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FUNCTIONAL INEQUALITIES CONNECTED WITH A DERIVATION AND A GENERALIZED DERIVATION

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Abstract. In this article, we investigate the functional inequalities concerned with a derivation and a generalized derivation.

1. Introduction

The stability problem of functional equations has originally been formulated by S. M. Ulam [6] in 1940: *Under what condition does there exists a homomorphism near an approximate homomorphism?* In following year, D. H. Hyers [2] answers the problem of Ulam under the assumption that the groups are Banach spaces. A generalized version of the theorem of Hyers for approximately linear mappings was given by Th. M. Rassias [4]. Since then, a great deal of work has been done by a number of authors.

Let A be an algebra. An additive mapping $\mu : A \to A$ is called a *ring derivation* if $\mu(xy) = x\mu(y) + \mu(x)y$ holds for all $x, y \in A$. An additive mapping $\mu : A \to A$ is said to be a *ring generalized derivation* if there exists a ring derivation $\delta : A \to A$ satisfying $\mu(xy) = x\mu(y) + \delta(x)y$ is fulfilled for all $x, y \in A$.

In particular, the stability result concerning derivations between operator algebras was first obtained by P. Šemrl [5]. Moreover, the Hyers-Ulam stability of ring derivations was studied and investigated by R. Badora and T. Miura *et al.* [1, 3].

The main purpose of this article is to establish the functional inequalities associated with the ring derivation and the ring generalized derivation.

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2. Stability of a derivation and a generalized derivation

Theorem 2.1. *Let A be a Banach algebra. Suppose that a function $f : A \to A$ satisfies the inequality*

$$\|f(x+y-zw) - f(x-y) + 2f(-y) + zf(w) + f(z)w\| \le \theta \tag{2.1}$$

for all $x, y, z, w \in A$ and $f(0) = 0$. Then there exists a ring derivation $d : A \to A$ such that $\|f(x) - d(x)\| \le 2\theta$ for all $x \in A$ and $x(f(y) - d(y)) = 0$ for all $x, y \in A$.

Proof. Let us take $w = 0$ in (2.1), then it becomes

$$\|f(x+y) - f(x-y) + 2f(-y)\| \le \theta. \tag{2.2}$$

If $x = 0$ and $y = -x$ in (2.2), it follows that

$$\|f(-x) + f(x)\| \le \theta. \tag{2.3}$$

Substituting x and y with $-x$ in (2.2), then we have

$$\|f(-2x) + 2f(x)\| \le \theta. \tag{2.4}$$

Combining (2.3) and (2.4), one can easily get that $\|2f(x) - f(2x)\| \le 2\theta$, and so we obtain the inequality $\|\frac{f(2x)}{2} - f(x)\| \le \theta$. An induction implies that

$$\Big\|\frac{f(2^n x)}{2^n} - f(x)\Big\| \le 2\Big(1 - \frac{1}{2^n}\Big)\theta. \tag{2.5}$$

For $n > m$, the relation (2.5) can be rewritten

$$\begin{aligned}\Big\|\frac{f(2^n x)}{2^n} - \frac{f(2^m x)}{2^m}\Big\| &= \frac{1}{2^m}\Big\|\frac{f(2^{n-m}\cdot 2^m x)}{2^{n-m}} - f(2^m x)\Big\| \\ &\le \frac{1}{2^{m-1}}\Big(1 - \frac{1}{2^{n-m}}\Big)\theta.\end{aligned}$$

As $m \to \infty$, it can be easily verified that $\{\frac{f(2^n x)}{2^n}\}$ is a Cauchy sequence. Since A is complete, the Cauchy sequence $\{\frac{f(2^n x)}{2^n}\}$ converges. Thus if $d(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ for all $x \in A$, then we have $||f(x) - d(x)|| \le 2\theta$ as $n \to \infty$ in (2.5). Replacing x and y with $2^n x$ and $2^n y$ in (2.2), respectively, and then dividing both sides by 2^n. We get

$$\Big\|\frac{f(2^n(x+y))}{2^n} - \frac{f(2^n(x-y))}{2^n} + \frac{2f(-2^n y)}{2^n}\Big\| \le \frac{\theta}{2^n}.$$

As $n \to \infty$ in the above inequality, we also get

$$d(x+y) - d(x-y) + 2d(-y) = 0. \tag{2.6}$$

The equation (2.6) can be more simplified as $d(y)+d(-y)=0$ if we take $x=0$. This implies that d is an odd function and $d(0)=0$. Similarly, substituting $y=x$ into (2.6), we obtain that $d(2x)=2d(x)$. Let $u=x+y$ and $v=x-y$ in the equation (2.6), then we can rewrite the equation (2.6) as $d(u)-d(v)+2d(-\frac{u-v}{2})=0$ and by putting $v:=-v$ in this equation, we have

$$2d\Big(\frac{u+v}{2}\Big)=d(u)+d(v). \tag{2.7}$$

Hence if we set $u=2x$ and $v=2y$ in (2.7) and use the equation $d(2x)=2d(x)$, we finally obtain $d(x+y)=d(x)+d(y)$, and so we can conclude that d is additive.

We claim that d is unique: Suppose that there exists another additive function $D:X\to Y$ satisfying the inequality $\|D(x)-d(x)\|\le 2\theta$. Since $D(2^nx)=2^nD(x)$ and $d(2^nx)=2^nd(x)$, we see that

$$\begin{aligned}\|D(x)-d(x)\|&=\frac{1}{2^n}\|D(2^nx)-d(2^nx)\|\\ &\le\frac{1}{2^n}[\|D(2^nx)-f(2^nx)\|+\|f(2^nx)-d(2^nx)\|]\le\frac{1}{2^{n-2}}\theta.\end{aligned}$$

By letting $n\to\infty$ in this inequality, we have $D=d$.

We finally assert that d is the derivation: Let us take $x=y=0$ in (2.1). Then it follows that

$$\|f(-zw)+zf(w)+f(z)w\|\le\theta. \tag{2.8}$$

Define $C(z,w)=f(-zw)+zf(w)+f(z)w$. Since C is bounded, we have $\lim_{n\to\infty}\frac{C(2^nz,w)}{2^n}=0$. Observed that

$$\begin{aligned}d(-zw)&=\lim_{n\to\infty}\frac{f(-2^nz\cdot w)}{2^n}\\ &=\lim_{n\to\infty}\frac{-2^nzf(w)-f(2^nz)w+C(2^nz,w)}{2^n}=-zf(w)-d(z)w.\end{aligned}$$

Based on the fact that d is a odd function, $d(zw)=zf(w)+d(z)w$. Now this equation can be rewritten as

$$d(2^nz\cdot w)=2^nzf(w)+2^nd(z)w,\ \ d(z\cdot 2^nw)=zf(2^nw)+2^nd(z)w.$$

Hence $zf(w)=z\frac{f(2^nw)}{2^n}$, and then we obtain $zf(w)=zd(w)$ as $n\to\infty$. So the assertion follows, which ends the proof of the theorem. □

From Theorem 2.1, we obtain the following corollary.

Corollary 2.2. *Let A be a Banach algebra with the unit. Suppose that a function $f : A \to A$ satisfies the inequality (2.1) and $f(0) = 0$. Then f is a ring derivation*

Theorem 2.3. *Let A be a Banach algebra with the unit. Suppose $f : A \to A$ is a function with $f(0) = 0$ for which there exists a function $g : A \to A$ such that*

$$\|f(x+y-zw) - f(x-y) + 2f(-y) + zf(w) + g(z)w\| \le \theta. \tag{2.9}$$

for all $x, y, z, w \in A$. Then f is a generalized derivation and g is a derivation.

Proof. Substituting $w = 0$ in (2.9), we get the inequality (2.2). Using the facts provided in the proof of Theorem 2.1, there is a unique additive mapping d satisfying $\|f(x) - d(x)\| \le 2\theta$, where $d(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$.

If we take $x = y = 0$ in (2.9), we also have

$$\|f(-zw) + zf(w) + g(z)w\| \le \theta. \tag{2.10}$$

Moreover, if we replace z and w with $2^n z$ and $2^n w$, respectively in (2.10) and then divide both sides by 2^{2n}, we get

$$\left\|\frac{f(-2^{2n}zw)}{2^{2n}} + z\frac{f(2^n w)}{2^n} + \frac{g(2^n z)}{2^n}w\right\| \le \frac{\theta}{2^{2n}} \to 0,$$

as $n \to \infty$. Hence it implies that

$$\lim_{n\to\infty} \frac{g(2^n z)}{2^n}w = -d(-zw) - zd(w) = d(zw) - zd(w),$$

because d is the odd function. Suppose that $w = e$(unit) in the above equation, then it follows that $\lim_{n\to\infty} \frac{g(2^n z)}{2^n} = d(z) - zd(e)$. Thus if $\delta(z) = d(z) - zd(e)$, we have

$$\delta(x+y) = d(x) + d(y) - xd(e) - yd(e) = \delta(x) + \delta(y).$$

Hence we show that δ is additive.

Let $C(z, w) = f(-zw) + zf(w) + g(z)w$. Since f and g satisfies the inequality given in (2.10), $\lim_{n\to\infty} \frac{C(2^n z, w)}{2^n} = 0$. We note that

$$\begin{aligned} d(-zw) &= \lim_{n\to\infty} \frac{f(-2^n z \cdot w)}{2^n} \\ &= \lim_{n\to\infty} \frac{-2^n zf(w) - g(2^n z)w + C(2^n z, w)}{2^n} = -zf(w) - \delta(z)w. \end{aligned}$$

Hence by the oddness of d, we obtain that

$$d(zw) = zf(w) + \delta(z)w. \tag{2.11}$$

Since δ is additive, we can rewrite the equation (2.11) as

$$d(2^n z \cdot w) = 2^n z f(w) + 2^n \delta(z) w, \; d(z \cdot 2^n w) = z f(2^n w) + 2^n \delta(z) w.$$

From the above relations, one can have $zf(w) = z\frac{f(2^n w)}{2^n}$. In addition, we can obtain $zf(w) = zd(w)$ as $n \to \infty$. If $z = e$, we also have that $f = d$. Therefore we have $f(zw) = zf(w) + \delta(z)w$.

We now want to show that δ is derivation using the equations developed in the previous. Indeed,

$$\delta(xy) = xf(y) + \delta(x)y - xyf(e) = x\delta(y) + \delta(x)y,$$

which means that f is the generalized derivation.

Our task is to prove that g is the derivation: Let us replace w by $2^n w$ in (2.10) and multiply by $\frac{1}{2^n}$. Then we have

$$\left\| \frac{f(-2^n zw)}{2^n} + z\frac{f(2^n w)}{2^n} + g(z)w \right\| \leq \frac{\theta}{2^n}.$$

As $n \to \infty$, we get $d(zw) = zd(w) + g(z)w$, and thus if $w = e$, we see that $g(z) = d(z) - zd(e) = \delta(z)$. Thus g is the derivation as well. The proof of the theorem is complete. $\square$

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