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h-STABILITY FOR LINEAR DIFFERENTIAL SYSTEMS VIA t_∞-QUASISIMILARITY

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Abstract. We study h-stability for linear differential systems by using t_∞-quasisimilarity and Gronwall's inequality.

1. Introduction

We consider two linear differential systems

$$x'(t) = A(t)x(t) \tag{1.1}$$

and

$$y'(t) = B(t)y(t), \tag{1.2}$$

where $A, B \in C(\mathbb{R}_+, M_n(\mathbb{R}))$. Here $\mathbb{R}_+ = [0, \infty)$ and $M_n(\mathbb{R})$ is the set of all $n \times n$ matrices over $\mathbb{R}$.

The concept of similarity is an effective tool to study the stability for differential systems. Markus [7] introduced the notion of kinematic similarity of matrices whose entries are complex-valued functions of a real variable t, continuous and bounded on $\mathbb{R}_+$ and showed that the relationship of kinematic similarity is an equivalence relation preserving the characteristic exponents and the type numbers of the linear differential systems. Conti [4] introduced the notion of t_∞-similarity in the set of all $n \times n$ continuous matrices $A(t)$ defined on $\mathbb{R}_+$ and showed that t_∞-similarity is an equivalence relation preserving strict, uniform and exponential stability of linear homogeneous differential systems. Trench [12] extended this notion to t_∞-quasisimilarity that is not symmetric or transitive, but still preserves stability properties. Hewer [6] studied the

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variational stability of nonlinear differential systems using the notion of t_∞-similarity.

The notion of h-stability was introduced by Pinto [8] with the intention of obtaining results about stability for a weakly stable system (at least, weaker than those given exponential stability and uniform stability) under some perturbations. Choi et al. [1] investigated h-stability for the nonlinear differential systems using the notions of t_∞-similarity and Liapunov functions. Recently, Choi et al. [2] studied h-stability for dynamic equations on time scales.

In this paper, it is shown that if (1.2) is t_∞-quasisimilar to (1.1) and (1.1) is h-stable, then (1.2) is also h-stable.

2. Main results

The symbol $|\cdot|$ will be used to denote any convenient vector norm on $\mathbb{R}^n$ or $\mathbb{R}^{n\times n}$.

We state the conditions for the stability of (1.1) in terms of its fundamental matrix.

LEMMA 2.1. [11] *Let $X(t)$ be a fundamental matrix of (1.1) with $X(t_0) = I$ (the identity matrix). Then (1.1) is*

(i) stable if and only if there exists a positive constant M such that

$$|X(t)| \le M \quad \text{for } t \ge t_0;$$

(ii) uniformly stable if and only if there exists a positive constant M such that

$$|X(t)X^{-1}(s)| \le M \quad \text{for } t_0 \le s \le t < \infty;$$

(iii) uniformly asymptotically stable if and only if there exist positive constants M and α such that

$$|X(t)X^{-1}(s)| \le Me^{-\alpha(t-s)} \quad \text{for } t_0 \le s \le t < \infty.$$

We see that uniform asymptotic stability and exponential asymptotic stability are equivalent for linear differential systems.

We recall the concept of h-stability introduced by Pinto [8].

DEFINITION 2.2. (1.1) is *h-stable* if there exist a constant $c > 0$ and a positive bounded function $h : \mathbb{R}_+ \to \mathbb{R}$ such that for any $t_0 \in \mathbb{R}_+$ and $x_0 \in \mathbb{R}^n$, the corresponding solution $x(t, t_0, x_0)$ satisfies

$$|x(t, t_0, x_0)| \le c|x_0|h(t)h(t_0)^{-1}, \; t \ge t_0, \tag{2.1}$$

where $h(t)^{-1} = \frac{1}{h(t)}$.

LEMMA 2.3. [8] *If (1.1) is h-stable if and only if there exist a positive bounded function h defined on $\mathbb{R}_+$ and a constant $c \geq 1$ such that*

$$|X(t)X^{-1}(t_0)| \leq ch(t)h(t_0)^{-1},\ t \geq t_0 \in \mathbb{R}_+,$$

where $X(t)$ is a fundamental matrix of (1.1) with $X(t_0) = I$.

We recall the notion of t_∞-quasisimilarity in [12].

Let $\mathcal{R}$ be the set of continuous $n \times n$ matrix-valued functions C on $\mathbb{R}_+$ such that $\int_a^\infty C(t)dt$ converges, and $\mathfrak{S}$ be the set of invertible $n \times n$ matrix-valued functions S such that S' is continuous, and S and S^{-1} are bounded on $\mathbb{R}_+$.

DEFINITION 2.4. B is t_∞-*quasisimilar* to A if there exits an $S \in \mathfrak{S}$ such that the function

$$S' + SB - AS = F_0 \tag{2.2}$$

is in $\mathcal{R}$, and either

$$\int_0^\infty |F_0(t)|dt < \infty \tag{2.3}$$

or, for some $k \geq 1$, there are matrix functions $F_1, \cdots, F_k, G_1, \cdots, G_k$, $M_0, \cdots, M_k, N_0, \cdots, N_k$ in $\mathcal{R}$ which satisfy

$$F_0 = M_0 + N_0, \tag{2.4}$$

$$G_j(t) = \int_t^\infty N_{j-1}(s)ds,\ 1 \leq j \leq k, \tag{2.5}$$

$$F_j = G_jB - AG_j = M_j + N_j,\ 1 \leq j \leq k, \tag{2.6}$$

such that

$$N_k = 0 \tag{2.7}$$

and

$$\int_0^\infty |\sum_{j=0}^k M_j(t)|dt < \infty. \tag{2.8}$$

We say that if B is t_∞-quasisimilar to A, then the system (1.2) is t_∞-*quasisimilar* to (1.1).

When $k = 0$, t_∞-quasisimilarity of systems reduces to t_∞-similarity as defined by Conti [4], who showed that the latter is an equivalence relation which preserves uniform and strict stability, but not stability. It can also be shown that it preserves uniform asymptotic stability, but not linear asymptotic equilibrium. Since the integrals in (2.5) may converge conditionally, t_∞-quasisimilarity is not symmetric or transitive if $k \geq 1$.

If assumption $\mathcal{A}_k(A,B,S)$ holds, i.e., B is t_∞-quasisimilar to A, let

$$\Gamma_0 = I;\ \Gamma_j = I + S^{-1}\sum_{i=1}^{j} G_i,\ 1 \le j \le k, \tag{2.9}$$

and

$$H_j = \sum_{i=0}^{j} M_i,\ 1 \le j \le k. \tag{2.10}$$

In terms of (2.10) we can combine (2.3) and (2.8)

$$\int_0^\infty |H_k(t)|dt < \infty. \tag{2.11}$$

The following lemma is basic for our results. The special case with $k=0$ is essentially due to Sansone and Conti [11, p. 492].

LEMMA 2.5. [12, Lemma 2] *If B is t_∞- quasisimilar to A with $S \in \mathfrak{S}$ for some $k \ge 0$, then there is an $a_1 \ge 0$ such that*

$$\begin{aligned} Y(t) &= \Gamma_k^{-1}(t)S^{-1}(t)X(t)[X^{-1}(\tau)S(\tau)\Gamma_k(\tau)Y(\tau) \\ &+ \int_\tau^t X^{-1}(s)H_k(s)Y(s)ds], t,\tau \ge a_1, \end{aligned} \tag{2.12}$$

where $X(t)$ and $Y(t)$ are fundamental matrices for (1.1) and (1.2), respectively.

The following theorem is our main result about the h-stability via t_∞-quasisimilarity.

THEOREM 2.6. *Suppose that B is t_∞-quasisimilar to A with $S \in \mathfrak{S}$ for some $k \ge 0$ and (1.1) is h-stable. Then (1.2) is also h-stable.*

Proof. Suppose that (1.1) is h-stable. Then there exist a positive function $h : \mathbb{R}_+ \to \mathbb{R}$ and a constant $c \ge 1$ such that

$$|x(t,t_0,x_0)| \le c|x_0|h(t)h(t_0)^{-1},\ t \ge t_0.$$

Multiplying (2.12) on the right by $Y^{-1}(\tau)$ and using the boundedness of $\Gamma_k, S, \Gamma_k^{-1}$, and S^{-1} yields the inequality, for $t \ge \tau \ge a_1$,

$$\begin{aligned} |Y(t)Y^{-1}(\tau)| &\le |\Gamma_k^{-1}(t)||S^{-1}(t)|[|X(t)X^{-1}(\tau)||S(\tau)||\Gamma_k(\tau)| \\ &+ \int_\tau^t |X(t)X^{-1}(s)||H_k(s)||Y(s)Y^{-1}(\tau)|ds] \\ &\le c_1 h(t)h^{-1}(\tau) \\ &+ c_2\int_\tau^t h(t)h^{-1}(s)|H_k(s)||Y(t)Y^{-1}(s)ds, \end{aligned} \tag{2.13}$$

where c_1 and c_2 are constants. Dividing (2.13) by $h(t)$ yields the inequality

$$\frac{|Y(t)Y^{-1}(\tau)|}{h(t)} \leq c_1 h^{-1}(\tau) + c_2 \int_\tau^t |H_k(s)| \frac{|Y(s)Y^{-1}(\tau)|}{h(\tau)} ds],\ t \geq \tau \geq a_1.$$

In view of the Gronwall's inequality, we obtain

$$\begin{aligned} |Y(t)Y^{-1}(\tau)| &\leq c_1 h(t) h^{-1}(\tau) \exp\left(c_2 \int_\tau^t |H_k(s)| ds\right) \\ &\leq d h(t) h^{-1}(\tau),\ t \geq \tau \geq a_1, \end{aligned}$$

where $d = \exp(c_2 \int_0^\infty |H_k(s)| ds)$. Hence (1.2) is h-stable. This completes the proof. □

Corollary 2.7. *Suppose that B is t_∞-similar to A with $S \in \mathfrak{S}$. (1.1) is h-stable if and only if (1.2) is also h-stable.*

We obtain the following two corollaries [12, Theorem 1] *from Theorem 2.6.*

Corollary 2.8. *If h is a constant function, then h-stability becomes uniformly stability.*

Corollary 2.9. *If h is a function given by $h(t) = \exp(-\alpha t)$ with a positive constant α, then h-stability becomes exponential stability.*

Theorem 2.10. *Suppose that*

$$\int_0^\infty |A(t)| dt < \infty \tag{2.14}$$

and there is an $S \in \mathfrak{S}$ such that the function

$$N_0 = S' + S(B - A)$$

is in $\mathcal{R}$. Suppose that for some $k \geq 1$ there are functions $K_1, \cdots, K_k$, $N_1, \cdots, N_k$ in $\mathcal{R}$ such that

$$\left(\int_t^\infty N_{j-1}(s) ds\right)(B(t) - A(t)) = K_j(t) + N_j(t),\ 1 \leq j \leq k,$$

where $N_k = 0$ and

$$\int_0^\infty |\sum_{j=1}^k K_j(t)| dt < \infty.$$

Then (1.2) is h-stable.

Proof. We see that system (1.2) is equivalent to

$$x(t) = x_0 + \int_{t_0}^{t} A(s)ds,\ t \geq t_0.$$

From the condition (2.14) of A and well-known Gronwall's inequality, we have

$$\begin{aligned} |x(t)| &\leq |x_0| \exp\left(\int_{t_0}^{t} |A(s)|ds\right) \\ &= |x_0| h(t) h^{-1}(t_0),\ t \geq t_0, \end{aligned}$$

where $h(t) = \exp(\int_0^t |A(s)|ds)$ is a positive bounded function on $\mathbb{R}_+$. Since (1.1) is t_∞-quasisimilar to (1.2) by Theorem 3 in [12], (1.2) is h-stable in view of Theorem 2.6. This completes the proof. □

COROLLARY 2.11. *Suppose that A satisfies (2.14) and*

$$\int_0^\infty |S'(t) + S(t)(B(t) - A(t))|dt < \infty. \tag{2.15}$$

Then (1.2) is h-stable.

Proof. Since $\int_0^\infty |F_0(t)|dt < \infty$, (1.1) is t_∞-similar to (1.2), and (1.1) is h-stable. Hence (1.2) is h-stable by Theorem 2.6. This completes the proof. □

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