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ON THE HIGH-ORDER CONVERGENCE OF THE k-FOLD PSEUDO-CAUCHY'S METHOD FOR A SIMPLE ROOT

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Abstract. In this study the k-fold pseudo-Cauchy's method of order $k+3$ is proposed from the classical Cauchy's method defined by an iteration $x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)} \cdot \left(1 - \sqrt{1 - 2f(x_n)f''(x_n)/f'(x_n)^2}\,\right)$. The convergence behavior of the asymptotic error constant is investigated near the corresponding simple zero. A root-finding algorithm with the k-fold pseudo-Cauchy's method is described and computational examples have successfully confirmed the current analysis.

1. Introduction

Let a function $f : \mathbb{R} \to \mathbb{R}$ have a simple real zero α and be sufficiently smooth in a neighborhood of α. The aim of this study is to locate α accurately with a high-order method. To this end we first transform the equation $f(x) = 0$ into an equivalent form $x - g(x) = 0$, where $g : \mathbb{R} \to \mathbb{R}$ is assumed to be sufficiently smooth in a neighborhood of α. Then an approximated α is to be sought within a prescribed error bound using an iterative scheme

$$x_{n+1} = g(x_n),\ n = 0,\ 1,\ 2, \cdots, \tag{1.1}$$

where $x_0 \in \mathbb{R}$ is given. For a given $p \in \mathbb{N}$, we further assume that

$$\begin{cases} \left|\frac{d^p}{dx^p} g(x)\right|_{x=\alpha} = |g^{(p)}(\alpha)| < 1, & \text{if } p = 1. \\ g^{(i)}(\alpha) = 0 \text{ for } 1 \le i \le p-1 \text{ and } g^{(p)}(\alpha) \ne 0, & \text{if } p \ge 2. \end{cases} \tag{1.2}$$

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Under the assumption that x_n belongs to a sufficiently small neighborhood of α for $n \in \mathbb{N} \cup \{0\}$, Taylor series[1,8] expansion about α yields

$$x_{n+1} = g(x_n) = g(\alpha) + g^{(p)}(\xi)\ (x_n - \alpha)^p/p!, \tag{1.3}$$

where $\xi \in (a, b)$ with $a = \min(\alpha, x_n)$ and $b = \max(\alpha, x_n)$. Since g is continuous at α, we find that for all given $\epsilon > 0$, there exists a number $\delta > 0$ such that

$$|x_{n+1} - \alpha| = |g(x_n) - g(\alpha)| = |g^{(p)}(\xi)|\ \frac{|(x_n - \alpha)^{p-1}|}{p!}\ |x_n - \alpha| < \epsilon, \tag{1.4}$$

whenever $|x_n - \alpha| < \delta$. Let $\boldsymbol{J} = \{x : |x - \alpha| \leq \delta\}$. The continuity of $g^{(p)}$ on $\boldsymbol{J}$ ensures the existence of a number $M > 0$ satisfying $|g^{(p)}(x)| \leq M$ for all $x \in \boldsymbol{J}$. Choose

$$\delta = \begin{cases} \min(\epsilon,\ 1/M), \text{ if } p = 1. \\ \{\min(\epsilon^{p-1},\ p!/M)\}^{1/(p-1)}, \text{ if } p \geq 2. \end{cases}$$

Then $|x_{n+1} - \alpha| = |g(x_n) - g(\alpha)| \leq |x_n - \alpha|$. Hence $g : \boldsymbol{J} \to \boldsymbol{J}$. Since $|x_n - \alpha| < \delta$, it follows from (1.4) that

$$|x_{n+1} - \alpha| \leq |g(x_n) - g(\alpha)| \leq K|x_n - \alpha|, \tag{1.5}$$

where $0 < K = \sup\{M\ |(x_n - \alpha)|^{p-1}/p! : n \in \mathbb{N} \cup \{0\}\} < M\delta^{p-1}/p!\ \leq 1$ for $p \geq 2$. If $p = 1$, then $K = M < 1$ can be chosen according to (1.2). Hence g is contractive on $\boldsymbol{J}$ for any $p \in \mathbb{N}$ and the sequence $\{x_n\}_{n=0}^{\infty}$ with $x_0 \in \boldsymbol{J}$ defined by (1.1) converges to a fixed point $\alpha \in \boldsymbol{J}$[6]. Now introducing $e_n = x_n - \alpha$ with the fact that $\lim_{n\to\infty} \xi = \alpha$, for the iterative method (1.1) we obtain the *asymptotic error constant* η and *order of convergence* p[2,9] as follows:

$$\eta = \lim_{n\to\infty} \left| \frac{e_{n+1}}{e_n{}^p} \right| = \ |g^{(p)}(\alpha)|\ /\ p!. \tag{1.6}$$

Now for an arbitrarily given $x \in \mathbb{R}$, under the further assumption that $f''(\alpha) \neq 0$ we define a function $F : \mathbb{R} \to \mathbb{R}$ by

$$F(w) = w - \frac{f'(x)}{f''(x)} \cdot \Big(1 - \sqrt{1 - \frac{2f(w)f''(x)}{f'(x)^2}}\ \Big) \tag{1.7}$$

such that (1.7) is well-defined in a sufficiently small neighborhood of α. Let $w_0 = F(x)$, and let $w_k(x)$ be recursively defined for $k \in \mathbb{N}$ by

$$w_k(x) = F(w_{k-1}) = w_{k-1} - \frac{f'(x)}{f''(x)} \cdot \Big(1 - \sqrt{1 - \frac{2f((w_{k-1})f''(x)}{f'(x)^2}}\ \Big) \tag{1.8}$$

Hence $w_k(x) = F^k(w_0) = F^{k+1}(x)$ for $k \in \mathbb{N}$, where we denote F^k by $F^k(w_0) = F(F(\cdots F(w_0)\cdots))$. Then the iterative scheme with $x_0 \in \mathbb{R}$

$$x_{n+1} = F^{k+1}(x_n) = g(x_n) \tag{1.9}$$

is called the *k-fold pseudo-Cauchy's method.* If $k = 0$, it is called the *Cauchy's method* and has the cubic convergence as shown in Laguerre-type numerical methods[6,7] including Halley's method and leap-frogging Newton's method[3,5]. If $k = 1$, it is simply called the *pseudo-Cauchy's method.*

2. Convergence analysis

By virtue of $f'(\alpha) \neq 0$ and $f''(\alpha) \neq 0$, Equation (1.8) shows that

$$w_k(\alpha) = \alpha, \text{ for all } k \in \mathbb{N} \cup \{0\}, \tag{2.1}$$

$$w_0{}'(\alpha) = w_0{}''(\alpha) = 0, \; w_0{}'''(\alpha) = -f'''(\alpha)/f'(\alpha). \tag{2.2}$$

Further analysis with the above results leads to Lemma 2.1:

Lemma 2.1. *Let $w_k^{(m)}(\alpha) = \frac{d^m}{dx^m} w_k(x)|_{x=\alpha}$ for any $k, m \in \mathbb{N} \cup \{0\}$. For the given function f with a simple zero α, as described in Section 1, let $c = f''(\alpha)/f'(\alpha)$ and $d = -f'''(\alpha)/f'(\alpha)$. In addition, let $H = f(w_{k-1}(x))$, $G = \sqrt{f'(x)^2 - 2H \cdot f''(x)}$ for $k \in \mathbb{N}$. Then the following relations hold:*

$$w_k^{(m)}(\alpha) = \begin{cases} \alpha, \; if \; m = 0. \\ 0, \; if \; 1 \le m \le k+2. \\ \frac{(k+3)!}{3!} c^k d, \; if \; m = k+3. \end{cases} \tag{2.3}$$

$$H^{(m)}(\alpha) = \begin{cases} 0, \; if \; 0 \le m \le k+2. \\ f'(\alpha) \; w_{k-1}^{(m)}(\alpha), \; if \; m = k+2, k+3. \end{cases} \tag{2.4}$$

(2.5)

$$G^{(m)}(\alpha) = \begin{cases} f^{(m+1)}(\alpha) \;, \; if \; 0 \le m \le k+1. \\ f^{(k+3)}(\alpha) - f''(\alpha) \; w_{k-1}^{(k+2)}(\alpha) \;, \; if \; m = k+2. \\ f^{(k+4)}(\alpha) - f''(\alpha) \; w_{k-1}^{(k+3)}(\alpha) \\ \quad + (k+3) \; w_{k-1}^{(k+2)}(\alpha) \Big(\frac{f''(\alpha)^2}{f'(\alpha)} - f'''(\alpha) \Big), \; if \; m = k+3. \end{cases}$$

Proof. If $k = 0$, then the assertion (2.3) is clear from (2.1) and (2.2). Hence it suffices to consider $k \in \mathbb{N}$. If $m = 0$, then equation (2.1) immediately gives the assertion. The remaining proof will be given based on induction on $m \geq 0$. We rewrite (1.8) to obtain for $k \in \mathbb{N}$

$$\eta_k \cdot f'' + f' = G, \tag{2.6}$$

where $\eta_k = w_k(x) - w_{k-1}(x),\ f' = f'(x),\ f'' = f''(x)$. It is easily shown that $G(\alpha) = f'(\alpha)$.

Suppose now (2.3), (2.4) and (2.5) hold for $m \geq 0$. As a result, we assume the following:

$$w_{k-1}^{(m)}(\alpha) = \begin{cases} \alpha,\ if\ m = 0. \\ 0,\ if\ 1 \leq m \leq k+1. \\ \frac{(k+2)!}{3!} c^{k-1} d,\ if\ m = k+2. \end{cases} \tag{2.7}$$

By differentiating $(m+1)$ times both sides of (2.6) with respect to x via Leibnitz Rule[8] and evaluating at $x = \alpha$ we obtain

$$\sum_{r=0}^{m+1} {}_{m+1}C_r \cdot {\eta_k}^{(r)}(x) \cdot f^{(m+3-r)}(x)\Big|_{x=\alpha} + f^{(m+2)}(\alpha) = G^{(m+1)}(\alpha), \tag{2.8}$$

where ${}_mC_r = \frac{m!}{(m-r)!r!}$. Since $w_k^{(r)}(\alpha) - w_{k-1}^{(r)}(\alpha) = 0$ for $0 \leq r \leq m-1 \leq k+1$, the leftmost side of (2.8) has possible nonvanishing terms for $r = m$ and $r = m+1$ as follows:

$$\begin{aligned} -(m+1)f'''(\alpha) \cdot w_{k-1}^{(m)}(\alpha) \ + \ f''(\alpha) \cdot \Big(w_k^{(m+1)}(\alpha) - w_{k-1}^{(m+1)}(\alpha)\Big) \\ + f^{(m+2)}(\alpha) \ = \ G^{(m+1)}(\alpha) \end{aligned} \tag{2.9}$$

in view of the induction hypothesis that $w_k^{(m)}(\alpha) = 0$ for $1 \leq m \leq k+2$. To get the right side of (2.8), we consider the following identity:

$$G^2 = f'(x)^2 - 2\ H\ f''(x). \tag{2.10}$$

Now suppose (2.4) holds for $m \geq 0$. Since $H'(x) = f'\Big(w_{k-1}(x)\Big)\ {w_{k-1}}'(x)$, we find that

$$\begin{aligned} H^{(m+1)}(\alpha) &= {H'}^{(m)}(x)\Big|_{x=\alpha} \\ &= \sum_{r=0}^{m} {}_mC_r \cdot w_{k-1}^{(r+1)}(\alpha) \cdot \Big[f'\Big(w_{k-1}(x)\Big)\Big]^{(m-r)}\ \Big|_{x=\alpha}. \end{aligned} \tag{2.11}$$

If $m = 0$, then (2.11) holds by the induction hypothesis. If $1 \leq m \leq k$, then $w_{k-1}^{(r+1)}(\alpha) = 0$ holds for $0 \leq r \leq m-1 \leq k-1$ by the induction hypothesis and yields $H^{(m+1)}(\alpha) = 0$. If $m = k+1$, then the only term for $r = m$ on the right side of (2.11) gives us $H^{(m+1)}(\alpha) = w_{k-1}^{(m+1)}(\alpha)\ f'(\alpha)$. If $m = k+2$, then the two terms for $r = m-1$ and $r = m$ on the right side of (2.11) gives us $H^{(m+1)}(\alpha) = w_{k-1}^{(m+1)}(\alpha)\ f'(\alpha) + m\ w_{k-1}^{(m)}(\alpha) \left[f'\Big(w_{k-1}(x)\Big)\right]'\Big|_{x=\alpha} = f'(\alpha)\ w_{k-1}^{(m+1)}(\alpha)$ in view of the induction hypothesis ${w_{k-1}}'(\alpha) = 0$ for $k \geq 1$. Hence (2.4) holds for $m+1 \in \mathbb{N}$, which completes the induction proof to (2.4).

We next wish to prove (2.10). To this end, we first let $F(x) = f'(x)$ to obtain the identity:

$$G^2 = F(x)^2 - 2\ H\ F'(x). \tag{2.12}$$

Differentiating both sides of (2.12) $m+1$ times with respect x via Leibnitz Rule[8] and evaluating at $x = \alpha$, we get the following identity:

$$\begin{aligned} &\sum_{r=0}^{m+1} {}_{m+1}C_r \left(G^{(r)} G^{(m+1-r)} - F^{(r)} F^{(m+1-r)}\right) \\ &\qquad = -2 \sum_{r=0}^{m+1} {}_{m+1}C_r\ H^{(r)}\ F^{(m+2-r)}, \end{aligned} \tag{2.13}$$

where $G^{(j)} = G^{(j)}(\alpha),\ F^{(j)} = F^{(j)}(\alpha)$ and $H^{(j)} = H^{(j)}(\alpha)$ for $0 \leq j \leq m+1$ and further $G^{(0)} = G, F^{(0)} = F$ and $H^{(0)} = H$. Let T_m denote the left side of (2.13). Combining the first with last terms from T_m and denoting the sum of remaining terms by S_m, we find

$$T_m = 2\ G\ \left(G^{(m+1)} - F^{(m+1)}\right) + S_m, \tag{2.14}$$

where

(2.15)

$$S_m = \begin{cases} 0, \text{ if } m = 0. \\ \sum_{r=1}^{m} {}_{m+1}C_r \left(G^{(r)} G^{(m+1-r)} - F^{(r)} F^{(m+1-r)}\right), \text{ if } 1 \leq m \leq k. \end{cases}$$

Using (2.4) in the right side of (2.13), we find the proof to (2.5) as follows:

(1) if $m = 0$, then $0 = T_0 = 2\ G\ \Big(G' - F'\Big)$, which yields $G' - F' = 0$.

(2) if $m = 1$, then $0 = T_1 = 2\ G\ \Big(G'' - F''\Big) + S_1$, where

$S_1 = {}_2C_1 \cdot G'\Big(G' - F'\Big) = 0.$

As a result, we find that $G'' - F'' = 0$.

(3) if $m = 2$, then $0 = T_2 = 2\ G\ \Big(G''' - F'''\Big) + S_2$, where

$S_2 = {}_3C_1 \cdot G'\Big(G'' - F''\Big) + {}_3C_2 \cdot G''\Big(G' - F'\Big) = 0$. As a result, we find that $G''' - F''' = 0$.

(4) Continuing in this manner, we obtain, for $0 \le m \le k$,

$$G^{(m+1)} = F^{(m+1)}. \tag{2.16}$$

(5) if $m = k+1$, then
$T_{k+1} = 2\ G\ \Big(G^{(k+2)} - F^{(k+2)}\Big) = -2H^{(k+2)} \cdot f''(\alpha)$, which yields

$$G^{(k+2)} = F^{(k+2)} - f''(\alpha)\ w_{k-1}^{(k+2)}(\alpha) \tag{2.17}$$

using $H^{(k+2)} = f'(\alpha)\ w_{k-1}^{(k+2)}(\alpha)$.

(6) if $m = k+2$, then $T_{k+2} = 2\ G\ \Big(G^{(k+3)} - F^{(k+3)}\Big) + S_{k+2}$.
By virtue of the identity $S_{k+2} = 2\ (k+3)\ G'\ \Big(G^{(k+2)} - F^{(k+2)}\Big)$, we finally obtain

$$\begin{aligned} G^{(k+3)} &= F^{(k+3)} - f''(\alpha)\ w_{k-1}^{(k+3)}(\alpha) \\ &\quad + (k+3)\ w_{k-1}^{(k+2)}(\alpha)\Big(\frac{f''(\alpha)^2}{f'(\alpha)} - f'''(\alpha)\Big). \end{aligned} \tag{2.18}$$

To prove (2.3), we first find that $w_k^{(m+1)}(\alpha) = 0$ for $0 \le m \le k$ by the induction hypothesis. Comparing (2.18) with the left side of (2.9) when $m = k+2$ and simplifying, we obtain:

$$w_k^{(m+1)}(\alpha) = c(m+1)\ w_{k-1}^{(m)}(\alpha).$$

Hence it follows that

$$w_k^{(m+1)}(\alpha) = \begin{cases} 0, \text{ if } 2 \le m+1 \le k+2. \\ c\ (m+1)\ w_{k-1}^{(m)}(\alpha), \text{ if } m+1 = k+3. \end{cases} \tag{2.19}$$

It is also found for $m+1 = k+3$ that

(2.20)

$$\begin{aligned} & w_k^{(m+1)}(\alpha) \\ &= w_k^{(k+3)}(\alpha) = c(k+3)\ w_{k-1}^{(k+2)}(\alpha) = c^2\ (k+3)(k+2)\ w_{k-2}^{(k+1)}(\alpha) \\ &= (k+3)(k+2)(k+1)\cdots 4\cdot 3 \cdots c^k \cdot {w_0}'''(\alpha) = \frac{(k+3)!}{3!}c^k d. \end{aligned}$$

Thus (2.3) also holds for $m+1 \in \mathbb{N}$, from which the induction proof is completed. □

As a consequence of the preceding analysis, we have proved the following theorem.

THEOREM 2.2. *Let $k \in \mathbb{N} \cup \{0\}$ be given and α be a simple zero of the smooth function f described in Section 1. Then the k-fold pseudo-Cauchy's method defined by (1.9) is at least of order $k+3$ and its asymptotic error constant η is given by $|c^k\ d|/6$, where $c = f''(\alpha)/f'(\alpha)$ and $d = -f'''(\alpha)/f'(\alpha)$.*

Proof. Let $g(x) = w_k(x) = F^{k+1}(x)$ and define the iteration $x_{n+1} = g(x_n)$ with $x_0 \in \boldsymbol{J}$ and the error $e_n = x_n - \alpha$ for $n \in \mathbb{N} \cup \{0\}$. Then g is contractive on $\boldsymbol{J}$ and Lemma 2.1 yields the asymptotic error constant η and the order of convergence p in view of (1.6)

$$\eta = \lim_{n\to\infty} \left| \frac{e_{n+1}}{{e_n}^{k+3}} \right| = \frac{1}{(k+3)!}\ |w_k^{(k+3)}(\alpha)| = \frac{1}{6}\ |c^k\ d|, \tag{2.21}$$

which completes the proof. □

3. Algorithm and numerical results

The theory presented in Sections 1 and 2 will be verified here with a zero-finding algorithm coded by *Mathematica*[10] along with some numerical experiments.

Algorithm 3.1 (Zero-Finding Algorithm)

Step 1. For $k \in \mathbb{N} \cup \{0\}$, construct the iteration function $g = F^{k+1}$ with the given function f having a simple real zero α, according to the description in Section 1.

Step 2. Set the minimum number of precision digits. With exact zero or most accurate zero α, supply the theoretical asymptotic error constant η. Set the error range ϵ, the maximum iteration number n_{max} and the initial value x_0. Compute $f(x_0)$ and $|x_0 - \alpha\ |$.

Step 3. Compute $x_{n+1} = g(x_n)$ for $0 \le n \le n_{max}$ and display the computed values of $n,\ x_n, f(x_n), |x_n - \alpha|,\ |e_{n+1}/{e_n}^{k+3}|$ and η.

To achieve sufficient accuracy, the minimum number of precision digits was selected between 250 and 700 by assigning the value of *$MinPrecision* in Mathematica. The error bound ϵ for $|\ x_n - \alpha\ | < \epsilon$ was chosen small enough up to 0.5×10^{-675} for the current experiment. Table 1 illustrates the order of convergence as well as the asymptotic error constant for a nonlinear function

$$f(x) = e^{-x} \sin x + \ln[1 + (x-\pi)^2]$$

that has a simple zero $\alpha = \pi$ and $f''(\alpha) \neq 0$. The number of computation is observed to get smaller as k increases due to high-order convergence.

TABLE 1. Convergence of k-fold pseudo-Cauchy's Method for $f(x) = e^{-x}\sin x + \ln[1 + (x - \pi)^2]$

k	n	x_n	$f(x_n)$	$\mid x_n - \alpha \mid$	$e_{n+1}/e_n{}^{k+3}$	η
	0	2.60000000000000	0.295503	0.541593		
	1	2.99547951767754	0.0284059	0.146113	0.9197524800	
	2	3.13204679372873	0.000507584	0.00954586	3.060182090	
0	3	3.14159265536030	-7.65103×10^{-11}	1.77050×10^{-9}	0.0020354069	0.3333333333
	4	3.14159265358979	7.99451×10^{-29}	1.84998×10^{-27}	0.3333333948	
	5	3.14159265358979	-9.12025×10^{-83}	2.11049×10^{-81}	0.3333333333	
	6	3.14159265358979	1.35410×10^{-244}	3.13349×10^{-243}	0.3333333333	
	7	3.14159265358979	$0. \times 10^{-300}$	$0. \times 10^{-299}$		
	0	2.60000000000000	0.295503	0.541593		
	1	3.02620460595904	0.0188104	0.115388	1.341126568	
	2	3.13790281432158	0.000173656	0.00368984	20.81441205	
1	3	3.14159265514911	-6.73841×10^{-11}	1.55931×10^{-9}	8.412089525	16.09379509
	4	3.14159265358979	-4.11166×10^{-36}	9.51466×10^{-35}	16.09379891	
	5	3.14159265358979	$-5.69974 \times 10^{-137}$	1.31896×10^{-135}	16.09379509	
	6	3.14159265358979	$0. \times 10^{-300}$	$0. \times 10^{-299}$		
	0	2.60000000000000	0.295503	0.541593		
	1	3.04643915130075	0.0135291	0.0951535	2.042024527	
	2	3.14012436140057	0.0000656997	0.00146829	188.2298946	
2	3	3.14159265359371	-1.69275×10^{-13}	3.91714×10^{-12}	573.9936360	777.0307211
	4	3.14159265358979	3.09677×10^{-56}	7.16615×10^{-55}	777.0307217	
	5	3.14159265358979	$-6.34586 \times 10^{-270}$	1.46848×10^{-268}	777.0307211	
	6	3.14159265358979	$0. \times 10^{-300}$	$0. \times 10^{-299}$		
	0	2.60000000000000	0.295503	0.541593		
	1	3.06094908935581	0.0102558	0.0806436	3.195458088	
	2	3.14100601394565	0.0000257100	0.000586640	2132.814690	
3	3	3.14159265358979	-5.70585×10^{-17}	1.32037×10^{-15}	32394.25364	37516.11961
	4	3.14159265358979	-8.59053×10^{-87}	1.98791×10^{-85}	37516.11961	
	5	3.14159265358979	$0. \times 10^{-300}$	$0. \times 10^{-299}$		
	0	2.60000000000000	0.295503	0.541593		
	1	3.07192827684424	0.00806645	0.0696644	5.096844722	
	2	3.14136073712516	0.0000100781	0.000231916	29124.35323	
4	3	3.14159265358979	-2.63627×10^{-21}	6.10051×10^{-20}	1690617.199	1811330.224
	4	3.14159265358979	2.46142×10^{-130}	5.69590×10^{-129}	1811330.224	
	5	3.14159265358979	$0. \times 10^{-300}$	$0. \times 10^{-299}$		
	0	2.60000000000000	0.295503	0.541593		
	1	3.08055308669244	0.00652095	0.0610396	8.245734090	
	2	3.14150292389386	3.88597×10^{-6}	0.0000897297	465633.9615	
5	3	3.14159265358979	-1.53968×10^{-26}	3.56293×10^{-25}	84784646.40	87453532.41
	4	3.14159265358979	$-9.81421 \times 10^{-190}$	2.27108×10^{-188}	87453532.41	
	5	3.14159265358979	$0. \times 10^{-300}$	$0. \times 10^{-299}$		

For each $0 \le k \le 5$, the order of convergence has been confirmed to be of at least $k+3$. As the second numerical example, we take $f(x) = \cos x - x$ with a simple zero

$$\alpha = 0.739085133215160641655312087673873404013 4 \cdots\cdots 002409803355672730892,$$

which is found to be accurate up to 700 significant decimal digits from the Mathematica command *FindRoot with options WorkingPrecision* $\rightarrow$ 1400*, AccuracyGoal* $\rightarrow$ 700. Table 2 also reveals a good agreement with the theory discussed in this paper. The computed asymptotic error constants were found to be in good agreement with the theory. The

TABLE 2. Convergence of k-fold pseudo-Cauchy's Method for $f(x) = \cos x - x$

k	n	x_n	$f(x_n)$	$\mid x_n - \alpha \mid$	$\lvert e_{n+1}/e_n{}^{k+3}\rvert$	η
0	0	0.600000000000000	0.225336	0.139085		0.06708165905
	1	0.738926849807921	0.000264896	0.000158283	0.05882924272	
	2	0.739085133214895	4.45152×10^{-13}	2.65983×10^{-13}	0.06707292075	
	3	0.739085133215161	2.11261×10^{-39}	1.26230×10^{-39}	0.06708165905	
	4	0.739085133215161	2.25814×10^{-118}	1.34926×10^{-118}	0.06708165905	
	5	0.739085133215161	2.75769×10^{-355}	1.64775×10^{-355}	0.06708165905	
	6	0.739085133215161	$0. \times 10^{-699}$	$0. \times 10^{-700}$		
1	0	0.600000000000000	0.225336	0.139085		0.02962398456
	1	0.739096143389592	-0.0000184268	0.0000110102	0.02942194343	
	2	0.739085133215161	-7.28576×10^{-22}	4.35331×10^{-22}	0.02962396029	
	3	0.739085133215161	-1.78065×10^{-87}	1.06395×10^{-87}	0.02962398456	
	4	0.739085133215161	$-6.35317 \times 10^{-350}$	3.79609×10^{-350}	0.02962398456	
	5	0.739085133215161	$0. \times 10^{-699}$	$0. \times 10^{-700}$		
2	0	0.600000000000000	0.225336	0.139085		0.01308227128
	1	0.739084366346134	1.28344×10^{-6}	7.66869×10^{-7}	0.01473389746	
	2	0.739085133215161	5.80690×10^{-33}	3.46968×10^{-33}	0.01308228103	
	3	0.739085133215161	1.10099×10^{-164}	6.57855×10^{-164}	0.01308227128	
	4	0.739085133215161	$0. \times 10^{-699}$	$0. \times 10^{-700}$		
3	0	0.600000000000000	0.225336	0.139085		0.005777272176
	1	0.739085186623453	-8.93848×10^{-8}	5.34083×10^{-8}	0.007377758329	
	2	0.739085133215161	-2.24404×10^{-46}	1.34083×10^{-46}	0.005777271599	
	3	0.739085133215161	$-5.61859 \times 10^{-278}$	3.35716×10^{-278}	0.005777272176	
	4	0.739085133215161	$0. \times 10^{-699}$	$0. \times 10^{-700}$		
4	0	0.600000000000000	0.225336	0.139085		0.002551305739
	1	0.739085129495538	6.22521×10^{-9}	3.71962×10^{-9}	0.003694315300	
	2	0.739085133215161	4.20639×10^{-62}	2.51336×10^{-62}	0.002551305766	
	3	0.739085133215161	2.70522×10^{-434}	1.61640×10^{-434}	0.002551305739	
	4	0.739085133215161	$0. \times 10^{-699}$	$0. \times 10^{-700}$		
5	0	0.600000000000000	0.225336	0.139085		0.001126684147
	1	0.739085133474214	-4.33555×10^{-10}	2.59053×10^{-10}	0.001849878907	
	2	0.739085133215161	-3.82445×10^{-80}	2.28515×10^{-80}	0.001126684146	
	3	0.739085133215161	$-1.40207 \times 10^{-640}$	8.37751×10^{-641}	0.001126684147	
	4	0.739085133215161	$0. \times 10^{-699}$	$0. \times 10^{-700}$		

computed root was rounded to be accurate up to the 250 significant digits. The limited paper space lists only up to 15 significant digits.

The high-order convergence[4] stated in Theorem 1 has been further confirmed with other test functions such as $f(x) = x^2 \sin(\pi x/8) + e^{(x-2)^2} - 1 - 2\sqrt{2}$ $(\alpha = 2)$, $f(x) = e^{(x^2+7x-30)} - 1$ $(\alpha = 3)$, $f(x) = \sin(\pi x/(2\sqrt{2}) - x^4 + 3$ $(\alpha = \sqrt{2})$ and many additional numerical experiments. The current numerical scheme can be extended to accurately locate multiple zeros of a given nonlinear algebraic equation.

References

[1] R. G. Bartle, *The Elements of Real Analysis*, 2nd ed., John Wiley & Sons., New York, 1976.

[2] S. D. Conte and Carl de Boor, *Elementary Numerical Analysis*, McGraw-Hill Inc., 1980.

[3] Young Hee Geum, *The asymptotic error constant of leap-frogging Newtons method locating a simple real zero*, Applied Mathematics and Computation, **189**, Issue 1 (2007), 963-969.

[4] Young Hee Geum, Young Ik Kim and Min Surp Rhee, *High-order convergence of the k-fold pseudo-Newtons irrational method locating a simple real zero*, Applied Mathematics and Computation, **182**, Issue 1 (2006), 492-497.

[5] A. Bathi Kasturiarachi, *Leap-frogging Newton's Method*, Int. J. Math. Educ. Sci. Technol. **33** (2002), no. 4, 521-527.

[6] Qiang Du, Ming Jin, T. Y. Li and Z. Zeng, *The Quasi-Laguerre Iteration*, Mathematics of Computation, **66** (1997), no. 217, 345-361.

[7] L. D. Petkovic, M. S. Petkovic and D. Zivkovic, *Hansen-Patrick's Family Is of Laguerre's Type*, Novi Sad J. Math. **33** (2003), no. 1, 109-115.

[8] Kenneth A. Ross, *Elementary Analysis*, Springer-Verlag New York Inc., 1980.

[9] J. Stoer and R. Bulirsh, *Introduction to Numerical Analysis*, 244-313, Springer-Verlag New York Inc., 1980.

[10] Stephen Wolfram, *The Mathematica Book, 4th ed.*, Cambridge University Press, 1999.

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