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THE H_1-STIELTJES INTEGRAL OF BANACH-VALUED FUNCTIONS

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Abstract. In this paper, we define the H_1 -Stieltjes integral of Banach-valued functions which is a generalization of real-valued H_1 - Stieltjes integral and investigate some properties of H_1 - Stieltjes integral. Also we show that if $f : [a, b] \to X$ be a function with $\dim X < \infty$, then $f \in H_1LS([a, b], X, \alpha)$ if and only if $f \in H_1S([a, b], X, \alpha)$.

1. Introduction and preliminaries

In 1999, I.J.L. Garces, L.P. Yee and A.D. Sheng [2] defined the H_1 -integral by means of Moore-Smith limit of the Riemann sums using the directed set and investigated the relation of Henstock integral. They proved that a function f is Henstock integrable if and only if $f = g$ almost everywhere on $[a, b]$ for some H_1 integrable function g on $[a, b]$. J.H.Yoon and K.S. Eun [7] studied the basic properties of H_1 -integral. In this paper, we will define the H_1 -Stieltjes integral of Banach-valued functions with respect to a function of bounded variation which is an extention of real valued H_1 - Stieltjes integral with an increasing function. We will investigate some properties of the H_1 -Stieltjes integral of Banach-valued functions .

Let $\delta(x) > 0$ for $x \in [a, b]$ be given. A division $\mathcal{D} = \{([u, v], \zeta)\}$ of $[a, b]$ is said to be δ - *fine* if $\zeta \in [u, v] \subset (\zeta - \delta(\zeta), \zeta + \delta(\zeta))$ for each $([u, v], \zeta)$. Denote by $\mathcal{P}$ the family of all δ - fine divisions of $[a, b]$. For $\mathcal{D}_1, \mathcal{D}_2 \in \mathcal{P}$, we write $\mathcal{D}_2 \geq \mathcal{D}_1$ if for each $\{[s, t], \zeta)\} \in \mathcal{D}_2$ there exists $\{[u, v], \zeta)\} \in \mathcal{D}_1$ such that $[s, t] \subset [u, v]$ and $\{\zeta : ([u, v], \zeta) \in \mathcal{D}_1\} \subset \{\eta : ([s, t], \eta) \in \mathcal{D}_2\}$. Then $(\mathcal{P}, \geq)$ is a directed set.

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Throughout this paper, X is a real Banach space with dual X^* and α is a function of bounded variation.

A function $f : [a, b] \to R$ is H_1 - *integrable* to a real number A on $[a, b]$ if A is the Moore-Smith limit of the Riemann sums with respect to the directed set $(\mathcal{P}, \geq)$, for some positive function δ on $[a, b]$. More precisely , $f : [a, b] \to R$ is H_1 - integrable on A on $[a, b]$ if there exists a positive function δ on $[a, b]$ such that for each $\varepsilon > 0$ there exists a δ - fine division $\mathcal{D}_0$ such that for any δ - fine division $\mathcal{D} \geq \mathcal{D}_0$ of $[a, b]$. we have

$$\|(\mathcal{D}) \sum f(\zeta)(v - u) - A\| < \varepsilon$$

We say that A is the H_1 - *integral* of f on $[a, b]$ and that f is H_1 - integrable on $[a, b]$ using δ and $f : [a, b] \to R$ is H_1 - integrable on a set $E \subset [a, b]$ if f_{χ_E} is H_1 - integrable on $[a, b]$.

It is easy to show that every H_1 - integrable function on $[a, b]$ is Henstock integrable, but the converse don't know([2]).

2. The H_1 - Stieltjes integral of Banach-valued functions

We define the H_1 -Stieltjes integral of Banach-valued functions with respect to a function of bounded variation which is an extention of real-valued H_1 - Stieltjes integral with respect to an increasing function.

Definition 2.1. *Let α be a function of bounded variation on $[a, b]$. A function $f : [a, b] \to X$ is H_1- Stieltjes integrable to $A \in X$ with respect to α on $[a, b]$ if there exists a positive function δ on $[a, b]$ such that for each $\varepsilon > 0$ there exists a δ- fine division $\mathcal{D}_0$ such that for any δ - fine division $\mathcal{D} \geq \mathcal{D}_0$ of $[a, b]$. we have*

$$\|(\mathcal{D}) \sum f(\zeta)(\alpha(v) - \alpha(u)) - A\| < \varepsilon.$$

We say that A is H_1-Stieltjes integral of f on $[a, b]$ and that f is H_1-Stieltjes integrable on $[a, b]$ using δ, we write $A = \int_a^b f d\alpha$ and $f : [a, b] \to X$ is H_1- Stieltjes integrable on a set $E \subset [a, b]$ if f_{χ_E} is H_1- Stieltjes integrable on $[a, b]$ where χ_E denotes the characteristic function of E. We will also write $f \in H_1S([a, b], X, \alpha)$ to mean that f is X-valued H_1-Stieltjes integrable function on $[a, b]$ with respect to α.

If $X = R$, the Banach-valued H_1- Stieltjes integral is the real-valued H_1- Stieltjes integral.

Just as in the Henstock integral, Cauchy criterion holds for the H_1 - Stieltjes integral of Banach-valued functions.

THEOREM 2.2. *A function $f : [a, b] \to X$ is H_1 -Stieltjes integrable with respect to α on $[a, b]$ if and only if there exists a positive function δ such that for each $\varepsilon > 0$ there exists a δ - fine division $\mathcal{D}_0$ such that*

$$\|(\mathcal{D}_1)\sum f(\zeta)(\alpha(v)-\alpha(u)) - (\mathcal{D}_2)\sum f(\zeta)(\alpha(v)-\alpha(u))\| < \varepsilon,$$

for any δ - fine divisions $\mathcal{D}_1, \mathcal{D}_2 \geq \mathcal{D}_0$ of $[a, b]$

Proof. Suppose that f is H_1 - Stieltjes integrable with respect to α on $[a, b]$. There exists a positive function δ such that for any $\varepsilon > 0$ there exists a δ -fine division $\mathcal{D}_0$ such that for any δ - fine division $\mathcal{D} \geq \mathcal{D}_0$ of such that

$$\|(\mathcal{D})\sum f(\zeta)(\alpha(v)-\alpha(u)) - \int_a^b f d\alpha\| < \frac{\varepsilon}{2}.$$

For any δ - fine divisions $\mathcal{D}_1, \mathcal{D}_2 \geq \mathcal{D}_0$ of $[a, b]$,

$$\|(\mathcal{D}_1)\sum f(\zeta)(\alpha(v)-\alpha(u)) - (\mathcal{D}_2)\sum f(\zeta)(\alpha(v)-\alpha(u))\|$$

$$\leq \|(\mathcal{D}_1)\sum f(\zeta)(\alpha(v)-\alpha(u)) - \int_a^b f d\alpha\|$$

$$+\|(\mathcal{D}_2)\sum f(\zeta)(\alpha(v)-\alpha(u)) - \int_a^b f d\alpha)\| < \varepsilon.$$

Hence the Cauchy criterion is satisfied. Conversely, suppose that the Cauchy criterion is satisfied, there exists a positive function δ such that for each positive integer n there exists a δ -fine division $\mathcal{D}_n$ such that for any δ-fine division $\mathcal{D}_1, \mathcal{D}_2 \geq \mathcal{D}_n$ of $[a, b]$, we have

$$\|(\mathcal{D}_1)\sum f(\zeta)(\alpha(v)-\alpha(u)) - (\mathcal{D}_2)\sum f(\zeta)(\alpha(v)-\alpha(u))\| < \frac{1}{n}.$$

We may assume that the sequence $\{\mathcal{D}_n\}$ is nondecreasing since $(\mathcal{P}, \geq)$ is a direct set. For any $\varepsilon > 0$, choose n_0 so that $\frac{1}{n_0} < \varepsilon$. For $m > n \geq n_0$, we have $\mathcal{D}_m \geq \mathcal{D}_n \geq \mathcal{D}_{n_0}$ and

$$\|(\mathcal{D}_m)\sum f(\zeta)(\alpha(v)-\alpha(u)) - (\mathcal{D}_n)\sum f(\zeta)(\alpha(v)-\alpha(u))\| < \frac{1}{n_0} < \varepsilon.$$

Hence $\{(\mathcal{D}_n)\sum f(\zeta)(\alpha(v)-\alpha(u))\}$ is a Cauchy sequence. Since X is a Banach apace, there exists a limit of $\{(\mathcal{D}_n)\sum f(\zeta)(\alpha(v)-\alpha(u))\}$. Let

L be the limit of this sequence. Choose a positive integer N such that $\frac{1}{N} < \frac{\varepsilon}{2}$ and

$$\|(\mathcal{D}_n)\sum f(\zeta)(\alpha(v)-\alpha(u)) - L\| < \frac{\varepsilon}{2},$$

for all $n \geq N$. Let $\mathcal{D}_n$ be a δ-fine division with $\mathcal{D}_n \geq \mathcal{D}_N$, we have

$$\|(\mathcal{D}_n)\sum f(\zeta)(\alpha(v)-\alpha(u)) - L\|$$

$$\leq \|(\mathcal{D}_n)\sum f(\zeta)(\alpha(v)-\alpha(u)) - (\mathcal{D}_m)\sum f(\zeta)(\alpha(v)-\alpha(u))\|$$

$$\|(\mathcal{D}_m)\sum f(\zeta)(\alpha(v)-\alpha(u)) - L\| < \varepsilon.$$

Hence the function f is H_1 -Stieltjes integrable with respect to α on $[a, b]$. □

The following theorem is some basic properties of the H_1- Stieltjes integral. Their proofs follows naturally the real-valued case([5])

THEOREM 2.3. *Let f and g be H_1 -Stieltjes integrable with respect to α on $[a, b]$. Then*

(a) kf is H_1 - Stieltjes integrable with respect to α on $[a, b]$ and

$$\int_a^b kf d\alpha = k\int_a^b f d\alpha$$

(b) $f+g$ is H_1 - Stieltjes integrable with respect to α on $[a, b]$ and

$$\int_a^b (f+g)d\alpha = \int_a^b f d\alpha + \int_a^b f d\alpha.$$

DEFINITION 2.4. *A set $K \subset H_1S([a, b]), X, \alpha)$ is called H_1- Stieltjes equiintegrable with respect to α if there exists a positive function δ on $[a, b]$ such that for each $\epsilon > 0$ there exist a δ - fine division D_0 such that for any δ-fine division $D \geq D_0$ of $[a, b]$*

$$\|(\mathcal{D})\sum f(\zeta)(\alpha(v)-\alpha(u)) - \int_a^b f(x)d\alpha\| < \varepsilon,$$

for each $f \in K$.

Using the concept of H_1- Stieltjes Equiintegrability we have the following convergence theorem for the H_1-Stieltjes integral. Its proof is similar to those for the McShane integral([3])

THEOREM 2.5. *If the sequence* $\{f_n\}$ *of Banach-valued function* $f_n : [a,b] \to X$ *is* H_1*-Stieltjes Equiintegrable with respect to* α *on* $[a,b]$*and* $\lim_{n\to\infty} f_n = f$*, then* $f \in H_1S([a,b]), X, \alpha)$ *and* $\lim_{n\to\infty} \int_a^b f_n d\alpha = \int_a^b f d\alpha$.

THEOREM 2.6. *Let* $f : [a,b] \to X$ *is* H_1*-Stieltjes integrable with respect to* α *on* $[a,b]$*. Then*

(a) *for each* $x^* \in X^*, x^*f$ *is* H_1 *-Stieltjes integrable with respect to* α *on* $[a,b]$ *and*

$$\int_a^b x^* f d\alpha = x^* \int_a^b f d\alpha.$$

(b) $\{x^*f : x^* \in B(X^*)\}$ *is* H_1*- Stieltjes equiintegrable with respect to* α *on* $[a,b]$

(c) f *is weakly measurable.*

Proof. (a) Since $f : [a,b] \to X$ is H_1 - Stieltjes integrable with respect to α on $[a,b]$such that for each $\varepsilon > 0$ there exist a positive function δ on $[a,b]$ such that there exists a δ- fine division $\mathcal{D}_0$ such that

$$\|(\mathcal{D}) \sum f(\zeta)(\alpha(v) - \alpha(u)) - \int_a^b f d\alpha\| < \varepsilon,$$

for any δ -fine division $\mathcal{D} \geq \mathcal{D}_0$. Hence for any $x^* \in X^*$ we have

$$\|(\mathcal{D}) \sum x^* f(\zeta)(\alpha(v) - \alpha(u)) - x^* \int_a^b f d\alpha\|$$

$$\leq \|x^*\| \|(\mathcal{D}) \sum f(\zeta)(\alpha(v) - \alpha(u)) - \int_a^b f d\alpha\| < \|x^*\|\varepsilon,$$

for any δ -fine division $\mathcal{D} \geq \mathcal{D}_0$. Therefore (a) holds.

(b) If $x^* \in B(X^*)$, then the above inequality give

$$\|(\mathcal{D}) \sum x^* f(\zeta)(\alpha(v) - \alpha(u)) - x^* \int_a^b f d\alpha\| < \varepsilon,$$

for any $x^* \in B(X^*)$, so the set $\{x^*f : x^* \in B(X^*)\}$ is H_1- Stieltjes equiintegrable with respect to α on $[a,b]$.

(c) f is weakly measurable since for each $x^* \in X, x^*f$ is H_1 -Stieltjes integrable with respect to α on $[a,b]$ □

3. The H_1L - Stieltjes integral of Banach-valued functions

The Saks-Henstock Lemma holds for the real-valued Henstock integral, but is not satisfied for Banach-valued functions. So we define an integral which satisfies Saks-Henstock Lemma with $|\cdot|$ replaced $\|\cdot\|$.

DEFINITION 3.1. *Let α be a function of bounded variation on $[a,b]$. A function $f:[a,b]\to X$ is H_1L - Stieltjes integrable with respect to α on $[a,b]$ if there a function $F:[a,b]\to X$ defined on the subintervals of $[a,b]$ with the following property: there exists a positive function δ such that for each $\varepsilon>0$ there exists a δ -fine division $D\geq D_0$ of $[a,b]$, we have*

$$\|(\mathcal{D})\sum f(\zeta)(\alpha(v)-\alpha(u))-F^{\alpha}([u,v]])\|<\varepsilon,$$

where $F^{\alpha}([u,v]])=F(\alpha(v))-F(\alpha(u))$. In this case, we will write $f\in H_1LS([a,b],X,\alpha)$.

By definition, an H_1L-Stieltjes integrable function with primitive F^{α} satisfies Saks-Henstock Lemma with $|\cdot|$ replaced $\|\cdot\|$. We note that by the triangle inequality, $f\in H_1LS([a,b],X,\alpha)$ implies $f\in H_1S([a,b],X,\alpha)$. In general, the converse is not true. For real-valued functions, the two integrals are equivalent.

The next theorem says that if $\dim X<\infty$, then the two integrals are equivalent.

THEOREM 3.2. *Let $f:[a,b]\to X$ be a function with $\dim X<\infty$. Then $f\in H_1LS([a,b],X,\alpha)$ if and only if $f\in H_1S([a,b],X,\alpha)$.*

Proof. It suffices to prove that $f\in H_1S([a,b],X,\alpha)$ implies $f\in H_1LS([a,b],X,\alpha)$. If $f\in H_1S([a,b],X,\alpha)$, i.e. there exist real-valued functions $f_i(1\leq i\leq n)$ which are H_1 - Stieltjes integrable with respect to α and $f(t)=(f_1(t),\cdots,f_n(t))$ for each $t\in[a,b]$.

Since all norms in R^n are equivalent, we just one of them, say $\|X\|=\sum_{i=1}^{n}|x_i|$ where $X=(x_1,\cdots,x_n)\in R^n$. Now,$f\in H_1S([a,b],R^n,\alpha)$ impies that for each $\varepsilon>0$ there exists a δ - fine division $\mathcal{D}_0$ such that for any δ - fine division $D\geq D_0$ of $[a,b]$, we have

$$\|(\mathcal{D})\sum f(\zeta)(\alpha(v)-\alpha(u))-F^{\alpha}([u,v]])\|<\varepsilon.$$

This implies that $|\ (\mathcal{D})\sum f_i(\zeta)(\alpha(v)-\alpha(u))-F_i^{\alpha}([u,v]])\ |<\varepsilon$ for $i=1,\cdots,n$, where $F_i(1\leq i\leq n)$ are the primitive of f_i. By the Saks -Henstock Lemma for real-valued function, we have

$$(\mathcal{D})\sum \mid f_i(\zeta)(\alpha(v)-\alpha(u)) - F_i^{\alpha}([u,v]]) \mid < 2\varepsilon,$$

for $i = 1, \cdots, n$. This implies

$$(\mathcal{D})\sum \|f(\zeta)(\alpha(v)-\alpha(u)) - F^{\alpha}([u,v]])\|$$

$$= (\mathcal{D})\sum\sum_{i=1}^{n} \mid f_i(\zeta)(\alpha(v)-\alpha(u)) - F_i^{\alpha}([u,v]) \mid$$

$$= \sum_{i=1}^{n}(\mathcal{D})\sum \mid f_i(\zeta)(\alpha(v)-\alpha(u)) - F_i^{\alpha}([u,v]) \mid < 2n\varepsilon.$$

Hence $f : [a,b] \to X$ is H_1L - Stieltjes integrable with respect to α on $[a,b]$. □

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