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C-DUNFORD INTEGRAL AND C-PETTIS INTEGRAL

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Abstract. In this paper, we give the Riemann-type extensions of Dunford integral and Pettis integral, C-Dunford integral and C-Pettis integral. We prove that a function f is C-Dunford integrable if and only if x^*f is C-integrable for each $x^* \in X^*$ and prove the controlled convergence theorem for the C-Pettis integral.

1. Introduction

In 1996 B. Bongiorno provided a new solution to the problem of recovering a function from its derivative by integration by introducing a constructive minimal integration process of Riemann type, called C-integral, which includes the Lebesgue integral and also integrates the derivatives of differentiable function. B.Bongiorno and L.Di Piazza [1, 2, 4] discussed some properties of the C-integral of real-valued functions. In [9, 10, 11], we studied the Banach-valued C-integral.

The Dunford integral and the Pettis integral are generalizations of Lebegue integral to the Banach-valued functions. In this paper, we give the Riemann-type extensions of Dunford integral and Pettis integral, C-Dunford integral and C-Pettis integral. We prove that a function f is C-Dunford integrable if and only if x^*f is C-integrable for each $x^* \in X^*$, we also discuss the relationship between the C-Pettis integral and Pettis integral, if a function f is C-integrable on $[a, b]$ then f is C-Pettis integrable on $[a, b]$, but an example shows that the converse is not true. Finally, we prove the controlled convergence theorem for the C-Pettis integral.

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2. Definitions and basic properties

Throughout this paper, $[a, b]$ is a compact interval in R. X will denote a real Banach space with norm $\|\cdot\|$ and its dual X^*. A partition D is a finite collection of interval-point pairs $\{([u_i, v_i], \xi_i)\}_{i=1}^n$, where $\{[u_i, v_i]\}_{i=1}^n$ are non-overlapping subintervals of $[a, b]$. $\delta(\xi)$ is a positive function on $[a, b]$, i.e. $\delta(\xi) : [a, b] \to \mathrm{R}^+$. We say that $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$ is

(1) a partial partition of $[a, b]$ if $\bigcup_{i=1}^n [u_i, v_i] \subset [a, b]$,

(2) a partition of $[a, b]$ if $\bigcup_{i=1}^n [u_i, v_i] = [a, b]$,

(3) δ - fine *McShane partition* of $[a, b]$ if $[u_i, v_i] \subset B(\xi_i, \delta(\xi_i)) = (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ and $\xi_i \in [a, b]$ for all i=1,2,$\cdots$,n,

(4) δ - fine *C-partition* of $[a, b]$ if it is a δ - fine *McShane partition* of $[a, b]$ and satisfying the condition

$$\sum_{i=1}^{n} dist(\xi_i, [u_i, v_i]) < \frac{1}{\varepsilon},$$

here $dist(\xi_i, [u_i, v_i]) = inf\{|t_i - \xi_i| : t_i \in [u_i, v_i]\}$,

(5) δ - fine *Henstock* partition of I_0 if $\xi_i \in I_i \subset B(\xi_i, \delta(\xi_i))$ for all i=1,2,$\cdots$,n.

Given a δ - fine *C-partition* $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$ we write

$$S(f, D) = \sum_{i=1}^{n} f(\xi_i)(v_i - u_i)$$

for integral sums over D, whenever $f : [a, b] \to X$.

Definition 2.1. A function $f : [a, b] \to X$ is C-integrable (Henstock integrable) if there exists a vector $A \in X$ such that for each $\varepsilon > 0$ there is a positive function $\delta(\xi) : [a, b] \to R^+$ such that

$$\|S(f, D) - A\| < \epsilon$$

for each δ - fine *C-partition (Henstock partition)* $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$ of $[a, b]$. A is called the *C-integral (Henstock integral)* of f on $[a, b]$, and we write $A = \int_a^b f$ or $A = (C)\int_a^b f$ $(A = (H)\int_a^b f)$.

The function f is C-integrable on the set $E \subset [a, b]$ if the function $f\chi_E$ is C-integrable on $[a, b]$. We write$\int_E f = \int_a^b f\chi_E$.

The basic properties of the C-integral, for example, linearity and additivity with respect to intervals can be founded in [10]. We do not present them here. The reader is referred to [10] for the details.

Definition 2.2. A function $f : [a,b] \to X$ is C-Dunford integrable (Henstock-Dunford integrable) on $[a,b]$ if x^*f is C-integrable (Henstock integrable) on $[a,b]$ for each $x^* \in X^*$ and if for every subinterval $[c,d] \subset [a,b]$ there exists an element $x^{**}_{[c,d]} \in X^{**}$ such that $\int_c^d x^*f = x^{**}_{[c,d]}(x^*)$ for each $x^* \in X^*$. We write

$$(CD)\int_c^d f = x^{**}_{[c,d]} \in x^{**}$$

$$((HD)\int_c^d f = x^{**}_{[c,d]} \in x^{**}_{.})$$

Definition 2.3. A function $f : [a,b] \to X$ is C-Pettis integrable (Henstock-Pettis integrable) on $[a,b]$ if f is C-Dunford integrable (Henstock-Dunford integrable) on $[a,b]$ and $(CD)\int_c^d f \in X$ $((HD)\int_c^d f \in X)$ for every interval $[c,d] \subset [a,b]$. We write

$$(CP)\int_c^d f = (CD)\int_c^d f \in X$$

$$((HP)\int_c^d f = (HD)\int_c^d f \in X_{.})$$

The function f is C-Pettis integrable on the set $E \subset [a,b]$ if the function $f\chi_E$ is C-Pettis integrable on $[a,b]$. We write$(CP)\int_E f = (CP)\int_a^b f\chi_E$.

Theorem 2.4. *$f : [a,b] \to X$ is C-Dunford integrable on $[a,b]$ if and only if x^*f is C-integrable on $[a,b]$ for each $x^* \in X^*$.*

Proof. If f is C-Dunford integrable on $[a,b]$ for each $x^* \in X^*$, then x^*f is C-integrable on $[a,b]$.

Now we prove the " only if " part.

From [10,Theorem 3.3], x^*f is C-integrable on $[a,b]$ for each $x^* \in X^*$, then x^*f is Henstock integrable on $[a,b]$ and $(H)\int_a^b x^*f = (C)\int_a^b x^*f$. Consequently, we have that f is Henstock-Dunford integrable on $[a,b]$ and for every subinterval $[c,d] \subset [a,b]$ there exists an element $x^{**}_{[c,d]} \in X^{**}$ such that $(H)\int_c^d x^*f = x^{**}_{[c,d]}(x^*)$ for each $x^* \in X^*$ from [6,Theorem 8.2.26].

Since x^*f is C-integrable on $[c,d]$ and

$$(H)\int_c^d x^*f = (C)\int_c^d x^*f = x^{**}_{[c,d]}(x^*)$$

for each $x^* \in X^*$. Hence f is C-Dunford integrable on $[a,b]$. □

Similar to the case for the Henstock-Dunford and Henstock-Pettis integrable functions, we can get the following two Theorems.

THEOREM 2.5. *If the function $f : [a, b] \to X$ is C-Dunford integrable on $[a, b]$, then each perfect set in $[a, b]$ contains a portion on which f is Dunford integrable.*

THEOREM 2.6. *Suppose that X contains no copy of c_0. If the function $f : [a, b] \to X$ is C-Pettis integrable on $[a, b]$, then each perfect set in $[a, b]$ contains a portion on which f is Pettis integrable.*

From the definitions of Pettis integral and C-Pettis integral, we can easily get the following theorem.

THEOREM 2.7. *If a function $f : [a, b] \to X$ is Pettis integrable on $[a, b]$ then f is C-Pettis integrable on $[a, b]$.*

THEOREM 2.8. *If a function $f : [a, b] \to X$ is C-integrable on $[a, b]$ then f is C-Pettis integrable on $[a, b]$.*

Proof. f is C-integrable on $[a, b]$, then x^*f is C-integrable on $[a, b]$ for each $x^* \in X^*$ and $(C)\int_a^b x^*f = x^*((C)\int_a^b f)$ from [10,Theorem 2.7].

For each subinterval $[c, d] \subset [a, b]$, we have $(C)\int_c^d f \in X$. Then f is C-Pettis integrable on $[a, b]$ and

$$(CP)\int_a^b f = (C)\int_a^b f.$$

□

REMARK 2.9. The following example show that the converse of Theorem 2.5 is not true. In other words, there exists a function which is C-Pettis integrable but is not C-integrable.

EXAMPLE 2.10. (a) Define a function $f : [0, 1] \longrightarrow l_\infty(\omega_1)$ by

$$f(t)(\alpha) = \begin{cases} 1 & \text{if } t \in N_\alpha \backslash C_\alpha, \\ 0 & \text{if Otherwise.} \end{cases} \tag{2.1}$$

where ω_1 is the first uncountable ordinal. $\{N_\alpha\}_{\alpha\in\omega_1}$ and $\{C_\alpha\}_{\alpha\in\omega_1}$ be two collection of subsets of $[0, 1]$ satisfying the following properties:

(1) for each $\alpha \in \omega_1$, N_α is a set of zero Lebesgue measure,

(2) $N_\alpha \subset N_\beta$, if $\alpha < \beta$,

(3) every subset of $[0, 1]$ of zero Lebesgue measure is contained in some set N_α,

(4) for each $\alpha \in \omega_1$, C_α is a countable set,

(5) $C_\alpha \subset C_\beta$, if $\alpha < \beta$,

(6) every countable subset of $[0,1]$ is contained in some set C_α.

In [5,Example(CH)], L. Di Piazza and D.Preiss proved that f is Pettis integrable but is not McShane integrable on $[0,1]$. It is easy to know that f is C-Pettis integrable on $[0,1]$ from Theorem 2.4. In [10,Theorem 3.4], we proved that f is McShane integrable if and only if f is C-integrable and Pettis integrable. Suppose that f is C-integrable on $[0,1]$, then f is McShane integrable on $[0,1]$. This is a contradiction, so f is not C-integrable on $[0,1]$.

3. Convergence theorem for the C-Pettis integral

Definition 3.1. Let $F_n, F : [a,b] \to R$ and let E be a subset of $[a,b]$.

(a) F is said to be AC_c on E if for each $\varepsilon > 0$ there is a constant $\eta > 0$ and a positive function $\delta(\xi) : E \to R^+$ such that $\sum_i |F([u_i, v_i])| < \epsilon$ for each δ - fine partial C-partition $D = \{([u_i, v_i], \xi_i)\}$ of $[a,b]$ satisfying $\xi_i \in E$ for each i and $\sum_i (v_i - u_i) < \eta$.

(b) F_n is said to be UAC_c on E if for each $\varepsilon > 0$ there is a constant $\eta > 0$ and a positive function $\delta(\xi) : E \to R^+$ such that $\sum_i |F_n([u_i, v_i])| < \epsilon$ for all n and for each δ - fine partial C-partition $D = \{([u_i, v_i], \xi_i)\}$ of $[a,b]$ satisfying $\xi_i \in E$ for each i and $\sum_i (v_i - u_i) < \eta$.

(c) F is said to be ACG_c on E if F is continuous on E and E can be expressed as a countable union of sets on each of which F is AC_c.

(d) F is said to be $UACG_c$ on E if F is continuous on E and E can be expressed as a countable union of sets on each of which F is UAC_c.

Theorem 3.2. *Let $f : [a,b] \to X$ and assume that $\{f_n\}$ be a sequence of C-integrable functions. Assume that the following conditions are satisfied:*

(1) $f_n \to f$ almost everywhere on $[a,b]$.

(2) F_n are $UACG_c$ on $[a,b]$.

Then f is C-integrable on $[a,b]$ and

$$\lim_{n\to\infty} (C) \int_a^b f_n = (C) \int_a^b f.$$

Proof. The proof is standard and similar to [7, *Theorem* 5.5.2]. □

Theorem 3.3. *(Controlled Convergence Theorem) Let $f : [a,b] \to X$ and assume that $\{f_n\}$ be a sequence of C-Pettis integrable functions. Assume that the following conditions are satisfied:*

(1) for each $x^* \in X^*$, $x^* f_n \to x^* f$ almost everywhere on $[a,b]$.

(2) *the family* $\{x^*F_n : x^* \in X^*, n \in \mathbb{N}\}$ *is* $UACG_c$ *on* $[a,b]$.
Then f *is C-Pettis integrable on* $[a,b]$ *and*

$$\lim_{n\to\infty}(CP)\int_a^b f_n = (CP)\int_a^b f \quad (weakly).$$

Proof. We will prove the Theorem in two steps.

Step 1. The sequence $\{f_n\}$ is C-Pettis integrable on $[a,b]$, then for each $x^* \in X^*$, x^*f_n is C-integrable on $[a,b]$. From Theorem 3.1 we have that x^*f is C-integrable on $[a,b]$ and

$$\lim_{n\to\infty}(C)\int_a^b x^*f_n = (C)\int_a^b x^*f.$$

Step 2. Assume $[c,d]$ is an arbitrary subinterval of $[a,b]$. Let $\mathcal{C}$ denote the weak closure of $\{(CP)\int_c^d f_n : n \in \mathbb{N}\}$. It is easy to see that $\mathcal{C}$ is bounded and that $\mathcal{C}\backslash\{(CP)\int_c^d f_n : n \in \mathbb{N}\}$ contains at most one point. We claim that $\mathcal{C}$ is weakly compact.

Suppose $\mathcal{C}$ is not weakly compact, then there exists a bounded sequence $(x_k^*) \subset X^*$, a sequence $(x_n) \subset \mathcal{C}$ and $\epsilon > 0$ such that

$$\begin{cases} x_k^*(x_n) = 0 & \text{if } k > n, \\ x_k^*(x_n) > \epsilon & \text{if } k \le n. \end{cases} \tag{3.1}$$

We can take subsequence $(g_n) \subset (f_n)$ and a sequence $(y_k^*) \subset x_k^*$ such that

$$\begin{cases} (C)\int_c^d y_k^* g_n = 0 & \text{if } k > n, \\ (C)\int_c^d y_k^* g_n > \epsilon & \text{if } k \le n, \\ \lim_{n\to\infty}(C)\int_c^d x^*g_n = (C)\int_c^d x^*f, & \text{for each } x^* \in X^*. \end{cases} \tag{3.2}$$

From [3,Lemma 1], we can find a subsequence $(y_{k_j}^*) \subset (y_k^*)$ such that $\lim_{j\to\infty} y_{k_j}^* f$ exists almost everywhere. Assume y_0^* is a *weak** cluster point of $(y_{k_j}^*) \subset (y_k^*)$, then we have

$$\lim_{j\to\infty} y_{k_j}^* f = y_0^* f$$

almost everywhere on $[a,b]$. It is not difficult to get that

$$\lim_{j\to\infty}(C)\int_c^d y_{k_j}^* f = (C)\int_c^d y_0^* f.$$

To force a contradiction, note that for each j, we have that

$$\lim_{n\to\infty}(C)\int_c^d y_{k_j}^* g_n = (C)\int_c^d y_{k_j}^* f.$$

When $n \geq k_j$, from (3) we have that $(C)\int_c^d y_{k_j}^* g_n > \epsilon$ and $(C)\int_c^d y_{k_j}^* f \geq \epsilon$. Therefore

$$\lim_{j\to\infty}(C)\int_c^d y_{k_j}^* f = (C)\int_c^d y_0^* f \geq \epsilon.$$

On the other hand, g_n is C-Pettis integrable for each n, the functional $x^* \longrightarrow (C)\int_c^d x^* g_n$ is $weak^*-$ continuous. Then if (y_α^*) is a subset of $(y_{k_j}^*)$ $weak^*$ converging to y_0^*, by (3) and passing to the limit with $n \to \infty$ we have that

$$\begin{aligned}\lim_{n\to\infty}\lim_{\alpha}(C)\int_c^d y_\alpha^* g_n &= \lim_{n\to\infty}\lim_{\alpha} y_\alpha^*(CP)\int_c^d g_n \\ &= \lim_{n\to\infty} y_0^*(CP)\int_c^d g_n \\ &= \lim_{n\to\infty}(C)\int_c^d y_0^* g_n \\ &= (C)\int_c^d y_0^* f = 0\end{aligned}$$

which contradicts the inequality $(C)\int_c^d y_0^* f \geq \epsilon$. Therefore the set $\mathcal{C}$ is weakly compact.

Since $\lim_{n\to\infty}(C)\int_c^d x^* f_n = (C)\int_c^d x^* f$, the sequence $\{(CP)\int_c^d f_n\}$ is weak Cauchy. It follows from the weak compactness of $\mathcal{C}$ that

$$\lim_{n\to\infty}(CP)\int_c^d f_n$$

exists weakly in X. Moreover by $[c,d]$ is an arbitrary subinterval of $[a,b]$, then f is C-Pettis integrable on $[a,b]$ and

$$\lim_{n\to\infty}(CP)\int_a^b f_n = (CP)\int_a^b f \quad (weakly).$$

□

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