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THE INTEGRALS OF BANACH SPACE-VALUED FUNCTIONS

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Abstract. In this paper, we define the ap-Henstock integral and the ap-Denjoy integral of Banach-valued functions, and we investigate some properties of these two integrals. In particular, we show that the ap-Henstock integral is equivalent to the ap-Denjoy integral.

1. Introduction and preliminaries

The ap-Denjoy integral of real valued functions was introduced in [13]. It is known [13] that the ap-Denjoy integral is equivalent to the ap-Henstock integral.

In this paper, we define the ap-Henstock integral and ap-Denjoy integral of Banach-valued functions, and we investigate the relationship of these two integrals.

Throughout this paper, X is a Banach space with dual X^*.

For a measurable set E of real numbers we denote by $|E|$ its Lebesgue measure. Let E be a measurable set and let c be a real number. The *density* of E at c is defined by

$$d_c E = \lim_{h \to 0^+} \frac{|E \cap (c-h, c+h)|}{2h}$$

provided the limit exists. The point c is called *a point of density* of E if $d_c E = 1$. The set E^d represents the set of all points $x \in E$ such that x is a point of density of E. A function $F : [a, b] \to X$ is said to be *approximately differentiable* at $c \in [a, b]$ if there exists a measurable set $E \subseteq [a, b]$ such that $c \in E^d$ and

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$$\lim_{\substack{x \to c \\ x \in E}} \frac{F(x) - F(c)}{x - c}$$

exists. The approximate derivative of F at c is denoted by $F'_{ap}(c)$.

An *approximate neighborhood*(of ap-nbd) of $x \in [a, b]$ is a measurable set $S_x \subseteq [a, b]$ containing x as a point of density. For every $x \in E \subseteq [a, b]$, choose an ap-nbd $S_x \subseteq [a, b]$ of x. Then we say that $S = \{S_x : x \in E\}$ is a *choice* on E. A tagged interval $(x, [c, d])$ is said to be *subordinate* to the choice $S = \{S_x\}$ if $c, d \in S_x$. Let $\mathcal{P} = \{(x_i, [c_i, d_i]) : 1 \leq i \leq n\}$ be a finite collection of non-overlapping tagged intervals. If $(x_i, [c_i, d_i])$ is subordinate to a choice S for each i, then we say that P is subordinate to S. Let $E \subseteq [a, b]$. If $\mathcal{P}$ is subordinate to S and each $x_i \in E$, then $\mathcal{P}$ is called E-subordinate to S. If $\mathcal{P}$ is subordinate to S and $[a, b] = \bigcup_{i=1}^{n} [c_i, d_i]$, then we say that $\mathcal{P}$ is a tagged partition of $[a, b]$ that is subordinate to S.

2. The ap-Henstock integral of Banach-valued functions

We introduce the ap-Henstock integral of Banach-valued functions.

Definition 2.1. A function $f : [a, b] \to X$ is *ap-Henstock integrable* on $[a, b]$ if there exists a vector $A \in X$ with the following property: for each $\epsilon > 0$ there exists a choice S on $[a, b]$ such that $\|f(\mathcal{P}) - A\| < \epsilon$ whenever $\mathcal{P}$ is a tagged partition of $[a, b]$ that is subordinate to S, where $f(\mathcal{P}) = (\mathcal{P}) \sum f(x)|I|$. The vector A is called the ap-Henstock integral of f on $[a, b]$ and is denoted by $(AH)\int_a^b f$.

If f is ap-Henstock integrable on $[a, b]$, then f is also ap-Henstock integrable on any subinterval of $[a, b]$. Hence, an interval function F can be defined by $F(I) = (AH) \int_I f$. The function F is called the primitive of f.

It is easy to show the following theorem.

Theorem 2.1. *Let f and g be functions mapping $[a, b]$ into X.*
(a) If f is ap-Henstock integrable on each $[a, c]$ and $[c, b]$, then f is ap-Henstock integrable on $[a, b]$ and

$$(AH)\int_a^b f = (AH)\int_a^c f + (AH)\int_c^b f$$

(b) If f and g are ap-Henstock integrable on $[a,b]$ and $\alpha, \beta \in \mathbb{R}$, then $\alpha f + \beta g$ is ap-Henstock integrable on $[a,b]$ and

$$(AH)\int_a^b (\alpha f + \beta g) = \alpha (AH)\int_a^b f + \beta (AH)\int_a^b g$$

THEOREM 2.2. *Let $f : [a,b] \to X$ be ap-Henstock integrable on $[a,b]$. Then for each $x^* \in X^*$ the function $x^* f$ is ap-Henstock integrable on $[a,b]$ and*

$$x^*(AH)\int_a^b f = (AH)\int_a^b x^* f$$

Proof. Since f is ap-Henstock integrable on $[a,b]$, for every $\epsilon > 0$ there exists a choice S on $[a,b]$ such that for any partition $\mathcal{P} = \{(x,I)\}$ that is subordinate to S we have

$$\|f(\mathcal{P}) - (AH)\int_a^b f\| < \epsilon$$

For any $x^* \in X^*$, we have

$$|x^* f(\mathcal{P}) - x^*(AH)\int_a^b f| \le \|x^*\| \|f(\mathcal{P}) - (AH)\int_a^b f\| < \|x^*\|\epsilon.$$

Hence, $x^* f$ is ap-Henstock integrable on $[a,b]$ and

$$(AH)\int_a^b x^* f = x^*(AH)\int_a^b f.$$

□

THEOREM 2.3. *Let $f : [a,b] \to X$ be a function. If $f = 0$ almost everywhere on $[a,b]$, then f is ap-Henstock integrable on $[a,b]$ and $(AH)\int_a^b f = 0$.*

Proof. Let $E = \{x \in [a,b] : \|f(x)\| \neq 0\}$ and for each positive integer n, let $E_n = \{x \in E : n-1 \le \|f(x)\| < n\}$. Let $\epsilon > 0$. For each n, choose an open set O_n such that $E_n \subseteq O_n$ and $|O_n| < \epsilon / n2^n$.

Define a choice $S = \{S_x : x \in [a,b]\}$ by

$$S_x = \begin{cases} [a,b] & \text{if} \quad x \in [a,b] - E, \\ O_n & \text{if} \quad x \in E_n. \end{cases}$$

Suppose that $\mathcal{P}$ is a tagged partition of $[a,b]$ that is subordinate to S. For each n, let $\mathcal{P}_n$ be the subset of $\mathcal{P}$ that has tag in E_n. We have

$$\|f(\mathcal{P})\| \le \sum_{n=1}^{\infty} \|f(\mathcal{P}_n)\| < \sum_{n=1}^{\infty} n|O_n| < \sum_{n=1}^{\infty} \epsilon/2^n = \epsilon.$$

Hence, f is ap-Henstock integrable on $[a, b]$ and $(AH)\int_a^b f{=}0$. □

THEOREM 2.4. *Let $f : [a, b] \to X$ be ap-Henstock integrable on $[a, b]$. Then*

(a) the function f is weakly measurable.

(b) If $g = f$ almost everywhere on $[a, b]$, then g is ap-Henstock integrable on $[a, b]$ and

$$(AH)\int_a^b f = (AH)\int_a^b g.$$

Proof. (a) Since x^*f is ap-Henstock integrable for each $x^* \in X^*$, x^*f is measurable. Hence f is weakly measurable.

(b) Since $f - g = 0$ almost everywhere on $[a, b]$, $f - g$ is ap-Henstock integrable on $[a, b]$ and $(AH)\int_a^b (f - g) = 0$ by Theorem 2.3. Hence, $g = f - (f - g)$ is ap-Henstock integrable on $[a, b]$ and

$$(AH)\int_a^b f - (AH)\int_a^b g = (AH)\int_a^b (f - g) = 0$$

□

DEFINITION 2.2. A function $F : [a, b] \to X$ is AC_s on a measurable set $E \subseteq [a, b]$ if for each $\epsilon > 0$ there exist a positive number δ and a choice S on E such that $\|(\mathcal{P}) \sum F(I)\| < \epsilon$ for every finite collection $\mathcal{P}$ of non-overlapping tagged intervals that is subordinate to S and satisfies $(P) \sum |I| < \delta$. The function F is ACG_s on E if E can be expressed as a countable union of measurable sets on each of which F is AC_s.

LEMMA 2.5. *Suppose that $f : [a, b] \to X$ and let $E \subseteq [a, b]$. If $|E| = 0$, then for each $\epsilon > 0$ there exists a choice S on E such that $\|f\|(\mathcal{P}) < \epsilon$ whenever $\mathcal{P}$ is E-subordinate to S, where $\|f\|(\mathcal{P}) = (\mathcal{P}) \sum \|f(x)\||I|$.*

Proof. For each positive integer n, let $E_n = \{x \in E : n-1 \leq \|f(x)\| < n\}$. Then $E = \bigcup_{n=1}^{\infty} E_n$. Let $\epsilon > 0$. For each n, let $S^n = \{S^n_x : x \in E_n\}$ be a choice on E_n and choose an open set O_n such that $E_n \subseteq O_n$ and $|O_n| < \epsilon/n2^n$. For each $x \in E_n$, let $S_x = S^n_x \cap O_n$. Then $S = \{S_x : x \in E\}$ is a choice on E.

Suppose that $\mathcal{P}$ is E-subordinate to S. Let $\mathcal{P}_n$ be the subset of $\mathcal{P}$ that tags in E_n. Then we have

$$\|f\|(\mathcal{P}) = \sum_{n=1}^{\infty} \|f\|(\mathcal{P}_n) < \sum_{n=1}^{\infty} n|O_n| < \sum_{n=1}^{\infty} \epsilon/2^n = \epsilon.$$

□

LEMMA 2.6. *Suppose that $F : [a,b] \to X$ is ACG_s on $[a,b]$ and let E be a subset of $[a,b]$. If $|E| = 0$, then for each $\epsilon > 0$ there exists a choice S of E such that $\|(\mathcal{P})\sum F(I)\| < \epsilon$ for each $\mathcal{P}$ that is subordinate to S.*

Proof. Suppose that $E \subseteq [a,b]$ is a measurable set of measure zero. Let $E = \bigcup_{n=1}^{\infty} E_n$, where $\{E_n\}$ is a sequence of disjoint measurable sets and F is AC_s on each E_n. Let $\epsilon > 0$. For each positive integer n, there exist a choice $S^n = \{S_x^n : x \in E_n\}$ on E_n and a positive number δ_n such that $\|(\mathcal{P})\sum F(I)\| < \epsilon/2^n$ whenever $\mathcal{P}$ is E_n-subordinate to S^n and $(\mathcal{P})\sum |I| < \delta_n$. For each positive integer n, choose an open set O_n such that $E_n \subseteq O_n$ and $|O_n| < \delta_n$. Let $S_x = S_x^n \cap O_n$ for each $x \in E_n$. Then $S = \{S_x : x \in E\}$ is a choice on E. Suppose that $\mathcal{P}$ is E-subordinate to S. Let $\mathcal{P}_n$ be a subset of $\mathcal{P}$ that has tags in E_n and note that $(\mathcal{P}_n)\sum |I| < |O_n| < \delta_n$. Hence, we have

$$\|(\mathcal{P})\sum F(I)\| \le \sum_{n=1}^{\infty} \|(\mathcal{P}_n)\sum F(I)\| < \sum_{n=1}^{\infty} \epsilon/2^n = \epsilon.$$

□

THEOREM 2.7. *If there exists an ACG_s function F on $[a,b]$ such that $F'_{ap} = f$ almost everywhere on $[a,b]$, then the function f is ap-Henstock integrable on $[a,b]$.*

Proof. Suppose that there exists an ACG_s function F on $[a,b]$ such that $F'_{ap} = f$ almost everywhere on $[a,b]$. Let

$$E = \{x \in [a,b] : F'_{ap}(x) \ne f(x)\}.$$

Then $|E| = 0$. Let $D = [a,b] - E$ and let $\epsilon > 0$. For each $x \in D$, there exists a measurable set $D_x \subseteq [a,b]$ such that $x \in D_x^d$ and

$$F'_{ap}(x) = \lim_{\substack{y \to x \\ y \in D_x}} \frac{F(y) - F(x)}{y - x}.$$

Hence, there exists $\delta_x > 0$ such that for every $y \in D_x \cap (x - \delta_x, x + \delta_x) = S_x$

$$\|F(y) - F(x) - F'_{ap}(x)(y - x)\| \le \epsilon |y - x|.$$

If $(x, [u,v])$ is a tagged interval with $u, v \in S_x$, then

$$\begin{aligned} &\|F(v) - F(u) - F'_{ap}(x)(v - u)\| \\ &\quad \le \|F(v) - F(x) - F'_{ap}(x)(v - x)\| + \|F(x) - F(u) - F'_{ap}(x)(x - u)\| \\ &\quad < \epsilon(v - x) + \epsilon(x - u) = \epsilon(v - u). \end{aligned}$$

Hence, there exists a choice $S' = \{S_x : x \in D\}$ on D such that $\|f(\mathcal{P}) - F(\mathcal{P})\| < \epsilon(\mathcal{P}) \sum |I|$ whenever $\mathcal{P}$ is a collection of tagged intervals that is subordinate to S'.

By Lemmas 2.5 and 2.6, there exists a choice S'' on E such that $\|f(\mathcal{P})\| < \epsilon$ and$\|F(\mathcal{P})\| < \epsilon$ whenever $\mathcal{P}$ is subordinate to S''. let $S = S' \cup S''$. Then S is a choice on $[a, b]$.

Suppose that $\mathcal{P}$ is a tagged partition of $[a, b]$ that is subordinate to S. Let $\mathcal{P}_E$ be the subset of $\mathcal{P}$ that has tags in E and let $\mathcal{P}_D = \mathcal{P} - \mathcal{P}_E$. Then we have

$$\begin{aligned} &\|f(\mathcal{P}) - F(\mathcal{P})\| \\ &\quad \leq \|f(\mathcal{P}_D) - F(\mathcal{P}_D)\| + \|f(\mathcal{P}_E)\| + \|F(\mathcal{P}_E)\| \\ &\quad < \epsilon(b - a + 2). \end{aligned}$$

Hence, f is ap-Henstock integrable on $[a, b]$. □

The ap-Henstock integral has the following gemetric property.

THEOREM 2.8. *If $f : [a, b] \to X$ is ap-Henstock integrable on $[a, b]$, then for every integral $[c, d] \subseteq [a, b]$,*

$$\frac{1}{d-c}(AH)\int_c^d f \in \overline{co} f\big([c, d]\big),$$

where $\overline{co} f\big([c, d]\big)$ is the closed convex hull of $f\big([c, d]\big)$.

Proof. Suppose that there exists an interval $[c, d]$ such that

$$\frac{1}{d-c}(AH)\int_c^d f \notin \overline{co} f\big([c, d]\big).$$

By the Hahn-Banach Theorem, we can select $x^* \in X^*$ and a real number α such that

$$x^*\Big(\frac{1}{d-c}(AH)\int_c^d f\Big) < \alpha \leq x^* f(t)$$

for all $t \in [c, d]$. Then

$$\frac{1}{d-c}(AH)\int_c^d x^* f < \alpha \leq x^* f(t).$$

Integrating over $[c, d]$, we have

$$(AH)\int_c^d x^* f < \alpha(d - c) \leq (AH)\int_c^d x^* f.$$

This is a contradiction. □

3. The ap-Denjoy integral of Banach-valued functions.

We introduce the notion of the approximate Lusin function. This function is used to define the ap-Denjoy integral.

For a function $F : [a, b] \to X$, F can be treated as a function of intervals be defining $F([c, d]) = F(d) - F(c)$.

DEFINITION 3.1. Let $F : [a, b] \to X$ be a function. The function F is *an approximate Lusin function*(or F is an AL function) on $[a, b]$ if for every measutable set $E \subseteq [a, b]$ of measure zero and for every $\epsilon > 0$ there exists a choice S on E such that $\|(\mathcal{P}) \sum F(I)\| < \epsilon$ for every finite collection $\mathcal{P}$ of non-overlapping tagged intervals that is E-subordinate to S.

From Lemma 2.6, we get the following theorem

THEOREM 3.1. *If $F : [a, b] \to X$ is ACG_s on $[a, b]$, then F is an AL function on $[a, b]$.*

DEFINITION 3.2. A function $f : [a, b] \to X$ is *ap-Denjoy integrable* on $[a, b]$ if there exists an AL function F on $[a, b]$ such that F is approximately differentiable almost everywhere on $[a, b]$ and $F'_{ap} = f$ almost everywhere on $[a, b]$. The function f is ap-Denjoy integrable on a measurable set $E \subseteq [a, b]$ if $f\chi_E$ is ap-Denjoy integrable on $[a, b]$.

If we add the condition $F(a) = 0$, then the function F is unique. We will denote this function $F(x)$ by $(AD)\int_a^x f$.

It is easy to show that if $f : [a, b] \to X$ is ap-Denjoy integrable on $[a, b]$, then f is ap-Denjoy integrable on every subinterval of $[a, b]$. This gives rise to an interval function F such that $F(I) = (AD)\int_I f$ for every subinterval $I \subseteq [a, b]$. The function F is called the primitive of f.

From the definition of the ap-Denjoy integral, we get the following theorem.

THEOREM 3.2. *Let $f : [a, b] \to X$ be ap-Denjoy integrable on $[a, b]$ and let $F(x) = (AD)\int_a^x f$ for each $x \in [a, b]$. Then the function F is approximately differentiable almost everywhere on $[a, b]$ and $F'_{ap} = f$ almost everywhere on $[a, b]$.*

THEOREM 3.3. *Let $f : [a, b] \to X$ and let $c \in (a, b)$.*

(a) If f is ap-Denjoy integrable on $[a, b]$, then f is ap-Denjoy integrable on every subinterval of $[a, b]$.

(b) If f is ap-Denjoy integrable on each of the intervals $[a, c]$ and $[c, b]$, then f is ap-Denjoy integrable on $[a, b]$ and

$$(AD)\int_a^b f = (AD)\int_a^c f + (AD)\int_c^b f.$$

Proof. (a) Let $[c, d]$ be any subinterval on $[a, b]$. Let $F(x) = (AD)\int_a^x f$ for each $x \in [a, b]$. Since F is an AL function on $[a, b]$ and $F'_{ap} = f$ almost everywhere on $[a, b]$, F is an AL function on $[c, d]$ and $F'_{ap} = f$ almost everywhere on $[c, d]$. Hence, f is ap-Denjoy integrable on $[c, d]$.

(b) Since f is ap-Denjoy integrable on each of intervals $[a, c]$ and $[c, b]$, there exist AL functions F and G such that $F'_{ap} = f$ almost everywhere on $[a, c]$ and $G'_{ap} = f$ almost everywhere on $[c, b]$ respectively. Define $H : [a, b] \to X$ by

$$H(x) = \begin{cases} F(x), & \text{if } x \in [a, c]; \\ F(c) + G(x), & \text{if } x \in (c, b]. \end{cases}$$

Then H is an AL function on $[a, b]$ and $H'_{ap} = f$ almost everywhere on $[a, b]$. Hence f is ap-Denjoy integrable on $[a, b]$ and $H(b) = F(c) + G(b)$, i.e.,

$$(AD)\int_a^b f = (AD)\int_a^c f + (AD)\int_c^b f$$

□

We can easily get the following theorem.

THEOREM 3.4. *Suppose that f and g are ap-Denjoy integrable on $[a, b]$. Then*

(a) kf is ap-Denjoy integrable on $[a, b]$ and $(AD)\int_a^b kf = k(AD)\int_a^b f$ for each $k \in \mathbb{R}$,

(b) $f + g$ is ap-Denjoy integrable on $[a, b]$ and $(AD)\int_a^b (f + g) = (AD)\int_a^b f + (AD)\int_a^b g$.

THEOREM 3.5. *A function $f : [a, b] \to X$ is ap-Denjoy integrable on $[a, b]$ if and only if there exists an ACG_s function F on $[a, b]$ such that $F'_{ap} = f$ almost everywhere on $[a, b]$.*

Proof. Suppose that there exists an ACG_s function F on $[a, b]$ such that $F'_{ap} = f$ almost everywhere on $[a, b]$. Then F is an AL function by Theorem 3.1. Hence, f is ap-Denjoy integrable on $[a, b]$.

Conversely, suppose that f is ap-Denjoy integrable on $[a, b]$ and let $F(x) = (AD)\int_a^x f$ for each $x \in [a, b]$. Then F is an AL function such that $F'_{ap} = f$ almost everywhere on $[a,b]$. Let $E = \{x \in [a,b] : F'_{ap}(x) \neq f(x)\}$. Then $|E| = 0$. Since F is an AL function, F is AC_s on E. For each positive integer n, let

$$E_n = \{x \in [a,b] - E : n-1 \leq \|f(x)\| < n\}.$$

Fix n and let $\epsilon > 0$. Since F is approximately differentiable for each $x \in E_n$, there exist a measurable set A_x containing x as a point of density and a positive number δ_x such that

$$\left\| \frac{F(y) - F(x)}{y - x} - f(x) \right\| < \epsilon$$

i.e.,

$$\|F(y) - F(x) - f(x)(y - x)\| < \epsilon |y - x|,$$

if $y \in A_x \cap (x - \delta_x, x + \delta_x)$. For each $x \in E_n$, let

$$S_x = A_x \cap (x - \delta_x, x + \delta_x)$$

Then $S = \{S_x : x \in E_n\}$ is a choice E_n. Suppose that $\mathcal{P}$ is a finite collection of non-overlapping tagged intervals that is E_n-subordinate to S and satisfies $\mu(\mathcal{P}) < \frac{\epsilon}{n}$. Then since $\|F(\mathcal{P}) - f(\mathcal{P})\| < \epsilon\mu(\mathcal{P})$, we have

$$\begin{aligned} \|F(\mathcal{P})\| &\leq \|F(\mathcal{P}) - f(\mathcal{P})\| + \|f(\mathcal{P})\| \\ &< \epsilon\mu(\mathcal{P}) + n\mu(\mathcal{P}) \\ &< (b - a + 1)\epsilon \end{aligned}$$

Hence, F is AC_s on E_n. Since $[a,b] = [\cup_{n=1}^{\infty} E_n] \cup E$, F is ACG_s on $[a,b]$. □

THEOREM 3.6. *Let $f : [a,b] \to X$ be ap-Denjoy integrable on $[a,b]$ and let $F(x) = (AD)\int_a^x f$ for each $x \in [a,b]$. Then F is approximately continuous on $[a,b]$.*

Proof. From the definition of the ap-Denjoy integral, F is approximately differentiable almost everywhere on $[a,b]$. Let E be the set of all non-differentiable points in $[a,b]$. Then E is a measurable set of measure zero. Since F is approximately continuous on $[a,b] - E$, it is sufficient to show that F is approximately continuous on E. Let $c \in E$ and let $\epsilon > 0$. Since F is an AL function, there exists a choice $S = \{S_x : x \in E\}$ such that $\|(\mathcal{P})\sum F(I)\| < \epsilon$ for every finite collection $\mathcal{P}$ of non-overlapping tagged intervals that is E-subordinate to S. If $x \in S_c \cap (c - \eta, c + \eta)$ for some $\eta > 0$, then the tagged interval $(c, [c,x])(\text{or}(c, [x,c]))$ is E-subordinate to S. Hence, $\|F(x) - F(c)\| = \|F([c,x])\| < \epsilon$. This shows that F is approximately continuous on E. □

From Theorem 2.7 and 3.5, we can get the following result.

THEOREM 3.7. *If a function $f : [a, b] \to X$ is ap-Denjoy integrable on $[a, b]$, then f is ap-Henstock integrable on $[a, b]$.*

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