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CIRCULAR UNITS IN A BICYCLIC FUNCTION FIELD

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Abstract. For a real subextension of some cyclotomic function field with a non-cyclic Galois group order l^2, l being a prime different from the characteristic of function field, we compute the index of the Sinnott group of circular units.

1. Introduction

The group E of units of an abelian number field K, though finitely generated, is very difficult to compute. However, it contains the explicitly described group C of circular units, which has a finite index in E. In 1980 Sinnott defined his generalisation of a group of circular units to an abelian number field and found a general formula for the index ([4]):

$$[E:C] = h_K 2^{[K:\mathbb{Q}]-1} \frac{\prod_{p|f_K}[K_{p^e}:\mathbb{Q}]}{[K:\mathbb{Q}]}(R:U)$$

where $f_K = \prod p^e$ is the decomposition of the conductor f_K of K into prime factors, $K_{p^e} = K \cap \mathbb{Q}_{p^e}$, and $(R:U)$ is the Sinnott index of the field K. The definition of $(R:U)$ is quite complicated, it is only known to very special cases. In [2], Kraemer computed the index $[E:C]$ explicitly and easily got the value of the Sinnott's index $(R:U)$ from the Sinnott's formula.

From [1], we see that

$$[\mathcal{O}_F^*:C_F] = \big(\frac{q-1}{\delta_F}\big)^{[F^+:k]-1} Q_0 h(\mathcal{O}_{F^+}) \frac{\prod_{i=1}^{s}[F_{\mathfrak{p}_i^{e_i}}:F_{\mathfrak{e}}]}{[F:F_{\mathfrak{e}}][K_{\mathfrak{e}}:F_{\mathfrak{e}}]} \frac{(e^+R:e^+U)}{d(F)}.$$

for a subfield F of some cyclotomic function field over a global function field k. Suppose l is a prime not equal to $char(k)$ and F a real abelian extension of degree l^2 over k with a non-cyclic Galois group and F is

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contained in some cyclotomic function field over $\mathbb{F}_q(T)$. Then we have $\delta_F = 1, Q_0 = 1$ ([1, Lemma 2.2]), $d(F) = 1$. Therefore we have

$$[E : C] = (q-1)^{l^2-1} h(\mathcal{O}_F) l^{v-2} (R : U), \tag{1.1}$$

where v is the number of subfields of degree l of F, which conductor is a power of irreducible polynomials. As in the number field case, the Sinnott index $(R : U)$ is difficult to compute and known only some special cases.

In this paper we will consider the case of a real subextension F of a cyclotomic function field with a non-cyclic Galois group of order l^2, where l is a prime different from the characteristic of k. First, we find a basis of C/B, where B is the product of the groups of circular units of all proper subfields of F. Next, we directly compute $[E : B]$, which gives us the index $[E : C]$. From the index-formula (1.1), we easily compute the Sinnot's index $(R : U)$.

Notation

$A = \mathbb{F}_q[T], k = \mathbb{F}_q(T)$

F : a real abelian extension of degree l^2 over k with a non-cyclic Galois group. Here l is a prime, not equal to $char(k)$. We assume that F is contained in some cyclotomic function field.

$G = \mathrm{Gal}(F/k)$, the Galois group of F over k.

R_F : the regulator of F

$h(F)$ (resp. $h(\mathcal{O}_F)$) : the divisor (resp. ideal) class number of F

F_i, f_i : subfields of F of degree l and their conductors $(0 \le i \le l)$

$G_i = Gal(F/F_i)$

λ_N : the primitive N-torsion point. Here $N \in \mathbb{A}$

$K_{P_i^{e_i}} = K(\lambda_{P_i^{e_i}})$ $P_i^{e_i}$-th cyclotomic function field

X_L : the group of Dirichlet characters corresponding to a field L

f_χ : the conductor of a character χ

$I_P \subseteq G$: the inertia group of P

E, D, C : the group of units, circular numbers and circular units of F

2. Circular units

Since $(\mathbb{A}/P)^*$ is cyclic for any irreducible polynomials P and G is not cyclic, the conductor f of F can not be a power of an irreducible polynomial. Therefore f is divisible by at least two different monic irreducible polynomials. The group $G = \mathrm{Gal}(F/k) \cong \mathbb{F}_l \times F_l$ has exactly

$l+1$ subgroups G_i of order l. Let $F_i \subset F$ be the subfield of degree l corresponding to G_i. Let δ_i be a fixed generator of $\mathrm{Gal}(F_i/k)$.

LEMMA 2.1. *The conductor $f = cond(F)$ of F has no square factors.*

Proof. If $P^2|f$, then $P^2|f_\chi$ for some $\chi \in X_F$. If $\chi = \chi_P \chi_{P_1} \cdots \chi_{P_t}$ is a decomposition of χ, then $P^2|f_{\chi_P}$. Then $p = char(k)||\langle\chi_P\rangle|$. Therefore p must divide the ramification index of p in F([5, Theorem 3.5].) Hence $p|[F:k] = l^2$, which is a contradiction. □

From the same argument in [2], we have

PROPOSITION 2.2. *Let $P_1, \ldots, P_s$ be all the monic irreducible polynomials ramifying in F. If $Q_i = \{j | P_j \nmid f_i, 1 \le j \le s\}$ is the index set for the monic irreducibles unramified in F_i, the the sets $Q_0, \ldots, Q_l$ are mutually disjoint proper subsets of the set $Q = \{1, \ldots, s\}$ and we have*

$$Q = \bigcup_{i=0}^{l} Q_i$$

The group D of cyclotomic numbers of F is the group generated by $\mathbb{F}_q^*, \varepsilon = N_{K_f/F}(\lambda_f), \varepsilon_i = N_{K_{f_i}/F_i}(\lambda_{f_i})$.

PROPOSITION 2.3. *$C/\mathbb{F}_q^*$ is a G-module generated by*

$$\eta = N_{K_f/F}(\lambda_f),$$

$$\eta_i = \begin{cases} N_{K_{f_i}/F_i}(\lambda_{f_i}) & \text{for } f_i \text{ composite,} \\ N_{K_{f_i}/F_i}(\lambda_{f_i})^{(1-\delta_i)} & \text{for } f_i \text{ irreducible.} \end{cases}$$

Notation : For $P_j \nmid f_i$,

$$(\delta_i|_{F_i})^{k_{ij}} = \mathrm{Frob}^{-1}(P_j, F_i),$$

$k_{ij} \in \mathbb{F}_l$

From the almost same argument in [2],

PROPOSITION 2.4. *We have*

$$\eta^l = \prod_{i=0}^{l} \varepsilon_i^{\prod_{j \in Q_i}(1-\delta^{k_{ij}})}.$$

COROLLARY 2.5.

$$\eta^l = \prod_{i=0}^{l} \eta_i^{\beta_i},$$

where $\beta_i \in \mathbb{Z}[\delta_i]$ are defined as follows:

$$\beta_i = \begin{cases} \prod_{j \in Q_i}(1-\delta_i^{k_{ij}}) & \textit{if } f_i \textit{ composite,} \\ \prod_{j \in Q_i}(1-\delta_i^{k_{ij}})/(1-\delta_i) & \textit{if } f_i \textit{ irreducible,} \end{cases}$$

Let B be the subgroup of $C/\mathbb{F}_q^*$ as a G-module generated by $\{\eta_i\}$. We see that $B = \prod_{i=1}^{l+1} C_i/\mathbb{F}_q^*$, where C_i is the group of circular units of F_i.

PROPOSITION 2.6. *The set $Z = \{\eta_i^{\delta_i^j} : 0 \leq i \leq l, 0 \leq j \leq l-2\}$ is a $\mathbb{Z}$-basis of B.*

Definition : For $0 \leq i \leq l$ we define:

$$a_i = \begin{cases} -\infty & \text{if there is a } j \in Q_i \text{ such that } k_{ij} = 0, \\ l-1-|Q_i| & \text{if there is no such } j \in Q_i \text{ and } f_i \text{ is composite,} \\ l-|Q_i| & \text{if there is no such } j \in Q_i \text{ and } f_i \text{ is irreducible,} \end{cases}$$

We may assume that G_i, K_i are chosen such that

$$a_0 \geq a_1 \geq \ldots \geq a_l.$$

Let σ (resp. τ) be a fixed generator of G_0 (resp. G_1). For $2 \leq i \leq l$, let $\tau\sigma^{l-n_i}$ be a fixed generator of G_i, which defines $n_i \in \mathbb{F}_l^*$ in a unique way. If we define $n_1 = 0$, $\tau|_{F_i} = \sigma^{n_i}|_{F_i}$ for $1 \leq i \leq l$. We may assume that $\delta_0 = \tau, \delta_i = \sigma$ for $1 \leq i \leq l$.

Notation : Let k (resp. q) be the smallest i such that $a_i \leq 0$ (resp. $a_i = -\infty$) (if no such i exists then k (resp. q) equals $l+1$.

Notation : Put $p_{ij} = \eta^{(1-\tau)^i(1-\sigma)^j}$.

$$S_0 = \{p_{ij} : j = 0, 0 \leq i < a_0\}, R_{m,n} = \{p_{ij} : i = m, 0 < j < n\},$$

$$U = \begin{cases} \emptyset & \text{for } k = 0, \\ S_0 & \text{for } k = 1, \\ S_0 \cup R_{0,a_1} \cup R_{1,a_2-1} \cup \cdots \cup R_{b-1,a_b-(b-1)} & \text{for } k \geq 2, \end{cases}$$

where $b = 1$ for $k = 2$, for $k > 2$, it is the greatest index $i < k$ such that $a_i > i$.

THEOREM 2.7. *The set U is a basis of C/B.*

Proof. The proof is the almost same as in the number field case ([2, Section 4]). □

COROLLARY 2.8.

$$[C:B] = \begin{cases} 1 & \text{for } k=0, \\ l^{a_0} & \text{for } k=1, \\ l^{a_0+(a_1-1)+\cdots+(a_b-b)} & \text{for } k\geq 2, \end{cases}$$

where b is defined as above.

3. Computation of the index [E:C]

Let ∞_F be a fixed place of F lying above ∞. From [6, Proposition 4.1], we have

LEMMA 3.1. *Let $\chi \in X_{F_i}$. Then we have*

$$\sum_{g\in G} \chi(g) \operatorname{ord}_{\infty_F}(\eta_i^g) = \begin{cases} l(q-1)L_k(0,\chi) & \text{if } f_i \text{ composite,} \\ l(q-1)(1-\chi(\delta_i)^{-1})L_k(0,\chi) & \text{if } f_i \text{ irreducible.} \end{cases}$$

First we compute $[E:B]$.

PROPOSITION 3.2. *Let v be the number of subfields of F having an irreducible conductor. Then we have*

$$[E:B] = l^{v-2+\frac{l(l-1)}{2}}(q-1)^{l^2-1}h(\mathcal{O}_F).$$

Proof. The proof is similar to the number field case([2, Proposition 5.1]). The regulator of the basis Z of B is

$$R_B = |\det(\operatorname{ord}_{\infty_F}(\eta_i^{\delta^j g}))|_{0\leq i\leq l, 0\leq j\leq l-2, 1\neq g\in G}.$$

We enlarge the matrix to

$$M = \begin{pmatrix} 1 & \cdots & 1 & \cdots \\ \vdots & & \vdots & \\ \operatorname{ord}_{\infty_F}(\eta_i^{\delta_j}) & \cdots & \operatorname{ord}_{\infty_F}(\eta_i^{\delta_j g}) & \cdots \\ \vdots & & \vdots & \end{pmatrix}$$

Put $Z = (\chi(g))_{g\in G,\chi\in X}$. Note that $|\det(M)| = l^2 R_B, |\det(Z)| = l^{l^2}$. We assume that the first row (resp. column) corresponds to the trivial automorphism (resp. character). The other rows (resp. columns) will be indexed with ordered pairs (i,j) (resp. (m,n)) with $0 \leq i,m \leq l$ and $1 \leq j,n \leq l-1$ and arranged in an increasing lexicographical order. The row (i,j) (resp. column (m,n)) corresponds to the automorphism g_i^j (resp. character χ_m^n), where g_i (resp. χ_m) denotes a fixed generator of G_i (resp. X_{K_m}).

Now we evaluate MZ:
1. In the first row we get the numbers $a_\chi = \sum_{g\in G}\chi(g)$, so we have $(l^2, 0, \ldots, 0)$. 2. At $(i,j),(m,n)$, with $i \neq m$ we get

$$a_{(i,j),(m,n)} = 0.$$

3. At $(i,j),(i,n)$ we get

$$a_{(i,j),(m,n)} = \sum_{g\in G}\chi_{i,n}(g)\,\mathrm{ord}_{\infty_F}(\eta_i^{\delta_i^j g}) = \chi_{i,n}(\delta_i)^{-j}c_{in},$$

where $c_{in} = \sum_{g\in G}\chi_{i,n}(g)\,\mathrm{ord}_{\infty_F}(\eta_i^g)$

Thus we have

$$\begin{aligned}|\det(MZ)| &= l^2\prod_{i=0}^{l}|\det(a_{(i,j),(i,n)})_{0\le j\le l-2, 1\le n\le l-1}| \\ &= l^2\prod_{i=0}^{l}(\prod_{n=1}^{l-1}|c_{in}|)\cdot|\det(\chi_{i,n}(\delta_i)^{-j})_{0\le j\le l-2,1\le n\le l-1}|\end{aligned}$$

By Lemma 3.1, we have

$$\prod_{n=1}^{l-1}|c_{in}| = \begin{cases} l^{l-1}(q-1)^{l-1}\prod_{n=1}^{l-1}|L_k(0,\chi_{i,n})| & (f_i \text{ comp.}), \\ l^{l-1}(q-1)^{l-1}\prod_{n=1}^{l-1}|1-\chi_{i,n}(\delta_i^{-1})||L_k(0,\chi_{i,n})| & (f_i \text{ irred.}),\end{cases}$$

and

$$\prod_{n=1}^{l-1}|1-\chi_{i,n}(\delta_i^{-1})| = l$$

Therefore we have

$$\begin{aligned}&|\det(MZ)| \\ &= l^2 l^{(l+1)(l-1)}(q-1)^{(l+1)(l-1)}l^v\prod_{i=0}^{l}\prod_{n=1}^{l-1}|L_k(0,\chi_{i,n})|\cdot\prod_{i=1}^{l}|\det(\chi_{i,n}(\delta_i)^{-j})| \\ &= l^{1+v+l^2}(q-1)^{l^2-1}h(\mathcal{O}_F)R_F\prod_{i=1}^{l}|\det(\chi_{i,n}(\delta_i)^{-j})|\end{aligned}$$

because

$$\prod_{i=0}^{l}\prod_{n=1}^{l-1}|L_k(0,\chi_{i,n})| = \prod_{\chi_0\neq\chi\in X_F}|L_k(0,\chi)| = h(F) = h(\mathcal{O}_F)R_F.$$

Let

$$F_i = (\chi_{i,n}(\delta_i)^{-j})_{0\le j\le l-2,1\le n\le l-1}$$

and

$$H_i = (\chi_i^n(\delta_i)^{-j})_{0\le j\le l-1, 0\le n\le l-1}.$$

Then $\det H_i = l\cdot \det F_i$. Thus

$$|\det F_i| = \frac{1}{l}|\det H_i| = \frac{1}{l}\cdot l^{\frac{l}{2}} = l^{\frac{l-2}{2}}$$

and so

$$|\det(MZ)| = l^{1+v+l^2}(q-1)^{l^2-1}h(\mathcal{O}_F)R_F l^{\frac{(l-2)(l+1)}{2}}$$

Since $|\det Z| = l^{l^2}$ and $|\det M| = l^2 R_B$, we have

$$R_B = l^{v-2+\frac{l(l-1)}{2}}(q-1)^{l^2-1}h(\mathcal{O}_F)R_F.$$

Since $[E:B] = R_B/R_F$, we have

$$[E:B] = l^{v-2+\frac{l(l-1)}{2}}(q-1)^{l^2-1}h(\mathcal{O}_F).$$

□

Finally we compute $[E:C]$ using $[E:B] = [E:C][C:B]$.

THEOREM 3.3. *Let v be the number of subfields of F having an irreducible conductor. Then we have*

$$[E:C] = \begin{cases} l^{v-2+\frac{l(l-1)}{2}}(q-1)^{l^2-1}h(\mathcal{O}_F) & \text{for } k=0, \\ l^{v-2+\frac{l(l-1)}{2}-a_0}(q-1)^{l^2-1}h(\mathcal{O}_F) & \text{for } k=1, \\ l^{v-2+\frac{l(l-1)}{2}-a_0-(a_1-1)-(a-2-2)-\cdots-(a_b-b)}(q-1)^{l^2-1}h(\mathcal{O}_F) & \text{for } k\ge 2. \end{cases}$$

COROLLARY 3.4. *For the Sinnott index $(R:U)$ we have*

$$(R:U) = \begin{cases} l^{\frac{l(l-1)}{2}} & \text{for } k=0, \\ l^{\frac{l(l-1)}{2}-a_0} & \text{for } k=1, \\ l^{\frac{l(l-1)}{2}-a_0-(a_1-1)-(a_2-2)-\cdots-(a_b-b)} & \text{for } k\ge 2. \end{cases}$$

Proof. Note that $\prod_{i=1}^{s}[F_{\mathfrak{p}_i^{e_i}}:k] = l^v$. □

COROLLARY 3.5. *Let r be the number of irreducible factors in the conductor of F and j the number of its nontrivial subfields K_i with* *conductor $f_{K_i} \neq f$, t the number of ramified irreducible polynomials whose decomposition group is bigger than its inertia group. Then,*

$$(R:U) = \begin{cases} 1 & \text{if } r = 2, \\ 1 & \text{if } r = 3 \text{ and } j = 2, \\ l & \text{if } r = 3 \text{ and } j = 3, \\ 1 & \text{if } r = 4 \text{ and } j = 2, \\ l & \text{if } r = 4 \text{ and } j = 3, \\ l^2 & \text{if } r = 4 \text{ and } j = 4, t \neq 0, \\ l^3 & \text{if } r = 4 \text{ and } j = 4, t = 0, \end{cases}$$

From the Theorem 3.3, we immediately have the following.

Corollary 3.6. *Assume that $l \nmid q-1$. If $(R:U) = 1, v \leq 1$ or $(R:U) = l, v = 0$, then l divides $h(\mathcal{O}_F)$.*

Remark 3.1. It is easy to check that the condition of the above corollary is satisfied when $r = 3$ or $(r = 4, j = 2)$, $(r = 4, j = 3)$ in the Corollary 3.5.

References

[1] J. Ahn, S. Bae and H. Jung, *Cyclotomic units and Stickelberger ideals of global function fields.* Trans. Amer. Math. Soc. **355** (2003), no. 5, 1803–1818.

[2] P. Kraemer, *Circular units in a bicyclic field.* J. Number Theory **105** (2004), no. 2, 302–321.

[3] M. Rosen, Number theory in function fields. Graduate Texts in Mathematics, 210. Springer-Verlag, New York, 2002.

[4] W. Sinnott, *On the Stickelberger ideal and the circular units of an abelian field.* Invent. Math. **62** (1980/81), 181–234.

[5] L. Washington, Introduction to Cyclotomic fields. Graduate Texts in Mathematics, 83. Springer-Verlag, New York, 1996.

[6] L. Yin, *Index-class number formulas over global function fields.* Compositio Math. **109** (1997), 49–66.

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