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ON THE STABILITY OF A MODIFIED JENSEN TYPE CUBIC MAPPING

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Abstract. In this paper we introduce a Jensen type cubic functional equation

$$f\left(\frac{3x+y}{2}\right)+f\left(\frac{x+3y}{2}\right) = 12f\left(\frac{x+y}{2}\right)+2f(x)+2f(y),$$

and then investigate the generalized Hyers–Ulam stability problem for the equation.

1. Introduction

In 1940, S. M. Ulam [11] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G_1 \to G_2$ satisfies $\rho(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then a homomorphism $h : G_1 \to G_2$ exists with $\rho(f(x), h(x)) < \epsilon$ for all $x \in G_1$?

In 1941, D. H. Hyers [2] considered the case of approximately additive mappings $f : E \to E'$, where E and E' are Banach spaces and f satisfies *Hyers inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon$$

for all $x, y \in E$. It was shown that the limit $L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in E$ and that $L : E \to E'$ is the unique additive mapping

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satisfying

$$\|f(x) - L(x)\| \leq \epsilon.$$

In 1978, Th. M. Rassias [8] provided a generalization of Hyers' Theorem which allows the *Cauchy difference operator* to be unbounded. Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then the limit $L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \leq \frac{2\epsilon}{2 - 2^p}\|x\|^p \tag{1.2}$$

for all $x \in E$. If $p < 0$ then inequality (1.1) holds for $x, y \neq 0$ and (1.2) for $x \neq 0$.

P. Găvruta [1] provided a further generalization of Th. M. Rassias' Theorem which allows the *Cauchy difference operator* to be controlled by general functions. In 1999 Y. Lee and K. Jun [7] have proved the Hyers–Ulam–Rassias stability of Jensen's equation $2f\left(\frac{x+y}{2}\right) = f(x) + f(y)$. Let X and Y be Banach spaces. Let $f : X \to Y$ be a mapping for which there exists a function $\varphi : E \setminus \{0\} \times E \setminus \{0\} \to [0, \infty)$ such that

$$\Phi(x, y) := \sum_{k=1}^{\infty} 3^{-k}\varphi(3^k x, 3^k y) < \infty,$$

$$\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\| \leq \varphi(x, y)$$

for all $x, y \in E \setminus \{0\}$. Then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - f(0) - T(x)\| \leq \frac{1}{3}(\Phi(x, -x) + \Phi(-x, 3x))$$

for all $x, y \in E \setminus \{0\}$.

During the last two decades a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers–Ulam stability to a number of functional equations and mappings. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem [3, 5, 6, 9, 10].

In [4], K. Jun and H. Kim determined the general solution of the cubic equation

$$f(2x+y)+f(2x-y) = 2f(x+y)+2f(x-y)+12f(x) \tag{1.3}$$

without assuming any regularity conditions on the unknown mapping f. It is easy to see that the function $f(x) = cx^3$ is a solution of the above functional equation. Thus, it is natural that equation (1.3) is called a cubic functional equation and every solution of the cubic functional equation (1.3) is said to be cubic function.

Now, we observe that the original cubic functional equation is equivalent to Jensen type cubic functional equations

$$4f\left(\frac{x+2y}{2}\right)+4f\left(\frac{x-2y}{2}\right)+3f(x) = 2f(x+y)+2f(x-y), \tag{1.4}$$

$$f\left(\frac{3x+y}{2}\right)+f\left(\frac{x+3y}{2}\right) = 12f\left(\frac{x+y}{2}\right)+2f(x)+2f(y). \tag{1.5}$$

In this paper, we are going to consider a modified Jensen type cubic functional equation (1.5) and then investigate the generalized Hyers–Ulam stability of the equation (1.5) using the direct method.

2. Stability of equation (1.5)

From now on, let X be a normed space and Y a Banach space. First of all, we are to establish the general solution of the functional equation (1.4) by elementary change of variables.

LEMMA 2.1. *Let X and Y be vector spaces. A mapping $f : X \to Y$ satisfies the functional equation (1.4) if and only if f is cubic.*

Proof. In fact, let f be a cubic function. Then it follows that $f(2x) = 8f(x)$ for all $x \in X$. Then replacing x by $\frac{x}{2}$ in (1.3) and then multiplying 2 on the resulting equation, we get the equation (1.4).

Conversely, let f be a solution of the equation (1.4). Then it is easy to see that $f(0) = 0$. Replacing x, y by $2x, 0$ in (1.4), respectively, we get

$f(2x) = 8f(x)$ for all $x \in X$. Thus substituting x for $2x$ in the equation (1.4), one obtains

$$\begin{aligned} &4f(x+y) + 4f(x-y) + 3f(2x) \\ &\quad = 2f(2x+y) + 2f(2x-y), \end{aligned}$$

which yields the equation (1.3). □

Lemma 2.2. *Let X and Y be vector spaces. A mapping $f : X \to Y$ satisfies the functional equation (1.5) if and only if f is cubic.*

In fact, let f be a cubic function. Then replacing x, y by $\frac{x+y}{2}, \frac{x-y}{2}$ in (1.3), respectively, we get the equation (1.5).

Conversely, let f satisfy the equation (1.5). Then replacing x, y by $x+y, x-y$ in (1.5), respectively, we get the equation (1.3).

Now, we are going to investigate the generalized Hyers–Ulam stability of the equation (1.5). For a given mapping $f : X \to Y$, we define a difference operator Df of f by

$$\begin{aligned} Df(x,y) := \quad & f\left(\frac{3x+y}{2}\right) + f\left(\frac{x+3y}{2}\right) \\ & -12f\left(\frac{x+y}{2}\right) - 2f(x) - 2(y) \end{aligned}$$

for all $x, y \in X$.

Theorem 2.3. *Let a function $f : X \to Y$ satisfy the functional inequality*

$$\|Df(x,y)\| \le \varphi(x,y) \tag{2.1}$$

and the function $\varphi : X^2 \to [0,\infty)$ satisfy

$$\Phi(x,y) := \sum_{i=0}^{\infty} \frac{\varphi(2^i x, 2^i y)}{8^i} < \infty \tag{2.2}$$

for all $x, y \in X$. Then there exists a unique cubic function $C : X \to Y$, defined by $C(x) = \lim_{n\to\infty} \frac{f(2^n x)}{8^n}$, satisfying the inequality

$$\|f(x) - C(x)\| \le \frac{1}{16}\Phi(x,x) \tag{2.3}$$

for all $x \in X$.

Proof. First, we observe that $\|f(0)\| \le \frac{\varphi(0,0)}{14}$. Put $y := x$ in (2.1) for any fixed $x \in X$. Then we obtain

$$\|2f(2x) - 16f(x)\| \le \varphi(x,x), \tag{2.4}$$

which yields

$$\left\|\frac{f(2x)}{8} - f(x)\right\| \leq \frac{1}{16}\varphi(x,x)$$

for all $x \in X$. Thus we have

$$\begin{aligned}
\text{(2.5)} \qquad \left\|\frac{f(2^m x)}{8^m} - \frac{f(2^n x)}{8^n}\right\| &\leq \sum_{i=m}^{n-1}\left\|\frac{f(2^i x)}{8^i} - \frac{f(2^{i+1}x)}{8^{i+1}}\right\| \\
&\leq \frac{1}{16}\sum_{i=m}^{n-1}\frac{\varphi(2^i x, 2^i x)}{8^i}
\end{aligned}$$

for $n > m \geq 0$. Since the right-hand side of the inequality (2.5) tends to 0 as $m \to \infty$ by the convergence of the series (2.2), the sequence $\left\{\frac{f(2^n x)}{8^n}\right\}$ is Cauchy in the Banach space Y. Therefore we may define a function $C : X \to Y$ by

$$C(x) = \lim_{n\to\infty}\frac{f(2^n x)}{8^n}$$

for all $x \in X$. By letting $n \to \infty$ in (2.5) with $m = 0$, we arrive at the formula (2.3). It follows from the definition of C and the convergence of the series (2.2) that

$$\begin{aligned}
\|DC(x,y)\| &= \lim_{n\to\infty}\left\|\frac{Df(2^n x, 2^n y)}{8^n}\right\| \\
&\leq \lim_{n\to\infty}\frac{1}{8^n}\varphi(2^n x, 2^n y) = 0
\end{aligned}$$

for all $x, y \in X$. Thus the function C is cubic.

Let $C' : X \to Y$ be another cubic function which satisfies the inequality (2.3). Since C' is cubic function, we can easily see that

$$C'(2^n x) = 8^n C'(x)$$

for any $n \in \mathbb{N}$. Thus it follows from (2.3) that

$$\begin{aligned}
\left\|\frac{f(2^n x)}{8^n} - C'(x)\right\| &= \left\|\frac{f(2^n x)}{8^n} - \frac{C'(2^n x)}{8^n}\right\| \\
&\leq \frac{1}{8^n}\sum_{i=0}^{\infty}\frac{\varphi(2^{i+n}x, 2^{i+n}x)}{8^i}
\end{aligned}$$

for all $x \in X$. By letting $n \to \infty$, then we get that $C(x) = C'(x)$ for all $x \in X$, which completes the proof of the theorem. $\square$

THEOREM 2.4. *Let a function $f : X \to Y$ satisfy the functional inequality*

$$\|Df(x,y)\| \leq \varphi(x,y)$$

for all $x, y \in X$. Suppose that there exists a constant L with $0 < L < 1$ such that the function $\varphi : X^2 \to [0, \infty)$ satisfies

$$\varphi(2x, 2y) \leq 8L\varphi(x,y) \tag{2.6}$$

for all $x, y \in X$. Then there exists a unique cubic function $C : X \to Y$, defined by $C(x) = \lim_{n\to\infty} \frac{f(2^n x)}{8^n}$, satisfying the inequality

$$\|f(x) - C(x)\| \leq \frac{\varphi(x,x)}{16(1-L)}$$

for all $x \in X$.

Proof. It follows from (2.6) that

$$\varphi(2^i x, 2^i y) \leq (8L)^i \varphi(x,y)$$

for all $x, y \in X$. Thus it follows from the inequality (2.5) that for all integers m, n with $n > m \geq 0$

$$\begin{aligned} \left\| \frac{f(2^m x)}{8^m} - \frac{f(2^n x)}{8^n} \right\| &\leq \frac{1}{16} \sum_{i=m}^{n-1} \frac{\varphi(2^i x, 2^i x)}{8^i} \\ &\leq \frac{1}{16} \sum_{i=m}^{n-1} L^i \varphi(x,x), \quad x \in X. \end{aligned} \tag{2.7}$$

Since the right-hand side of the inequality (2.7) tends to 0 as $m \to \infty$ by the convergence of the series $\sum_{i=0}^{\infty} L^i \varphi(x,x)$, the sequence $\left\{\frac{f(2^n x)}{8^n}\right\}$ is Cauchy in the Banach space Y. Therefore we may define a function $C : X \to Y$ by

$$C(x) = \lim_{n\to\infty} \frac{f(2^n x)}{8^n}$$

for all $x \in X$. By letting $n \to \infty$ in (2.7) with $m = 0$, we arrive at the desired estimation of f by a function C.

It follows from the definition of the function C that

$$\begin{aligned} \|DC(x,y)\| &= \lim_{n\to\infty} \left\| \frac{Df(2^n x, 2^n y)}{8^n} \right\| \\ &\leq \lim_{n\to\infty} \frac{1}{8^n} \varphi(2^n x, 2^n y) \\ &\leq \lim_{n\to\infty} L^n \varphi(x,y) = 0 \end{aligned}$$

for all $x, y \in X$. Thus the function C is cubic.

The proof of the remaining assertion in the theorem follows the same way as that of Theorem 2.3. $\square$

For a function $\varphi(t) := t^p, p < 3$, it follows easily that $\varphi(2t) = 8L\varphi(t)$, where $L := 2^{p-3} < 1$. Therefore, we have the following corollary by Theorem 2.3 and Theorem 2.4.

COROLLARY 2.5. *If a function $f : X \to Y$ satisfies the inequality*

$$\|Df(x, y)\| \leq \varepsilon(\|x\|^p + \|y\|^p) \tag{2.8}$$

for some $p < 3$ and for all $x, y \in X$, then there exists a unique cubic function $C : X \to Y$ such that

$$\|f(x) - C(x)\| \leq \frac{\varepsilon\|x\|^p}{8 - 2^p}$$

for all $x \in X$.

COROLLARY 2.6. *If a function $f : X \to Y$ satisfies the inequality*

$$\|Df(x, y)\| \leq \varepsilon$$

for all $x, y \in X$, then there exists a unique cubic function $C : X \to Y$ such that

$$\|f(x) - C(x)\| \leq \frac{\varepsilon}{14}$$

for all $x \in X$.

Now, we investigate another generalized Hyers–Ulam stability of the equation (1.5) controlled by a perturbing function $\varphi : X^2 \to [0, \infty)$.

THEOREM 2.7. *Let a function $f : X \to Y$ satisfy the functional inequality*

$$\|Df(x, y)\| \leq \varphi(x, y)$$

and the function $\varphi : X^2 \to [0, \infty)$ satisfy

$$\Phi_1(x, y) := \sum_{i=1}^{\infty} 8^i \varphi(\frac{x}{2^i}, \frac{y}{2^i}) < \infty$$

for all $x, y \in X$. Then there exists a unique cubic function $C : X \to Y$, defined by $C(x) = \lim_{n\to\infty} 8^n f(\frac{x}{2^n})$, satisfying the inequality

$$\|f(x) - C(x)\| \leq \frac{1}{16}\Phi_1(x, x)$$

for all $x \in X$.

Proof. We note that $\varphi(0,0)=0$ and so $f(0)=0$ by the convergence of $\Phi_1(0,0)$. Now it follows from the inequality (2.4) that

$$\begin{aligned} \left\|8^m f(\frac{x}{2^m}) - 8^n f(\frac{x}{2^n})\right\| &\leq \sum_{i=m+1}^{n} \left\|8^{i-1} f(\frac{x}{2^{i-1}}) - 8^i f(\frac{x}{2^i})\right\| \\ &\leq \frac{1}{16}\sum_{i=m+1}^{n} 8^i \varphi(\frac{x}{2^i}, \frac{x}{2^i}) \end{aligned} \tag{2.9}$$

for $n > m \geq 0$.

The rest of our proof is similar to the corresponding part of Theorem 2.3. □

THEOREM 2.8. *Let a function $f : X \to Y$ satisfy the functional inequality*

$$\|Df(x,y)\| \leq \varphi(x,y)$$

for all $x, y \in X$. Suppose that there exists a constant L with $0 < L < 1$ such that the function $\varphi : X^2 \to [0,\infty)$ satisfies

$$\varphi(x,y) \leq \frac{L}{8}\varphi(2x, 2y) \tag{2.10}$$

for all $x, y \in X$. Then there exists a unique cubic function $C : X \to Y$, defined by $C(x) = \lim_{n\to\infty} 8^n f(\frac{x}{2^n})$, satisfying the inequality

$$\|f(x) - C(x)\| \leq \frac{L\varphi(x,x)}{16(1-L)}$$

for all $x \in X$.

Proof. We note that $\varphi(0,0) = 0$ and so $f(0) = 0$ by the inequality $\varphi(0,0) \leq \frac{L}{8}\varphi(0,0)$. It follows from (2.10) that

$$\varphi(\frac{x}{2^i}, \frac{y}{2^i}) \leq (\frac{L}{8})^i \varphi(x,y)$$

for all $x, y \in X$. Thus it follows from the inequality (2.9) that for all integers m, n with $n > m \geq 0$

$$\begin{aligned} \left\|8^m f(\frac{x}{2^m}) - 8^n f(\frac{x}{2^n})\right\| &\leq \frac{1}{16}\sum_{i=m+1}^{n} 8^i \varphi(\frac{x}{2^i}, \frac{x}{2^i}) \\ &\leq \frac{1}{16}\sum_{i=m+1}^{n} L^i \varphi(x,x), \quad x \in X. \end{aligned}$$

The rest of the proof is similar to the corresponding part of Theorem 2.4. □

For a function $\varphi(t) := t^p, p > 3$, it follows easily that $\varphi(\frac{t}{2}) = \frac{L}{8}\varphi(t),\ L := 2^{3-p} < 1$.

Corollary 2.9. *If a function $f : X \to Y$ satisfies the inequality (2.8) for some $p > 3$ and for all $x, y \in X$, then there exists a unique cubic function $C : X \to Y$ such that*

$$\|f(x) - C(x)\| \le \frac{\varepsilon\|x\|^p}{2^p - 8}$$

for all $x \in X$.

Using the similar argument, we can obtain the generalized Hyers–Ulam stability result for the equation (1.4).

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