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RECURRENCE RELATIONS OF THE STANDARD GENERALIZED EXTREME VALUE DISTRIBUTION BY THE LOWER RECORD VALUES

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Abstract. In this paper, we establish some recurrence relations which is satisfied by the quotient moments of the lower record values from the standard generalized extreme value (GEV) distribution.

1. Introduction

The record value model was introduced by Chandler [6]. Let X_1, $X_2, \cdots$ be a sequence of independent and identically distributed (i.i.d.) random variables with a cumulative distribution function (cdf) $F(x)$ and a probability density function (pdf) $f(x)$. Suppose $Y_n = \min\{X_1, X_2, \cdots, X_n\}$ for $n \geq 1$. We say X_j, $j \geq 1$ is a lower record value of this sequence, if $Y_j < Y_{j-1}$ for $j > 1$. And we suppose that X_1 is a first lower record value. The indices at which the lower record values occur are given by record times $\{L(n), n \geq 1\}$, where $L(n) = \min\{j | j > L(n-1)\ ,\ X_j < X_{L(n-1)}\ ,\ n \geq 2\}$ with $L(1) = 1$.

The cdf of the standard GEV distribution is

$$F(x) = \begin{cases} e^{-(1-kx)^{\frac{1}{k}}}, \left\{ \begin{array}{l} x < \dfrac{1}{k} \ \text{ when } k > 0 \\ x > \dfrac{1}{k} \ \text{ when } k < 0 \end{array} \right\} \\ e^{-e^{-x}}, -\infty < x < \infty \text{ when } \ k = 0 \end{cases} \tag{1.1}$$

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and the pdf of the standard GEV distribution is

$$(1.2)\qquad f(x) = \begin{cases} (1-kx)^{\frac{1}{k}-1} e^{-(1-kx)^{\frac{1}{k}}}, & \left\{ \begin{array}{l} x < \frac{1}{k} \text{ when } k > 0 \\ x > \frac{1}{k} \text{ when } k < 0 \end{array} \right\} \\ e^{-x}\, e^{-e^{-x}}, -\infty < x < \infty \text{ when } k = 0 \end{cases}$$

A notation that designates that X has the cdf (1.1) is $X \sim GEV(0,1,k)$.

Some recurrence relations by the lower record values are known. Ahsanullah [1, 2] established recurrence relations for various distributions by record values. Also, Balakrishnan and Ahsanullah [3] have proved recurrence relations for the exponential distribution and the generalized Pareto distribution. Balakrishnan, Ahsanullah and Chan [4, 5] investigated recurrence relations for the Gumbel distribution and the generalized extreme value distribution. These recurrence relations satisfied by the single and product moments of record values.

In this paper, we will show some recurrence relations which is satisfied by the quotient moments of the lower record values from the standard GEV distribution.

2. Main theorems

THEOREM 2.1. *For $k \neq 0$, $1 \leq m < n$ and $r,\ s = 0, 1, 2, \cdots$,*

$$E\left(\frac{X^s_{L(n)}}{X^{r+1}_{L(m)}}\right) = \frac{(r+1)}{k(r+1)-m} E\left(\frac{X^s_{L(n)}}{X^{r+2}_{L(m)}}\right) - \frac{m}{k(r+1)-m} E\left(\frac{X^s_{L(n)}}{X^{r+1}_{L(m+1)}}\right).$$

Proof. The joint pdf $f_{(m),(n)}(x,y)$ of the lower record values $X_{L(m)}$ and $X_{L(n)}$ is given by

$$f_{(m),(n)}(x,y) = \frac{(H(x))^{m-1} h(x) (H(y) - H(x))^{n-m-1} f(y)}{\Gamma(m)\,\Gamma(n-m)},$$

where $H(x) = -ln(F(x))$ and $h(x) = \left(-\frac{d}{dx} H(x)\right) = \frac{f(x)}{F(x)}$.

We have that for the standard GEV in (1.1) and (1.2),

$$(2.1)\qquad (1-kx)\, f(x) = F(x)\, H(x).$$

Let us consider for $k \neq 0$, $1 \leq m < n$ and $r,\ s = 0, 1, 2, \cdots$,

$$E\left(\frac{X^s_{L(n)}}{k\,X^{r+2}_{L(m)}} - \frac{X^s_{L(n)}}{X^{r+1}_{L(m)}}\right)$$

$$= \int\int_{-\infty<y<x<\infty} \left(\frac{y^s}{k\,x^{r+2}} - \frac{y^s}{x^{r+1}}\right)\ f_{(m),(n)}(x,y)\,dx\,dy$$

$$= \int\int_{-\infty<y<x<\infty} \frac{y^s}{kx^{r+2}}(1-kx)$$

$$\times \frac{(H(x))^{m-1}h(x)(H(y)-H(x))^{n-m-1}f(y)}{\Gamma(m)\,\Gamma(n-m)}\,dx\,dy.$$

According to (3), we can rewrite the expectation expression as

$$E\left(\frac{X^s_{L(n)}}{k\,X^{r+2}_{L(m)}} - \frac{X^s_{L(n)}}{X^{r+1}_{L(m)}}\right) = \int_{-\infty}^{\infty} \frac{y^s f(y)}{k\,\Gamma(m)\Gamma(n-m)}$$

$$\times \left(\int_y^{\infty} \frac{1}{x^{r+2}}(H(x))^m(H(y)-H(x))^{n-m-1}dx\right)dy.$$

Using integrating by parts treating $\dfrac{1}{x^{r+2}}$ for integration and $(H(x))^m$ $(H(y)-H(x))^{n-m-1}$ for differentiation on the second integration, we obtain

$$\int_y^{\infty} \frac{1}{x^{r+2}}(H(x))^m(H(y)-H(x))^{n-m-1}dx$$

$$= \frac{(n-m-1)}{(r+1)} \int_y^{\infty} \frac{1}{x^{r+1}}(H(x))^m h(x)(H(y)-H(x))^{n-m-2}dx$$

$$- \frac{m}{(r+1)} \int_y^{\infty} \frac{1}{x^{r+1}}(H(x))^{m-1}h(x)(H(y)-H(x))^{n-m-1}dx.$$

By the above result, we can write

$$E\left(\frac{X^s_{L(n)}}{k\,X^{r+2}_{L(m)}} - \frac{X^s_{L(n)}}{X^{r+1}_{L(m)}}\right)$$

$$= \frac{1}{k(r+1)} \int\int_{-\infty<y<x<\infty} \frac{y^s}{x^{r+1}}$$

$$\times \frac{(H(x))^m h(x)(H(y)-H(x))^{n-m-2}f(y)}{\Gamma(m)\,\Gamma(n-m-1)}\,dx\,dy$$

$$
\begin{aligned}
&- \frac{m}{k(r+1)} \int\int_{-\infty<y<x<\infty} \frac{y^s}{x^{r+1}} \\
&\qquad \times \frac{(H(x))^{m-1} h(x) (H(y)-H(x))^{n-m-1} f(y)}{\Gamma(m)\,\Gamma(n-m)}\, dx\, dy \\
&= \frac{m}{k(r+1)} \int\int_{-\infty<y<x<\infty} \frac{y^s}{x^{r+1}} f_{(m+1),(n)}(x,y)\, dx\, dy \\
&\quad - \frac{m}{k(r+1)} \int\int_{-\infty<y<x<\infty} \frac{y^s}{x^{r+1}} f_{(m),(n)}(x,y)\, dx\, dy \\
&= \frac{m}{k(r+1)} E\left(\frac{X^s_{L(n)}}{X^{r+1}_{L(m+1)}}\right) - \frac{m}{k(r+1)} E\left(\frac{X^s_{L(n)}}{X^{r+1}_{L(m)}}\right).
\end{aligned}
$$

Hence we have

$$
E\left(\frac{X^s_{L(n)}}{X^{r+1}_{L(m)}}\right) = \frac{(r+1)}{k(r+1)-m} E\left(\frac{X^s_{L(n)}}{X^{r+2}_{L(m)}}\right) - \frac{m}{k(r+1)-m} E\left(\frac{X^s_{L(n)}}{X^{r+1}_{L(m+1)}}\right).
$$

This completes the proof. □

Corollary 2.2. *For $k \neq 0$, $m \geq 1$ and $r,\ s = 0, 1, 2, \cdots$,*

$$
E\left(\frac{X^s_{L(m+1)}}{X^{r+1}_{L(m)}}\right) = \frac{(r+1)}{k(r+1)-m} E\left(\frac{X^s_{L(m+1)}}{X^{r+2}_{L(m)}}\right) - \frac{m}{k(r+1)-m} E\left(X^{s-r-1}_{L(m+1)}\right).
$$

Proof. Upon substituting $n = m+1$ in Theorem 2.1 and simplifying, then we obtain that

$$
E\left(\frac{X^s_{L(m+1)}}{X^{r+1}_{L(m)}}\right) = \frac{(r+1)}{k(r+1)-m} E\left(\frac{X^s_{L(m+1)}}{X^{r+2}_{L(m)}}\right) - \frac{m}{k(r+1)-m} E\left(X^{s-r-1}_{L(m+1)}\right).
$$

This completes the proof. □

Corollary 2.3. *For $k \neq 0$ and $m \geq 1$,*

$$
\begin{aligned}
V(X_{L(m+1)}) &= \left(\frac{2}{m} E\left(\frac{X^4_{L(m+1)}}{X^3_{L(m)}}\right) - \frac{(2k-m)}{m} E\left(\frac{X^4_{L(m+1)}}{X^2_{L(m)}}\right)\right) \\
&\quad - \left(\frac{2}{m} E\left(\frac{X^3_{L(m+1)}}{X^3_{L(m)}}\right) - \frac{(2k-m)}{m} E\left(\frac{X^3_{L(m+1)}}{X^2_{L(m)}}\right)\right)^2.
\end{aligned}
$$

Proof. Since the variance $V(X_{L(m+1)})$ of the lower record value $X_{L(m+1)}$ is $V(X_{L(m+1)}) = E(X^2_{L(m+1)}) - \big(E(X_{L(m+1)})\big)^2$, for the case $r = 1$ and

$s = 3$ in Corollary 2.2, we get

$$E(X_{L(m+1)}) = \frac{2}{m} E\left(\frac{X^3_{L(m+1)}}{X^3_{L(m)}}\right) - \frac{(2k-m)}{m} E\left(\frac{X^3_{L(m+1)}}{X^2_{L(m)}}\right).$$

Also, for the case $r = 1$ and $s = 4$ in Corollary 2.2, we have

$$E(X^2_{L(m+1)}) = \frac{2}{m} E\left(\frac{X^4_{L(m+1)}}{X^3_{L(m)}}\right) - \frac{(2k-m)}{m} E\left(\frac{X^4_{L(m+1)}}{X^2_{L(m)}}\right).$$

Hence we simply obtain the variance $V(X_{L(m+1)})$ of the lower record value $X_{L(m+1)}$. That is,

$$\begin{aligned} V(X_{L(m+1)}) = &\left(\frac{2}{m} E\left(\frac{X^4_{L(m+1)}}{X^3_{L(m)}}\right) - \frac{(2k-m)}{m} E\left(\frac{X^4_{L(m+1)}}{X^2_{L(m)}}\right)\right) \\ &- \left(\frac{2}{m} E\left(\frac{X^3_{L(m+1)}}{X^3_{L(m)}}\right) - \frac{(2k-m)}{m} E\left(\frac{X^3_{L(m+1)}}{X^2_{L(m)}}\right)\right)^2. \end{aligned}$$

This completes the proof. □

THEOREM 2.4. *For $k = 0$, $1 \le m < n$, $r = 1, 2, \cdots$ and $s = 0, 1, 2, \cdots$,*

$$E\left(\frac{X^s_{L(n)}}{X^{r+1}_{L(m)}}\right) = \frac{m}{r} E\left(\frac{X^s_{L(n)}}{X^r_{L(m+1)}}\right) - \frac{m}{r} E\left(\frac{X^s_{L(n)}}{X^r_{L(m)}}\right).$$

Proof. In the same manner as Theorem 2.1, we have that for the standard GEV in (1.1) and (1.2),

$$h(x) = H(x). \tag{2.2}$$

Let us consider for $k = 0$, $1 \le m < n$, $r = 1, 2, \cdots$ and $s = 0, 1, 2, \cdots$,

$$\begin{aligned} E\left(\frac{X^s_{L(n)}}{X^{r+1}_{L(m)}}\right) &= \int\int_{-\infty<y<x<\infty} \frac{y^s}{x^{r+1}} f_{(m),(n)}(x,y)\, dx\, dy \\ &= \int\int_{-\infty<y<x<\infty} \frac{y^s}{x^{r+1}} \frac{(H(x))^{m-1} h(x) (H(y) - H(x))^{n-m-1} f(y)}{\Gamma(m)\,\Gamma(n-m)}\, dx\, dy. \end{aligned}$$

According to (2.2), we can rewrite the expectation expression as

$$\begin{aligned} E\left(\frac{X^s_{L(n)}}{X^{r+1}_{L(m)}}\right) = &\int_{-\infty}^{\infty} \frac{y^s f(y)}{\Gamma(m)\Gamma(n-m)} \\ &\times \left(\int_y^{\infty} \frac{1}{x^{r+1}} (H(x))^m (H(y) - H(x))^{n-m-1} dx\right) dy. \end{aligned}$$

Using integrating by parts treating $\frac{1}{x^{r+1}}$ for integration and $(H(x))^m$ $(H(y)-H(x))^{n-m-1}$ for differentiation on the second integration, we obtain

$$\begin{aligned}\int_y^\infty & \frac{1}{x^{r+1}}(H(x))^m(H(y)-H(x))^{n-m-1}dx \\ &= \frac{(n-m-1)}{r}\int_y^\infty \frac{1}{x^r}(H(x))^{m+1}(H(y)-H(x))^{n-m-2}dx \\ &\quad -\frac{m}{r}\int_y^\infty \frac{1}{x^r}(H(x))^m(H(y)-H(x))^{n-m-1}dx.\end{aligned}$$

By the above result, we can write

$$\begin{aligned}E&\left(\frac{X^s_{L(n)}}{X^{r+1}_{L(m)}}\right) \\ &= \frac{1}{r}\int\int_{-\infty<y<x<\infty}\frac{y^s}{x^r}\frac{(H(x))^{m+1}(H(y)-H(x))^{n-m-2}f(y)}{\Gamma(m)\,\Gamma(n-m-1)}\,dx\,dy \\ &\quad -\frac{m}{r}\int\int_{-\infty<y<x<\infty}\frac{y^s}{x^r}\frac{(H(x))^m(H(y)-H(x))^{n-m-1}f(y)}{\Gamma(m)\,\Gamma(n-m)}\,dx\,dy \\ &= \frac{m}{r}\int\int_{-\infty<y<x<\infty}\frac{y^s}{x^r}f_{(m+1),(n)}(x,y)\,dx\,dy \\ &\qquad -\frac{m}{r}\int\int_{-\infty<y<x<\infty}\frac{y^s}{x^r}f_{(m),(n)}(x,y)\,dx\,dy \\ &= \frac{m}{r}E\left(\frac{X^s_{L(n)}}{X^r_{L(m+1)}}\right)-\frac{m}{r}E\left(\frac{X^s_{L(n)}}{X^r_{L(m)}}\right).\end{aligned}$$

Hence we have

$$E\left(\frac{X^s_{L(n)}}{X^{r+1}_{L(m)}}\right)=\frac{m}{r}E\left(\frac{X^s_{L(n)}}{X^r_{L(m+1)}}\right)-\frac{m}{r}E\left(\frac{X^s_{L(n)}}{X^r_{L(m)}}\right).$$

This completes the proof. □

Corollary 2.5. *For* $k=0$, $1\le m<n$, $r=1,2,\cdots$ *and* $s=0,1,2,\cdots$,

$$E\left(\frac{X^s_{L(m+1)}}{X^{r+1}_{L(m)}}\right)=\frac{m}{r}E\left(X^{s-r}_{L(m+1)}\right)-\frac{m}{r}E\left(\frac{X^s_{L(m+1)}}{X^r_{L(m)}}\right).$$

Proof. Upon substituting $n = m+1$ in Theorem 2.4 and simplifying, then we obtain that

$$E\left(\frac{X^s_{L(m+1)}}{X^{r+1}_{L(m)}}\right) = \frac{m}{r}E\left(X^{s-r}_{L(m+1)}\right) - \frac{m}{r}E\left(\frac{X^s_{L(m+1)}}{X^r_{L(m)}}\right).$$

□

COROLLARY 2.6. *For $k = 0$ and $m \geq 1$,*

$$V(X_{L(m+1)}) = \left(\frac{1}{m}E\left(\frac{X^3_{L(m+1)}}{X^2_{L(m)}}\right) + E\left(\frac{X^3_{L(m+1)}}{X_{L(m)}}\right)\right) - \left(\frac{1}{m}E\left(\frac{X^2_{L(m+1)}}{X^2_{L(m)}}\right) + E\left(\frac{X^3_{L(m+1)}}{X_{L(m)}}\right)\right)^2.$$

Proof. In the same manner as Corollary 2.3, for the case $r = 1$ and $s = 2$ in Corollary 2.5, we get

$$E(X_{L(m+1)}) = \frac{1}{m}E\left(\frac{X^2_{L(m+1)}}{X^2_{L(m)}}\right) + E\left(\frac{X^2_{L(m+1)}}{X_{L(m)}}\right).$$

Also, for the case $r = 1$ and $s = 3$ in Corollary 2.5, we have

$$E(X^2_{L(m+1)}) = \frac{1}{m}E\left(\frac{X^3_{L(m+1)}}{X^2_{L(m)}}\right) + E\left(\frac{X^3_{L(m+1)}}{X_{L(m)}}\right).$$

Hence we simply obtain the variance $V(X_{L(m+1)})$ of the lower record value $X_{L(m+1)}$. That is,

$$V(X_{L(m+1)}) = \left(\frac{1}{m}E\left(\frac{X^3_{L(m+1)}}{X^2_{L(m)}}\right) + E\left(\frac{X^3_{L(m+1)}}{X_{L(m)}}\right)\right) - \left(\frac{1}{m}E\left(\frac{X^2_{L(m+1)}}{X^2_{L(m)}}\right) + E\left(\frac{X^3_{L(m+1)}}{X_{L(m)}}\right)\right)^2.$$

This completes the proof. □

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