*

$$\lim_{k\to\infty} (Pettis) \int_{X_k} f = (CP) \int_a^b f \quad weakly.$$

Proof. Since f is C-Pettis integrable on $[a,b]$, then f is C-Dunford integrable on $[a,b]$, and so by Theorem 3.4, there is a sequence $\{X_k\}$ of closed subsets such that $X_k \subset X_{k+1}$ for all k, $\bigcup_{k=1}^{\infty} X_k = [a,b]$, f is Dunford integrable on each X_k and

$$\lim_{n\to\infty} (L) \int_{X_k \bigcap [a,x]} x^*f(t)dt = (CD) \int_a^x x^*f(t)dt$$

uniformly on $[a,b]$ for each $x^* \in X^*$. Since X contains no copy of c_0 and f is measurable, it follows from [13] that f is Pettis integrable on X_k and

$$\lim_{k\to\infty} (Pettis) \int_{X_k \bigcap [a,x]} f(t)dt = (CP) \int_a^x f(t)dt$$

uniformly on each $[a,x]$, that is

$$\lim_{k\to\infty} (Pettis) \int_{X_k} f = (CP) \int_a^b f \quad weakly.$$

□

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