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HOMOGENIZATION FOR FISSURED MEDIUM EQUATIONS

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ABSTRACT. We introduce the homogenized differential systems for fissured medium equations representing the small temperature variation or densities of a fluid in a system consisting of two components.

1. Introduction

In this paper we consider the fissured medium equations:

$$\frac{\partial u}{\partial t} - \text{div } B(x)\nabla u + \frac{1}{\delta}(u - v) = f(t), \tag{1.1}$$

$$-\text{div } C(x)\nabla v + \frac{1}{\delta}(v - u) = 0 \tag{1.2}$$

on an anisotropic fissured medium consisting of a matrix of porous and permeable blocks or cells which are separated from one another by a highly developed system of fissures or flow paths through which the majority of diffusion occurs [10], [11]. The unknown u, v represent the *densities* (of a fluid) or *temperatures* obtained by averaging in the respective medium over a generic neighborhood sufficiently large to include many cells. The *anisotropic heterogeneity* of the medium gives rise to the *flux* terms $-B(x)\nabla u$ and $-C(x)\nabla v$ with the *conductivities* B, C of $n \times n$-matrices. One assumes that the flow rate (of a fluid) or the temperature exchange between the two components is proportional to the density difference $u - v$ with *resistance of the medium* δ. This produces the first diffusion equation with the exchange term $\frac{1}{\delta}(u - v)$. A possible *external force* is described by $f(x, t)$ on the right side of the

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first equation which describes the diffusion on the fissure. Compared to the fissure, the matrix is steady state which brings the second equation.

The double-porosity concept developed slowly during the first half of the 20th century, and one finds a good presentation of applications to diffusion through a heterogeneous medium in [1]. The special cases which are used as a model for diffusion through a slowly sorbing porous medium are studied in [6], [4], and various nonlinear extensions are studied in [2], [3], [12].

In this paper, we focus on the homogenization of fissured medium equations. In Section 3.1, the system of fissured medium equations is introduced and it is comprehensively expanded to get the homogenized (fissured medium) system in Section 3.2. The main technique is the *compactness argument* via a-priori estimates. The technique we introduce in this paper would be applicable for forthcoming homogenizations for nonlinear systems of fissured medium equations with appropriate energy estimates.

We observe that the homogenized system has the same form as the ε-equation, which is not always the case.

2. Remarks from functional analysis

We briefly review some basic notions from functional analysis. Let $\mathbf{V}$ be a Hilbert space which is dense in another Hilbert space $\mathbf{H}$ and assume the identity $\mathbf{V} \to \mathbf{H}$ is continuous. Let $a(\cdot,\cdot)$ be a continuous bilinear form on $\mathbf{V}$. Then we define D to be the set of all $u \in \mathbf{V}$ such that the function $v \mapsto a(u,v)$ is continuous on $\mathbf{V}$ *with the* $\mathbf{H}$*-norm*. For each such $u \in D$ there is then a unique $\mathbf{A}u \in \mathbf{H}$ such that

$$a(u,v) = \langle \mathbf{A}u, v \rangle_{\mathbf{H}} \ , \qquad u \in D \ , v \in \mathbf{V} \ ,$$

and this defines a linear operator $\mathbf{A} : D \to \mathbf{H}$. A form $a : \mathbf{V} \times \mathbf{V} \to \mathbf{R}$ is $\mathbf{V}$-*coercive* (or $\mathbf{V}$-elliptic) if there exists a constant $c_0 > 0$ for which

$$c_0 \|u\|^2_{\mathbf{V}} \leq a(u,u), \qquad u \in \mathbf{V}.$$

Equivalently, $\mathcal{A} \in \mathcal{L}(\mathbf{V}, \mathbf{V}')$ defined by $\mathcal{A}u(v) \equiv a(u,v)$, $(u, v \in \mathbf{V})$ is $\mathbf{V}$-coercive if $a(\cdot,\cdot)$ is. An unbounded operator $\mathbf{A} : D \to \mathbf{H}$ is *accretive* if

$$\langle \mathbf{A}x, x \rangle_{\mathbf{H}} \geq 0 \ , \qquad x \in D$$

and it is *m-accretive* if, in addition, $I + \mathbf{A}$ maps D onto $\mathbf{H}$. It is easy to see how the unbounded operator $\mathbf{A}$ with domain D in $\mathbf{H}$ constructed as above from a continuous bilinear form $a(\cdot,\cdot)$ on $\mathbf{V}$ is related to the continuous operator $\mathcal{A} \in \mathcal{L}(\mathbf{V}, \mathbf{V}')$ which is equivalent to $a(\cdot, \cdot)$. In fact, the graph of $\mathbf{A}$ is the restriction of the graph of $\mathcal{A}$ to $\mathbf{V} \times \mathbf{H}$. That is, note that we have the following inclusion $\mathbf{H}' \hookrightarrow \mathbf{V}'$ by the restriction to $\mathbf{V}$ of the functionals on $\mathbf{H}$, so if we take the domain $D = \{u \in \mathbf{V} : \mathcal{A}u \in \mathbf{H}'\}$, then $\mathbf{A}u \in \mathbf{H}$, which means that $\mathcal{A}u \in \mathbf{H}'$ through the identification of $\mathbf{H}$ with $\mathbf{H}'$ by its Riesz map. Thus, with this identification, it is clear that $\mathbf{A}$ is the restriction of $\mathcal{A}$ to $\mathbf{H} \subset \mathbf{V}'$; that is, if $\mathcal{R}$ is the Riesz map of $\mathbf{H}$ onto $\mathbf{H}'$, then $\mathbf{A} = \mathcal{R}^{-1}\mathcal{A}|_D$. We call such $\mathbf{A}$ (induced from $\mathcal{A}$) *regular*.

3. Fissured medium equations and its homogenized process

3.1. The ε-fissured medium equations

We consider the homogenization of the small *temperature variation* in a system consisting of two components which is introduced in [10] and [11];

$$\frac{\partial}{\partial t} u_\varepsilon(t) - \operatorname{div} B\left(\frac{x}{\varepsilon}\right) \nabla u_\varepsilon(t) + \frac{1}{\delta}(u_\varepsilon - v_\varepsilon) = f(t), \tag{3.1}$$

$$-\operatorname{div} C\left(\frac{x}{\varepsilon}\right) \nabla v_\varepsilon(t) + \frac{1}{\delta}(v_\varepsilon - u_\varepsilon) = 0. \tag{3.2}$$

We assume that *anisotropic conductivities* $B \equiv (b_{ij})$, $C \equiv (c_{ij})$ are $n \times n$-matrices satisfying: for $\xi = (\xi_1, \xi_2, \cdots, \xi_n) \in \mathbf{R}^n$,

$$c_0 \,|\xi|^2 \le \sum_{i,j=1}^{n} b_{ij}(x)\xi_i\xi_j \quad \text{and} \quad c_0 \,|\xi|^2 \le \sum_{i,j=1}^{n} c_{ij}(x)\xi_i\xi_j \tag{3.3}$$

for some strictly positive constant $c_0 > 0$, and assume that δ is strictly positive and represents the *resistance of the medium* to this exchange. The given medium Ω is presumed to be an open subset of $\mathbf{R}^n$(the same theory can be obtained even the case when Ω is a curved domain such as a Riemannian manifold). If there is a boundary, one poses Dirichlet boundary condition $u = 0$ on $\partial\Omega$, which naturally makes us consider the function space $\mathbf{V} \equiv H_0^1(\Omega)$. We also assume that the external source $f : [0, T] \to L^2(\Omega)$ is Hölder continuous, that is, for some $0 < \alpha < 1$,

$$\|f(t_1) - f(t_2)\|_{L^2(\Omega)} \le C|t_1 - t_2|^\alpha \quad \text{for } t_1, t_2 \in [0, T], \tag{3.4}$$

and $b_{ij}, c_{ij} \in C^\infty(\Omega)$ for all $1 \le i, j \le n$.

3.2. Homogenized process

3.2.1. Existence of the ε-solution.

We put the fissured medium equations (3.1) and (3.2) in functional analysis setting. To accomplish it, define bilinear forms $\mathcal{B}_\varepsilon, \mathcal{C}_\varepsilon \in \mathcal{L}(\mathbf{V}, \mathbf{V}')$ as follows:

$$\mathcal{B}_\varepsilon u_1(u_2) \equiv \int_\Omega B\left(\frac{x}{\varepsilon}\right) \nabla u_1 \cdot \nabla u_2 \, dx, \qquad u_1, u_2 \in \mathbf{V},$$

$$\mathcal{C}_\varepsilon v_1(v_2) \equiv \int_\Omega C\left(\frac{x}{\varepsilon}\right) \nabla v_1 \cdot \nabla v_2 \, dx, \qquad v_1, v_2 \in \mathbf{V}.$$

Then we have a variational formulation;

$$\frac{\partial u_\varepsilon}{\partial t} + \mathcal{B}_\varepsilon u_\varepsilon + \frac{1}{\delta}(u_\varepsilon - v_\varepsilon) = f(t) \ , \tag{3.5}$$

$$\mathcal{C}_\varepsilon v_\varepsilon + \frac{1}{\delta}(v_\varepsilon - u_\varepsilon) = 0 \qquad \text{in } \mathbf{V}', \tag{3.6}$$

with the identification $\frac{1}{\delta}(u_\varepsilon - v_\varepsilon)$ to its dual via the Riesz map $\mathcal{R}$ (introduced in Section 2). From equation (3.6), we have

$$v_\varepsilon = \left(\mathcal{C}_\varepsilon + \frac{1}{\delta}\right)^{-1} \frac{1}{\delta} u_\varepsilon. \tag{3.7}$$

This expression makes sense by virtue of Lax-Milgram theorem - in fact, the linear operator $\mathcal{C}_\varepsilon + \frac{1}{\delta} : \mathbf{V} \to \mathbf{V}'$ is $\mathbf{V}$-coercive (uniformly in $\varepsilon > 0$). Substituting (3.7) into equation (3.6), we get

$$\frac{\partial u_\varepsilon}{\partial t} + \left\{ \frac{1}{\delta}\left(I - \frac{1}{\delta}\left(\mathcal{C}_\varepsilon + \frac{1}{\delta}\right)^{-1}\right) + \mathcal{B}_\varepsilon \right\} u_\varepsilon = f \quad \text{in } \mathbf{V}'. \tag{3.8}$$

Clearly, we have $u_\varepsilon(t) \in \mathbf{V}$ at each $t \in [0, T]$. We define a continuous operator $\mathcal{A}_\varepsilon : \mathbf{V} \to \mathbf{V}'$ by

$$\mathcal{A}_\varepsilon \equiv \frac{1}{\delta}\left(I - \frac{1}{\delta}\left(\mathcal{C}_\varepsilon + \frac{1}{\delta}\right)^{-1}\right) + \mathcal{B}_\varepsilon, \tag{3.9}$$

and we put the corresponding unbounded operator $\mathbf{A}_\varepsilon : D(\mathbf{A}_\varepsilon) \to \mathbf{H}$ of $\mathcal{A}_\varepsilon$ on the Hilbert space $\mathbf{H} \equiv L^2(\Omega)$ with the domain $D(\mathbf{A}_\varepsilon) \equiv \{v \in \mathbf{V} : \mathbf{A}_\varepsilon v \in \mathbf{H}\}$ as we pointed out in Section 2. Then equation (3.8) reads

$$\frac{\partial u_\varepsilon}{\partial t} + \mathbf{A}_\varepsilon u_\varepsilon = f. \tag{3.10}$$

Note that each operator $\mathbf{A}_\varepsilon : D(\mathbf{A}_\varepsilon) \longrightarrow \mathbf{H}$ is self-adjoint and regular m-accretive. In fact, the first part of (3.9) is the *Yosida approximations* of an m-accretive operator $\mathcal{C}_\varepsilon$, so it is m-accretive (page 161 in

[10]). Hence by Hille-Yosida theorem, there exists the unique solution of initial value problem for (3.10) − in fact, we obtain a unique solution $u_\varepsilon \in C([0,T];L^2(\Omega)) \cap C^1((0,T);H_0^1(\Omega)\cap H^2(\Omega))$ for the Cauchy problem (3.10). Also, we define $v_\varepsilon(t)$ by (3.7), then $v_\varepsilon \in C([0,T];L^2(\Omega)) \cap C^1((0,T);H_0^1(\Omega)\cap H_{loc}^4(\Omega))$ is the unique solution to the system of equations (3.5) and (3.6).

3.2.2. Solution of the limt equation.

We will show that the sequence $\{u_\varepsilon\}$ of solutions for (3.10) converges to some u in some topological space, which shall be clarified in the following process. The limit u turns out to be a solution of an evolution equation. This equation could be described in the following. We first define the operators $\mathcal{B}$, $\mathcal{C}$ as

$$\mathcal{B}u_1(u_2) \equiv \int_\Omega \bar{B}\nabla u_1 \cdot \nabla u_2\,dx, \qquad \mathcal{C}u_1(u_2) \equiv \int_\Omega \bar{C}\nabla u_1 \cdot \nabla u_2\,dx,$$

for $u_1, u_2 \in \mathbf{V}$, where we set

$$\bar{B} \equiv \inf_{u\in H^1([0,1]^n)} \langle (I+\nabla u)\cdot A(I+\nabla u)\rangle,$$
$$\bar{C} \equiv \inf_{u\in H^1([0,1]^n)} \langle (I+\nabla u)\cdot B(I+\nabla u)\rangle.$$

Then as in the above, a continuous operator $\mathcal{A}:\mathbf{V}\to\mathbf{V}'$ can be chosen by

$$\mathcal{A} \equiv \frac{1}{\delta}\left(I - \frac{1}{\delta}\left(\mathcal{C}+\frac{1}{\delta}\right)^{-1}\right) + \mathcal{B}$$

and let the operator $\mathbf{A}: D(\mathbf{A})\to\mathbf{H}$ be the corresponding unbounded operator of $\mathcal{A}$ with $D(\mathbf{A}) \equiv \{v\in\mathbf{V} : \mathbf{A}v\in\mathbf{H}\}$. Since $\mathcal{A}$ is self-adjoint and $\mathbf{V}$-coercive, there exists a unique solution $u \in C([0,T];L^2(\Omega)) \cap C^1((0,T);H_0^1(\Omega)\cap H^2(\Omega))$ of the Cauchy problem

$$\frac{\partial u}{\partial t} + \mathbf{A}u = f. \tag{3.11}$$

3.2.3. A-priori estimates.

We put u_ε in the functional equation (3.6) and v_ε in the equation (3.5) and add the resulting two equations together to get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_\varepsilon\|^2_{L^2(\Omega)} + c_0\|\nabla u_\varepsilon\|^2_{L^2(\Omega)} + c_0\|\nabla v_\varepsilon\|^2_{L^2(\Omega)} + \frac{1}{\delta}\|u_\varepsilon - v_\varepsilon\|^2_{L^2(\Omega)}&\\ \le \int_\Omega f(x)u_\varepsilon(x)dx&\\ \le \|f\|_{L^2(\Omega)}\|u_\varepsilon\|_{L^2(\Omega)}.&\end{aligned}$$

This leads to

$$\frac{d}{dt}\|u_\varepsilon\|^2_{L^2(\Omega)} \le 2\|f\|_{L^2(\Omega)}\|u_\varepsilon\|_{L^2(\Omega)}. \tag{3.12}$$

Substituting $\|u_\varepsilon\|^2_{L^2(\Omega)} \equiv y$, we notice that (3.12) is an ordinary differential inequality $\dot{y}(t) \le \alpha(t)\sqrt{y(t)}$ with $\alpha(t) \equiv 2\|f(t)\|_{L^2(\Omega)}$ and $\dot{} \equiv \frac{d}{dt}$. Equivalently, we have $\frac{\dot{y}}{\sqrt{y}} \le \alpha$ or

$$\|u_\varepsilon(t)\|_{L^2(\Omega)} \le 4\int_0^T \|f(t)\|_{L^2(\Omega)}dt + \|u_\varepsilon(0)\|_{L^2(\Omega)}.$$

This shows $\max_{t\in[0,T]} \|u_\varepsilon(t)\|_{L^2(\Omega)}$ is bounded. We also have $\max_{t\in[0,T]} \|v_\varepsilon(t)\|_{L^2(\Omega)}$ is bounded, since the operator $\left(\mathcal{C}_\varepsilon + \frac{1}{\delta}\right)^{-1}$ is bounded on $L^2(\Omega)$ with respect to the operator norm and $v_\varepsilon(t) = \frac{1}{\delta}\left(\mathcal{C}_\varepsilon + \frac{1}{\delta}\right)^{-1} u_\varepsilon(t)$ for all $t \in [0,T]$. In turn, from the fact that

$$\begin{aligned} c_0\int_0^T \|\nabla u_\varepsilon(t)\|^2_{L^2(\Omega)}dt + c_0\int_0^T \|\nabla v_\varepsilon(t)\|^2_{L^2(\Omega)}dt \\ \le \max_{t\in[0,T]} \|u_\varepsilon(t)\|_{L^2(\Omega)}\int_0^T \|f(t)\|_{L^2(\Omega)}dt + \frac{1}{2}\|u_\varepsilon(0)\|^2_{L^2(\Omega)}, \end{aligned}$$

we observe that $\|\nabla u_\varepsilon\|_{L^2([0,T];L^2(\Omega))}$ and $\|\nabla v_\varepsilon\|_{L^2([0,T];L^2(\Omega))}$ are bounded.

3.2.4. Weak convergence and the homogenized equation.

For any $\phi \in C^1[0,T]$ with $\phi(T) = 0$, letting $\int_0^T u_\varepsilon(t)\phi(t)dt \equiv \bar{u}_\varepsilon$, we can notice that the sequence $\{\bar{u}_\varepsilon\}$ is in $\mathbf{V}$. It follows from the a-priori estimates that a subsequence $\{\bar{u}_{\varepsilon_j}\}$ converges weakly to some $\bar{u} \equiv \int_0^T u(t)\phi(t)dt$ in $\mathbf{V}$. Also, putting $\int_0^T v_\varepsilon(t)\phi(t)dt \equiv \bar{v}_\varepsilon$, a subsequence $\{\bar{v}_{\varepsilon_j}\}$ converges weakly to some $\bar{v} \equiv \int_0^T v(t)\phi(t)dt$ in $\mathbf{V}$. By heuristic homogenization methods, we can find that the corresponding flows converge weakly in $L^2(\Omega)$: for any $\psi \in L^2(\Omega)$, we have

$$\mathcal{B}_{\varepsilon_j}\bar{u}_{\varepsilon_j}(\psi) \to \mathcal{B}\bar{u}(\psi) \quad \text{and} \quad \mathcal{C}_{\varepsilon_j}\bar{v}_{\varepsilon_j}(\psi) \to \mathcal{C}\bar{v}(\psi)$$

(for example, we refer [13]).

Remark 3.1. *By virtue of Rellich-Kandorochov theorem, we obtain that a subsequence $\{\bar{u}_{\varepsilon_j}\}$ converges strongly to $\bar{u}$ in $L^2(\Omega)$, In particular, $\int_0^T u_{\varepsilon_j}(t)\,dt \to \int_0^T u(t)\,dt$ in $L^2(\Omega)$. This in turn implies that a subsequence $\{u_{\varepsilon_j}(t)\}$ converges strongly to $u(t)$ in $L^2(\Omega)$ for each $t \in [0,T]$.*

Therefore we have that

$$\begin{aligned}\int_0^T \frac{\partial}{\partial t} u_{\varepsilon_j}(t)\phi(t)dt &= -\int_0^T u_{\varepsilon_j}(t)\frac{\partial}{\partial t}\phi(t)dt + u_{\varepsilon_j}(0)\phi(0) \\ &\to -\int_0^T u(t)\frac{\partial}{\partial t}\phi(t)dt + u(0)\phi(0) \text{ strongly in } L^2(\Omega) \\ &= \int_0^T \frac{\partial}{\partial t} u(t)\phi(t)dt.\end{aligned}$$

Hence taking weak limits in (3.5) and (3.6) yields the limit system

$$\begin{aligned}\frac{\partial u}{\partial t} + \mathcal{B}u + \frac{1}{\delta}(u-v) &= f, \\ \mathcal{C}v + \frac{1}{\delta}(v-u) &= 0 \qquad \text{in } \mathbf{V}',\end{aligned}$$

which is equivalent to the limit equation

$$\frac{\partial u}{\partial t} + \left\{ \frac{1}{\delta}\left(I - \frac{1}{\delta}\left(\mathcal{C} + \frac{1}{\delta}\right)^{-1}\right) + \mathcal{B} \right\} u = f.$$

This produces the Cauchy problem (3.11) and the *homogenized fissured medium equation*;

$$\frac{\partial}{\partial t} u(t) - \text{div } \bar{B}\ \nabla u(t) + \frac{1}{\delta}(u-v) = f(t)\ , \tag{3.13}$$

$$-\text{div } \bar{C}\ \nabla v(t) + \frac{1}{\delta}(v-u) = 0. \tag{3.14}$$

Remark 3.2. *By the uniqueness of the solution, we can observe that the* original *sequence* $\{u_\varepsilon(t)\}$ *converges to* $u(t)$ *in* $L^2(\Omega)$ *for each* $t \in [0,T]$, *and* $\{v_\varepsilon(t)\}$ *converges to* $v(t)$ *in* $L^2(\Omega)$ *for each* $t \in [0,T]$.

We summarize the process as follows:

Theorem 3.3. *Let* Ω *be an open subset of* $\mathbb{R}^n$ *and a positive constant* $T>0$. *For a given initial source* $u_\varepsilon(0)=u_0\in L^2(\Omega)$ *and an external source* $f \in C^{0,\alpha}([0,T];L^2(\Omega))$, $0<\alpha<1$, *the sequences of solutions* u_ε *in* $C([0,T];L^2(\Omega)) \cap C^1((0,T);H_0^1(\Omega)\cap H^2(\Omega))$, v_ε *in* $C([0,T];L^2(\Omega)) \cap C^1((0,T);H_0^1(\Omega)\cap H^4_{loc}(\Omega))$ *for the system of fissured medium equations (3.1) and (3.2) satisfying the elliptic condition (3.3) converge strongly in* $L^2(\Omega)$ *together with the weak-convergence in* $L^2[0,T]$ *to the solutions* u *in* $C([0,T];L^2(\Omega)) \cap C^1((0,T);H_0^1(\Omega) \cap H^2(\Omega))$, v *in* $C([0,T];L^2(\Omega)) \cap C^1((0,T);H_0^1(\Omega) \cap H^4_{loc}(\Omega))$, *respectively, of the fissured medium equations (3.13) and (3.14) with the same initial source* $u(0)=u_0$.

REMARK 3.4. *We note that the homogenized fissured medium equations have the same form as the ε-fissured medium equations.*

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