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LIE BIALGEBRA ARISING FROM POISSON BIALGEBRA $U(\mathfrak{sp}_4)^\circ$

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Abstract. Let $U(\mathfrak{sp}_4)$ be the universal enveloping algebra of the symplectic Lie algebra $\mathfrak{sp}_4$. Then the restricted dual $U(\mathfrak{sp}_4)^\circ$ becomes a Poisson Hopf algebra with the Sklyanin Poisson bracket determined by the standard classical r-matrix. Here we illustrate a method to obtain the Lie bialgebra from a Poisson bialgebra $U(\mathfrak{sp}_4)^\circ$.

1. Introduction

Let $\mathfrak{g}$ be a Lie algebra associated to a connected and simply connected Lie group G and let $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$. Then the coordinate ring $\mathcal{O}(G)$ is isomorphic as a Hopf algebra to the restricted dual $U(\mathfrak{g})^\circ$ of $U(\mathfrak{g})$ and it is sometimes more convenient to work on $U(\mathfrak{g})^\circ$ than to do on $\mathcal{O}(G)$. Moreover, if $\mathfrak{g}$ is a coboundary Lie bialgebra then the Sklyanin Poisson bracket on $U(\mathfrak{g})^\circ$ is shown explicitly in [4, 2.2]. Thus $U(\mathfrak{g})^\circ$ becomes a Poisson Hopf algebra and the pair $((\mathfrak{m}/\mathfrak{m}^2)^*, \mathfrak{m}/\mathfrak{m}^2)$ is a natural Lie bialgebra by [4, 1.5], where $\mathfrak{m}$ is the kernel of the counit in $U(\mathfrak{g})^\circ$. Here, as an example, we calculate these algorithms for the symplectic Lie algebra $\mathfrak{sp}_4$ with the standard Lie bialgebra and recover it.

2. Example

Recall that the definition for Lie bialgebra in [1, 1.3] and [3, 2.1.1]. A Lie bialgebra is a pair $(\mathfrak{g}, \psi)$, where $\mathfrak{g}$ is a Lie algebra and $\psi : \mathfrak{g} \longrightarrow \mathfrak{g}\wedge\mathfrak{g}$, called cobracket, satisfying the following conditions:

(a) The dual map $\psi^* : \mathfrak{g}^* \wedge \mathfrak{g}^* \longrightarrow \mathfrak{g}^*$ makes $\mathfrak{g}^*$ a Lie algebra.

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(b) The cobracket $\psi : \mathfrak{g} \longrightarrow \mathfrak{g} \wedge \mathfrak{g}$ is a 1-cocycle on $\mathfrak{g}$, the $\mathfrak{g}$-module structure on $\mathfrak{g} \wedge \mathfrak{g}$ given by the adjoint action. In other words, we have that for any $a, b \in \mathfrak{g}$,

$$\psi([a,b]) = a \cdot \psi(b) - b \cdot \psi(a),$$

where

$$a \cdot (b \otimes c) = [a \otimes 1 + 1 \otimes a, b \otimes c] = [a,b] \otimes c + b \otimes [a,c].$$

The symplectic Lie algebra $\mathfrak{sp}_4$ is the Lie algebra consisting of $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where A, B, C, D are 2×2-matrices such that

$$A^t = -D, B^t = B, C^t = C.$$

In the symplectic Lie algebra $\mathfrak{sp}_4$, set

$$\begin{array}{llll} h_1 = E_{11} - E_{22} - E_{33} + E_{44} & h_2 = E_{22} - E_{44} & & \\ e_1 = E_{12} - E_{43} & e_2 = E_{24} & e_3 = E_{14} + E_{23} & e_4 = E_{13} \\ f_1 = E_{21} - E_{34} & f_2 = E_{42} & f_3 = E_{41} + E_{32} & f_4 = E_{31}. \end{array}$$

(See [2, 8.3] for $\mathfrak{sp}_4$.) Let H be the subspace of $\mathfrak{sp}_4$ spanned by h_1, h_2 and let $\alpha_1, \alpha_2 \in H^*$ be defined by

$$\begin{array}{ll} \alpha_1(h_1) = 2, & \alpha_2(h_1) = -2 \\ \alpha_1(h_2) = -1, & \alpha_2(h_2) = 2. \end{array}$$

Then $e_1, e_2, e_3, e_4, f_1, f_2, f_3, f_4$ are weight vectors with weights

$$\begin{array}{lll} \mathrm{wt}(e_1) = \alpha_1, & \mathrm{wt}(e_2) = \alpha_2, & \mathrm{wt}(e_3) = \alpha_1 + \alpha_2, \\ \mathrm{wt}(e_4) = 2\alpha_1 + \alpha_2 & & \\ \mathrm{wt}(f_1) = -\alpha_1, & \mathrm{wt}(f_2) = -\alpha_2, & \mathrm{wt}(f_3) = -(\alpha_1 + \alpha_2), \\ \mathrm{wt}(f_4) = -(2\alpha_1 + \alpha_2). & & \end{array}$$

Hence α_1, α_2 are positive simple roots. It is well-known that

$$r = e_1 \wedge f_1 + 2e_2 \wedge f_2 + e_3 \wedge f_3 + 2e_4 \wedge f_4 \in \mathfrak{sp}_4 \wedge \mathfrak{sp}_4$$

satisfies the modified classical Yang-Baxter equation and gives the standard Lie bialgebra structure ψ in $\mathfrak{sp}_4$ such that

$$\begin{array}{ll} \psi(h_1) = 0, & \psi(h_2) = 0 \\ \psi(e_1) = e_1 \wedge h_1, & \psi(e_2) = 2e_2 \wedge h_2 \\ \psi(e_3) = e_3 \wedge h_1 + 2e_3 \wedge h_2 - 4e_1 \wedge e_2, & \\ \psi(e_4) = 2e_4 \wedge h_1 + 2e_4 \wedge h_2 - 2e_1 \wedge e_3 & \\ \psi(f_1) = f_1 \wedge h_1, & \psi(f_2) = 2f_2 \wedge h_2 \\ \psi(f_3) = f_3 \wedge h_1 + 2f_3 \wedge h_2 - 4f_1 \wedge f_2, & \\ \psi(f_4) = 2f_4 \wedge h_1 + 2f_4 \wedge h_2 - 2f_1 \wedge f_3. & \end{array}$$

(See [3, Exercise 4.1.11].)

The weight lattice $\mathbf{P}$ in $\mathfrak{sp}_4$ is a free abelian group with basis consisting of the fundamental dominant integral weights λ_1, λ_2, where $\lambda_i(h_j) = \delta_{ij}$ for $i, j = 1, 2$. Hence

$$\alpha_1 = 2\lambda_1 - \lambda_2, \quad \alpha_2 = -2\lambda_1 + 2\lambda_2.$$

The natural $\mathfrak{sp}_4$-module $V = \mathbf{k}^4$ is an irreducible highest weight module with highest weight λ_1. In fact, set

$$v_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, v_2 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, v_3 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}, v_4 = \begin{pmatrix}0\\0\\-1\\0\end{pmatrix} \in V.$$

Then v_1 is a highest weight vector with highest weight λ_1 and

$$v_1 \in V_{\lambda_1}, v_2 = f_1 v_1 \in V_{-\lambda_1+\lambda_2}, v_3 = f_2 v_2 \in V_{\lambda_1-\lambda_2}, v_4 = f_1 v_3 \in V_{-\lambda_1}.$$
$$(v_1 \xrightarrow{f_1} v_2 \xrightarrow{f_2} v_3 \xrightarrow{f_1} v_4)$$

Here we simply write $c_{f,v}$ for $c^V_{f,v}$, $v \in V, f \in V^*$. Observe that

$$v_1^* = (V^*)_{-\lambda_1}, v_2^* = v_1^* e_1 \in (V^*)_{\lambda_1-\lambda_2},$$
$$v_3^* = v_2^* e_2 \in (V^*)_{-\lambda_1+\lambda_2}, v_4^* = v_3^* e_1 \in (V^*)_{\lambda_1},$$
$$(v_1^* \xrightarrow{e_1} v_2^* \xrightarrow{e_2} v_3^* \xrightarrow{e_1} v_4^*)$$

$$\begin{array}{llll}
h_1^* = \overline{c_{v_1^*,v_1} - 1} & h_2^* = \overline{c_{v_2^*,v_2} + c_{v_1^*,v_1} - 2} & & \\
e_1^* = \overline{c_{v_1^*,v_2}} & e_2^* = \overline{c_{v_2^*,v_3}} & e_3^* = \overline{c_{v_1^*,v_3}} & e_4^* = -\overline{c_{v_1^*,v_4}} \\
f_1^* = \overline{c_{v_2^*,v_1}} & f_2^* = \overline{c_{v_3^*,v_2}} & f_3^* = \overline{c_{v_3^*,v_1}} & f_4^* = -\overline{c_{v_4^*,v_1}}
\end{array}$$

and

$$\begin{array}{ll}
c_{v_1^*,v_1} \in U(\mathfrak{sp}_4)^\circ_{-\lambda_1,\lambda_1} & c_{v_2^*,v_2} \in U(\mathfrak{sp}_4)^\circ_{\lambda_1-\lambda_2,-\lambda_1+\lambda_2} \\
c_{v_1^*,v_2} \in U(\mathfrak{sp}_4)^\circ_{-\lambda_1,-\lambda_1+\lambda_2} & c_{v_2^*,v_3} \in U(\mathfrak{sp}_4)^\circ_{\lambda_1-\lambda_2,\lambda_1-\lambda_2} \\
c_{v_1^*,v_3} \in U(\mathfrak{sp}_4)^\circ_{-\lambda_1,\lambda_1-\lambda_2} & c_{v_1^*,v_4} \in U(\mathfrak{sp}_4)^\circ_{-\lambda_1,-\lambda_1} \\
c_{v_2^*,v_1} \in U(\mathfrak{sp}_4)^\circ_{\lambda_1-\lambda_2,\lambda_1} & c_{v_3^*,v_2} \in U(\mathfrak{sp}_4)^\circ_{-\lambda_1+\lambda_2,-\lambda_1+\lambda_2} \\
c_{v_4^*,v_1} \in U(\mathfrak{sp}_4)^\circ_{\lambda_1,\lambda_1}. &
\end{array}$$

Let S be the antipode of $U(\mathfrak{sp}_4)^\circ$. Since $V^* \cong V$ as a $U(\mathfrak{sp}_4)$-module and

$$m \circ (\mathrm{id}_{U(\mathfrak{sp}_4)^*} \otimes S) \circ \Delta = \epsilon 1,$$

we have

$$\begin{array}{llll}
\overline{c_{v_3^*,v_4}} = -\overline{c_{v_1^*,v_2}}, & \overline{c_{v_2^*,v_4}} = -\overline{c_{v_1^*,v_3}}, & \overline{c_{v_2^*,v_3}} = 0, & \overline{c_{v_1^*,v_4}} = 0 \\
\overline{c_{v_4^*,v_3}} = -\overline{c_{v_2^*,v_1}}, & \overline{c_{v_3^*,v_1}} = -\overline{c_{v_4^*,v_2}}, & \overline{c_{v_3^*,v_2}} = 0, & \overline{c_{v_4^*,v_1}} = 0.
\end{array}$$

Moreover $U(\mathfrak{sp}_4)^\circ$ is a Poisson bialgebra by [4, 2.2]. The Lie bracket in $\mathfrak{m}/\mathfrak{m}^2 = \mathfrak{sp}_4^*$ is as follows:

$$\begin{array}{lll}
[h_1^*, h_2^*] = 0 & [h_1^*, e_1^*] = -e_1^* & [h_1^*, e_2^*] = 0 \\
[h_1^*, e_3^*] = -e_3^* & [h_1^*, e_4^*] = -2e_4^* & [h_1^*, f_1^*] = -f_1^* \\
[h_1^*, f_2^*] = 0 & [h_1^*, f_3^*] = -f_3^* & [h_1^*, f_4^*] = -2f_4^* \\
[h_2^*, e_1^*] = 0 & [h_2^*, e_2^*] = -2e_2^* & [h_2^*, e_3^*] = -2e_3^* \\
[h_2^*, e_4^*] = -2e_4^* & [h_2^*, f_1^*] = 0 & [h_2^*, f_2^*] = -2f_2^* \\
[h_2^*, f_3^*] = -2f_3^* & [h_2^*, f_4^*] = -2f_4^* & [e_1^*, e_2^*] = -4e_3^* \\
[e_1^*, e_3^*] = -2e_4^* & [e_1^*, e_4^*] = 0 & [e_2^*, e_3^*] = 0 \\
[e_2^*, e_4^*] = 0 & [e_3^*, e_4^*] = 0 & [f_1^*, f_2^*] = -4f_3 \\
[f_1^*, f_3^*] = -2f_4^* & [f_1^*, f_4^*] = 0 & [f_2^*, f_3^*] = 0 \\
[f_2^*, f_4^*] = 0 & [f_3^*, f_4^*] = 0 & [e_i^*, f_j^*] = 0, \text{ all } i, j.
\end{array}$$

This recovers the standard Lie bialgebra in $\mathfrak{sp}_4$.

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