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HOMOGENEOUS REAL HYPERSURFACES IN A COMPLEX HYPERBOLIC SPACE WITH FOUR CONSTANT PRINCIPAL CURVATURES

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Abstract. We deal with the classification problem of real hypersurfaces in a complex hyperbolic space. In order to classify real hypersurfaces in a complex hyperbolic space we characterize a real hypersurface M in $H_n(\mathbb{C})$ whose structure vector field is not principal. We also construct extrinsically homogeneous real hypersurfaces with four distinct curvatures and their structure vector fields are not principal.

1. Introduction

Since E. Cartan's work in the late 30's classification problem of hypersurfaces with constant principal curvatures is known to be far from trivial. Among the many great deals some differential geometers have studied the classification problem of real hypersurfaces in a complex hyperbolic space, however, a complete classification has not been obtained until now. This paper deals with the classification problem in a complex hyperbolic space.

Let $H_n(\mathbb{C})$ be a complex hyperbolic space of complex dimension $n(\geqq 2)$ endowed with the metric of constant holomorphic sectional curvature -4, and M be a real hypersurface in $H_n(\mathbb{C})$.

S. Montiel ([7]) gave the following classification theorem.

Theorem A. *Let M be a connected real hypersurface of $H_n(\mathbb{C})$ $n(\geqq 3)$ with two distinct constant principal curvatures, then M is holomorphic congruent to one of the model spaces of A_0 and A_1.*

Moreover, J. Berndt[2] proved the following theorem.

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Theorem B. *Let M be a connected real hypersurface of $H_n(\mathbb{C})$ with constant principal curvatures. If M has a principal structure vector field, then M is holomorphic congruent to an open part of well-known homogeneous model spaces A_0, A_1, A_2, and B type.*

The homogeneous model spaces of A_0 are horospheres, and have two distinct principal curvatures 2 and 1 with multiplicities 1 and $2n-2$ respectively. Those of type A_1 are geodesic spheres (resp. tubes over totally geodesic complex hyperbolic hyperplanes), and have two distinct principal curvatures $2\coth 2t$ and $\tanh t$ (resp. $\coth t$) with multiplicities 1 and $2n-2$ respectively. The A_2 types are tubes over totally geodesic $H_k(\mathbb{C})$ $(1 \leq k \leq n-2)$, and have three distinct principal curvatures $2\coth 2t$, $\tanh t$ and $\coth t$ with multiplicities 1, $2p$ and $2q$ respectively, where $p > 0,\ q > 0$ and $p+q = n-1$. The B types are tubes over totally real hyperbolic space $H_n(\mathbb{R})$, and have three distinct principal curvatures $2\tanh 2t$ of multiplicity 1, $\coth t$ of multiplicity $n-1$ and $\tanh t$ of multiplicity $n-1$, unless $\coth t = \sqrt{3}$. When $\coth t = \sqrt{3}$, they have two distinct principal curvatures with multiplicities n and $n-1$.

An orbit in $H_n(\mathbb{C})$ is said to be *extrinsically homogeneous* if it is an orbit under a closed subgroup L of the identity component G of the group of all isometries of $H_n(\mathbb{C})$. For such orbits, the orbit with the maximal dimension is called *principal.* Moreover, if such an principal orbit is a real hypersurface in $H_n(\mathbb{C})$ with r distinct constant principal curvatures, then this orbit is said to be of *r-type.*

As proposed also in R. Niebergall and P. J. Ryan([5]), the following is an open problem : *Classify all extrinsically homogeneous real hypersurfaces in $H_n(\mathbb{C})$*. From theorem A it is well known that every extrinsically homogeneous real hypersurface M in $H_n(\mathbb{C})$ of 2-type is congruent to one of the model spaces of A_0 and A_1. Recently when an extrinsically real hypersurface M is of 3-type in a $H_n(\mathbb{C})$ and its structure vector field is not principal, some differential geometers have studied this problem. For this problem J. Saito ([6]) proved that every extrinsically homogeneous real hypersurface of

3-type in $H_n(\mathbb{C})$ has a principal structure vector field. However there is a mistake in deduction to lead a certain formula. In fact J. Berndt ([1]) and I.-B. Kim, H.S. Kim and R.Takagi([4]) showed that there are real hypersurfaces with three distinct constant principal curvatures on which structure vector fields are not principal in $H_n(\mathbb{C})$. In addition, for these real hypersurfaces I.-B. Kim, H.S. Kim and R.Takagi([4]) classified all homogeneous real hypersurfaces in the case where: the multiplicities of the principal curvatures with respect to nonzero components of structure vector field are 1 and 1.

Recently, in [1], J. Berndt constructed subgroups B_n and L_n of the connected component of the group of isometries of $H_n(\mathbb{C})$ for each n such that:
(1) a certain orbit $B_n(o)$ under B_n has three distinct principal curvatures 1,-1 and 0 with multiplicities $1, 1, 2n-3$ respectively and the structure vector field on $B_n(o)$ is not principal. These the number of distinct principal curvatures does not depend on all points. Such orbit $B_n(0)$ is said to be a *Be type of first kind.*
(2) a certain orbit $L_n(o)$ under B_n has three distinct principal curvatures or four distinct principal curvatures, i.e., the number of distinct principal curvatures depend on a point. Such orbit $L_n(o)$ is said to be a *Be type of second kind.*

Through this paper we assume that:

(C) Every real hypersurface M has three distinct constant principal curvatures and its structure vector field is not principal.

In this paper, we shall deal with an extrinsically homogeneous real hypersurface M such that the structure vector field on M is not principal and M has four distinct principal curvatures in $H_n(\mathbb{C})$. First, we investigate some properties of multiplicities of the principal curvatures with respect to nonzero components of structure vector field. Next, we show that there is a real hypersurface M in $H_n(\mathbb{C})$ for the case where: the multiplicities of the principal curvatures with respect to nonzero components of structure vector field are 1 and $k(\geq 2)$. Last, we construct extrinsically homogeneous real hypersurfaces with four distinct curvatures and their structure vector fields are not principal.

2. Preliminaries

Let $H_n(\mathbb{C})$ be a complex hyperbolic space of complex dimension $n(\geqq 3)$ with the metric of constant holomorphic sectional curvature -4, and M be a real hypersurface in $H_n(\mathbb{C})$ with the induced metric. Choose a local field $\{e_1, \cdots, e_{2n}\}$ of orthonormal frame in a way that, restricted to M, the vectors $e_1, \cdots, e_{2n-1}$ are tangent to M. Hereafter let the indices i, j, k, l run through from 1 to $2n-1$ unless otherwise stated. We denote by θ^i, θ^i_j and Θ^i_j the canonical 1-forms, the connection forms and curvature form of M respectively. Then they satisfy

$$
\begin{aligned}
&d\theta^i + \sum_j \theta^i_j \wedge \theta^j = 0, \qquad \theta^i_j + \theta^j_i = 0, \\
&d\theta^i_j + \sum_k \theta^i_k \wedge \theta^k_j = \Theta^i_j
\end{aligned} \tag{2.1}
$$

We denote by $\widetilde{J}$ the natural complex structure of $H_n(\mathbb{C})$ and $(J^i_j,\ \xi_i)$ be the almost contact structure of M, i.e., $\widetilde{J}(e_i) = \sum_j J^j_i e_j + \xi_i e_{2n}$. Then $(J^i_j,\ \xi_i)$ satisfies

$$
\sum_k J^i_k J^k_j = -\delta^i_j + \xi_i \xi_j, \quad \sum_j J^j_i \xi_j = 0, \quad \sum_i \xi_i \xi_i = 1, \tag{2.2}
$$

where $\xi = \sum \xi_i e_i$ is said to be the *structure vector field* of M and ξ_i is called the *components* of ξ. Let ϕ_i be 1-forms of M such that $\sum_i \phi_i \theta^i$ is the second fundamental form of M for e_{2n}. Then the parallelism of $\widetilde{J}$ of $H_n(\mathbb{C})$ implies

$$
d J^i_j = \sum_k (J^i_k \theta^k_j - J^j_k \theta^k_i) - \xi_i \phi_j + \xi_j \phi_i, \tag{2.3}
$$

$$
d\xi_i = \sum_k (\xi_k \theta^k_i - J^k_i \phi_k). \tag{2.4}
$$

The equation of Gauss is given by

$$
\Theta^i_j = \phi_i \wedge \phi_j - \theta^i \wedge \theta^j - \sum_{k,l} (J^i_k J^j_l + J^i_j J^k_l) \theta^k \wedge \theta^l. \tag{2.5}
$$

The equation of Codazzi is given by

$$d\phi_i = -\sum_j \phi_j \wedge \theta_i^j - \sum_{j,k} (\xi_j J_k^i + \xi_i J_k^j)\theta^j \wedge \theta^k. \tag{2.6}$$

We assume that all principal curvatures $\lambda_1, \cdots, \lambda_{2n-1}$(not necessarily distinct) of M for e_{2n} are constant. We may set $\phi_i = \lambda_i \theta^i$. For an index i, we denote by $[i]$ the set of indices j with $\lambda_i = \lambda_j$. Then it is obvious that the vector $V_i = \sum_{j \in [i]} \xi_j e_j$ is independent of the choice of orthonormal frame $\{e_j \,|\, j \in [i]\}$ for the eigenspace belonging to ξ_i. Therefore for any index i we can indicate a special index i' so that the vector V_i linearly depends on $e_{i'}$. In other words, we can choose an orthonormal frame for the eigenspace belonging to λ_i so that $\xi_j = 0$ for $j \in [i] \backslash \{i'\}$. In the same way, for J_k^j(j is any index and fixed and k is the index that $\lambda_k \neq \lambda_{i'}$), we can indicate a special index k' and choose an orthonormal frame for the eigenspace belonging $\lambda_{k'}$ so that $J_l^j = 0$ for $l \in [k] \backslash \{k'\}$.

Then by (2.1) and (2.6) we can write the connection forms θ_j^i in the form

$$(\lambda_i - \lambda_j)\theta_j^i = -\sum_k (A_{ijk} + \xi_i J_k^j + \xi_j J_k^i)\theta^k, \tag{2.7}$$

where $A_{ijk} = A_{jik} = A_{ikj}$. From (2.7), it is easily seen that

$$A_{ijk} = -\xi_i J_k^j - \xi_j J_k^i \qquad \text{if } \lambda_i = \lambda_j, \tag{2.8}$$

$$\xi_i J_k^j = 0 \qquad \text{if } \lambda_i = \lambda_j = \lambda_k. \tag{2.9}$$

We quote an important formula,

$$\begin{aligned} &2 \sum_k^{\lambda_k \neq \lambda_i} \frac{(A_{ijk} + \xi_k J_j^i + \xi_i J_j^k)^2}{\lambda_k - \lambda_i} \\ &-2 \sum_k^{\lambda_k \neq \lambda_j} \frac{(A_{ijk} + \xi_k J_i^j + \xi_j J_i^k)^2}{\lambda_k - \lambda_j} \\ &+6(\lambda_i - \lambda_j){J_j^i}^2 - 3(\xi_j^2 \lambda_i - \xi_i^2 \lambda_j) - (\lambda_i - \lambda_j)(c + \lambda_i \lambda_j) \\ &= 0. \end{aligned} \tag{2.10}$$

Hereafter we assume that M has three distinct constant principal curvatures x, y, and z. Let $m(x), m(y)$ and $m(z)$ be the multiplicities of x, y and z respectively. We shall make use of the following convention on the range of indices:

$$1 \le a, b, c \le m(x), \quad m(x) + 1 \le r, s, t \le m(x) + m(y)$$
$$m(x) + m(y) + 1 \le u, v, w \le 2n - 1.$$

From now on $m(x), m(y)$, and $m(z)$ are called the *multiplicities with respect to components* ξ_a , ξ_r and ξ_u, respectively.

3. Properties of the structure vector field on M

From now on we assume that the structure vector field ξ is not principal and we investigate its properties.

Now we note the following fact.

LEMMA ([6]). *If M has three distinct constant principal curvatures and its structure vector field is not principal, then there exists an orthonormal frame $\{e_1, e_2, \cdots, e_{2n-1}\}$ on M such that $\xi_u = 0$ and $\xi_a \xi_r \neq 0$.*

There are three cases for multiplicities as follows .
(1) $m(x) = m(y) = 1$,
(2) $m(x), m(y) \ge 2$.
(3) $m(x) = 1, m(y) \ge 2$.

Case (1): In this case, we have $m(z) \ge 3$. Moreover $J_{12} = 0$. Hence $J_{uv} \neq 0$ since rank$J = 2n - 2$. Now we can choose an orthonormal frame $\{e_u\}$ so that

$$J_{13} = -\xi_2, J_{23} = \xi_1 \quad \text{and} \quad J_{1u} = J_{2u} = J_{3u} = 0 (u \neq 3).$$

Let us take the exterior derivative of $\xi_u = 0$. Then we have

$$\frac{\xi_1^2}{x - z} + \frac{\xi_2^2}{y - z} = z \tag{03.1}$$

$$\frac{3\xi_1^2}{x - z} + \frac{A_{123} + \xi_2^2}{y - z} = x \tag{03.2}$$

$$\frac{A_{123} - \xi_1^2}{x - z} - \frac{3\xi_2^2}{y - z} = -y. \tag{03.3}$$

$$A_{12u} = 0 \ \text{if} a \neq 3. \tag{03.4}$$

It follows from (03.1) and the relation $\xi_1^2 + \xi_2^2 = 1$ that ξ_1^2 is constant. Taking account of the coefficient of θ_3 in $d\xi_1 = 0$, from (03.2) and (03.3) we have

$$3z^2 - 2z(x + y) + xy + 1 = 0. \tag{03.5}$$

We put $i = 1$ and $j = u$ in (2.10), then from (03.1) and (03.4) we have

$$\frac{2\xi_2^2}{y - z} - 3z\xi_2^2 + (y - z)(yz - 1) = 0. \tag{03.6}$$

Similarly, putting $i = 2$ and $j = u$ in (2.10), then we get

$$\frac{2\xi_1^2}{y - z} - 3z\xi_1^2 + (x - z)(xz - 1) = 0. \tag{03.7}$$

Cancelling ξ_1^2 and ξ_2^2 from (03.6) and (03.7) we have

$$z - (x + y) + yz(y - z) + xz(x - z) = 0.$$

From the above equation and (03.5) we have

$$3z = x + y \quad \text{and} \quad xy = 3z^2 - 1. \tag{03.8}$$

Using (03,8) I.-B. Kim, H.S. Kim and R.Takagi([4]) classified all extrinsically homogeneous real hypersurfaces, more precisely,

THEOREM C ([4]). *If M is an extrinsically homogeneous real hypersurface in $H_n(\mathbb{C})$ with case* (1), *then M is congruent to Be type of first kind.*

Case (2): In this case, we shall prove that this case does not occur. Now we have from (2.9) $J_{ab} = J_{rs} = 0$. Here we indicate a special index 1(resp. r') and choose an orthonormal frame $\{e_a\}$ (resp, $\{e_r\}$) so that $\xi_a = 0$ if $a \neq 1$ (resp. $\xi_r = 0$ if $r \neq r'$), Then we from (2.2)

$$J_r^1 = 0 = J_{r'}^a \tag{3.1}$$

For indices u, we choose an orthonormal frame $\{e_u\}$ so that $J^1_{u'} = \xi_{r'}$, $J^1_u = 0$ if $u \neq u^{'}$. Then, from (2.2) and (3.1)

$$J^{u'}_{r'} = \xi_1, \quad J^a_{u'} = J^r_{u'} = J^{r'}_u = 0 \quad \text{if } u \neq u^{'} \text{ and } a \neq 1 \tag{3.2}$$

Taking the exterior derivative of $J^1_r = 0$ and $\xi_a = 0$ $(a \neq 1)$, we have

$$\sum_k \frac{\xi_{r'} A_{u'rk}\theta^k}{y - z} - \sum_a J^r_a \theta^a_1 - \sum_{v \neq u'} \sum_k \frac{J^r_v(A_{1vk} + \xi_1 J^v_k)\theta^k}{x - z} - y\xi_1\theta^r = 0,$$

$$\sum_k \frac{\xi_{r'}(A_{ar'k} + \xi_{r'} J^a_k)\theta^k}{x - y} + \xi_1\theta^1_a + y\sum_r J^a_r\theta^r + z\sum_v J^a_v\theta^v = 0.$$

Taking account of the coefficient of θ_b in the equations and using (2.2), (3.1) and (3.2) we have

$$A_{aru'} = 0. \quad (a \neq 1 \, \text{or} \, r \neq r') \tag{3.3}$$

Now we put $i = u'$, $j = a$ $(a \neq 1)$ in (2.10). Then using (2.8), (3.1), (3.2) and (3.3) we have

$$yz - 1 = 0. \tag{3.4}$$

Moreover put $i = u'$, $j = r$ $(r \neq r')$. Then using (2.8), (3.1), (3.2) and (3.3) we have

$$xz - 1 = 0. \tag{3.5}$$

From (3.4) and (3.5) we have $z = 0$ which contradicts $-1 \neq 0$. Thus we have the following Theorem 1.

THEOREM 1. *There is no real hypersurface in $H_n(\mathbb{C})$ with three distinct principal curvatures such that multiplicities of the principal curvatures with respect to nonzero components of structure vector field are greater than 2.*

Case (3): In this case we can choose $\{e_r\}$ so that $\xi_r = 0$ if $r \neq 2$. Then we have

$$J^1_r = 0. \tag{3.6}$$

From (2.3) and (3.6) it follows that

$$m(y) + \sum J_v^{u\,2} = m(z).$$

This implies that $m(z) \geq m(y)$. For simplicity we put $m(y) = k$. Now we choose an orthonormal frame $\{e_u\}$ so that

$$J_{u'}^2 = -\xi_1, J_v^2 = 0 \quad \text{if}\, v \neq u',$$

where $u' = k + 2$. Then from (2.2) and (3.6)

$$J_{u'}^1 = -\xi_2,\ J_v^{u'} = 0,\ J_v^1 = 0,\ J_{u'}^r = 0.\,(r \neq 2, v \neq u') \tag{3.7}$$

Taking the exterior derivative of $\xi_{u'} = 0$, we have

$$3(\frac{y-z}{x-z})\xi_1^2 + 3(\frac{x-z}{y-z})\xi_2^2 = x(y-z) + y(x-z) - 1. \tag{3.8}$$

$$A_{1ru'} = 0 \quad \text{if}\, r \neq 2. \tag{3.9}$$

Hence it follows from (3.8) and the equation $\xi_1^2 + \xi_2^2 = 1$ that

$$-3\frac{(x-y)(x+y-2z)}{(x-z)(y-z)}\xi_1^2 = x(y-z) + y(x-z) - 1 - 3(\frac{x-z}{y-z}). \tag{3.10}$$

If $x + y - 2z \neq 0$, then ξ_1 is constant. Taking account of the coefficient of θ^{k+2} in $d\xi_1 = 0$, we have

$$3(\frac{x-y}{x-z})\xi_1^2 = z(x-y) - x(y-z) + 2$$
$$3(\frac{x-y}{y-z})\xi_2^2 = z(x-y) + y(x-z) - 2.$$

From the equations above, we have

$$z(x+y-2z) - (x-z)(y-z) - 1 = 0. \tag{3.11}$$

If $x + y - 2z = 0$ in the equation (3.10), then using $x - z = z - y$ and $y = 2z - x$ we have

$$(y-z)(x-z) + 1 = 0.$$

This implies that the equation (3.11) holds if $x+y-2z=0$. Put $i=r(r\neq 2), j=k+2$ in (2.10), the using (2.8), (3.7) and (3.9) we have

$$yz-1=0. \tag{3.12}$$

Taking the account of the coefficient of θ^r in $dJ_r^2=0$ and using (2.2), (2.8) and (3.7), we have $y^2-yz=2$. From this equation, (3.11) and (3.12) we get

$$y^2=3, y=3z, \quad \text{and} \quad x=0. \tag{3.13}$$

From this equation we have $y=\pm\sqrt{3}$ and $z=\pm\frac{1}{\sqrt{3}}$. There is no loss of generality such that we may assume that

$$y=\sqrt{3} \quad \text{and} \quad z=\frac{1}{\sqrt{3}}.$$

Let E_{ij} denote a square matrix with entry 1 where the ith row and the jth column meet. From the above results and (2.2) we can write

$$\begin{aligned}
\xi =&\frac{1}{3}e_1+\frac{2\sqrt{2}}{3}e_2,\\
J =&-\frac{2\sqrt{2}}{3}(E_{1,k+2}-E_{k+2,1})+\frac{1}{3}(E_{2,k+2}-E_{k+2,2})\\
&-\sum_{\alpha=1}^{k-1}(E_{2+\alpha,k+2+\alpha}-E_{k+2+\alpha,2+\alpha})\\
&-\sum_{p=1}^{n-k-1}(E_{2k+1+2p-1,2k+1+2p}-E_{2k+1+2p,2k+1+2p-1}).
\end{aligned} \tag{3.14}$$

From (2.1), (2.7) and (3.14) we have

$$\begin{aligned}
&\theta_2^1=-\frac{1}{\sqrt{3}}\theta^{u'}, \quad \theta_{2+\alpha}^1=-\frac{1}{\sqrt{3}}\theta^{\alpha+u'}, \quad \theta_{u'}^1=\frac{1}{\sqrt{3}}(2\sqrt{2}\theta^1-\theta^2),\\
&\theta_{\alpha+u'}^1=-\frac{1}{\sqrt{3}}\theta^{2+\alpha}, \quad \theta_{2k+2p}^1=-z\theta^{2k+2p+1},\\
&\qquad\qquad \theta_{2k+2p+1}^1=z\theta^{2k+2p}\\
&\theta_{2+\alpha}^2=\sqrt{\frac{2}{3}}\theta^{\alpha+u'}, \quad \theta_{u'}^2=\frac{1}{\sqrt{3}}(-\theta^1+\sqrt{2}\theta^2),
\end{aligned} \tag{3.15}$$

$$\theta^2_{\alpha+u'} = -2\sqrt{\frac{2}{3}}\theta^{2+\alpha}, \quad \theta^2_{2k+2p} = \sqrt{\frac{2}{3}}\theta^{2k+2p+1},$$
$$\theta^2_{2k+2p+1} = -\sqrt{\frac{2}{3}}\theta^{2k+2p}$$
$$\theta^{2+\alpha}_{2+\beta} = \theta^{u'+\alpha}_{u'+\beta}, \quad \theta^{2+\alpha}_{u'+\beta} = \delta_{\alpha\beta}\frac{1}{\sqrt{3}}(\theta^1 - \sqrt{2}\theta^2)$$
$$\theta^{2+\alpha}_{2k+2p} = \theta^{2+\alpha}_{2k+2p+1} = \theta^{2+\alpha}_{u'} = 0, \quad \theta^{u'}_{u'+\alpha} = -\sqrt{\frac{2}{3}}\theta^{u'+\alpha}$$
$$\theta^{u'}_{2k+2p} = -\sqrt{\frac{2}{3}}\theta^{2k+2p}, \quad \theta^{u'}_{2k+2p+1} = -\sqrt{\frac{2}{3}}\theta^{2k+2p+1},$$
$$\theta^{u'+\alpha}_{2k+2p} = \theta^{u'+\alpha}_{2k+2p+1} = 0,$$

where $1 \leq \alpha, \beta \leq k-1$ and $1 \leq p \leq n-k-1$.

For later use we choose a new orthonormal frame field $\hat{e} = \{\hat{e}_1, \dots, \hat{e}_{2n-1}\}$ in such a way that

$$\begin{aligned} &\hat{e}_1 = \xi = \frac{1}{3}e_1 + \frac{2\sqrt{2}}{3}e_2, \quad \hat{e}_2 = -\frac{2\sqrt{2}}{3}e_1 + \frac{1}{3}e_2, \\ &\hat{e}_{2+\alpha} = -e_{2+\alpha}, \quad \hat{e}_{k+1+s} = e_{2n-s}, \quad (1 \leq s \leq m(z)-k), \\ &\hat{e}_{k+1+m(z)-k+\alpha} = e_{2k+2-\alpha} \quad (1 \leq \alpha \leq k-1) \quad \hat{e}_{2n-1} = e_{u'}. \end{aligned} \tag{3.16}$$

Then the transition matrix F from the frame $\hat{e}$ to the frame e is

$$F = \frac{1}{3}(E_{11} + E_{22}) + \frac{2\sqrt{2}}{3}(E_{21} - E_{12}) - \sum_{r\neq 2} E_{rr} + \sum_{i=1}^{m(z)} E_{k+1+i,2n-i}. \tag{3.17}$$

Let $\hat{J}$ and $\hat{A}$ denote the almost contact structure and the shape operator of M with respect to the frame $\hat{e}$, respectively. Then from (3.14) and (3.17) we have

$$\begin{aligned} &\hat{J}\hat{e}_i = \hat{e}_{2n+1-i}, \quad (1 \leq i \leq n) \\ &\hat{A} = \frac{8}{3\sqrt{3}}E_{11} + \frac{1}{3\sqrt{3}}E_{22} + \frac{2\sqrt{2}}{3\sqrt{3}}(E_{12} + E_{21}) \\ &\qquad + \sqrt{3}\sum_{r\neq 2} E_{rr} + \frac{1}{\sqrt{3}}\sum_u E_{uu}. \end{aligned} \tag{3.18}$$

For the corresponding dual 1-form $\hat{\theta}^i$ and connection form $\hat{\theta}^i_j$ to the fame $\hat{e}$, it follows from (3.15) and (3.16)

$$
\begin{aligned}
&\hat\theta^1_2 = -\frac{1}{\sqrt3}\hat\theta^{2n-1}, \quad \hat\theta^1_{2+\alpha} = -\frac{1}{\sqrt3}\hat\theta^{2n-1-\alpha},\\
&\hat\theta^1_{k+1+2p} = \frac{1}{\sqrt3}\hat\theta^{k+2p}, \quad \hat\theta^1_{k+2p} = -\frac{1}{\sqrt3}\hat\theta^{k+2p+1},\\
&\hat\theta^2_{2+\alpha} = -\sqrt{\frac{2}{3}}\theta^{2n-1-\alpha}, \quad \hat\theta^2_{k+2p+1} = \sqrt{\frac{2}{3}}\hat\theta^{k+2p},\\
&\qquad\qquad\qquad\qquad \hat\theta^2_{k+2p} = -\sqrt{\frac{2}{3}}\hat\theta^{k+2p+1},\\
&\hat\theta^1_{2n-1-k-\alpha} = \sqrt3\hat\theta^{u'-\alpha}, \quad \hat\theta^2_{2n-1-k-\alpha} = 0\\
&\hat\theta^1_{2n-1} = \frac{1}{3\sqrt3}(2\sqrt2\hat\theta^1 + \hat\theta^2) \quad \hat\theta^2_{2n-1} = \frac{1}{3\sqrt3}\hat\theta^1 + \frac{7\sqrt2}{3\sqrt3}\hat\theta^2\\
&\hat\theta^{2+\alpha}_{2+\beta} = \theta^{2+\alpha}_{2+\beta}, \quad \hat\theta^{2+\alpha}_{k+1+2p} = \hat\theta^{2+\alpha}_{k+2p} = \hat\theta^{2+\alpha}_{2n-1} = 0,\\
&\hat\theta^{2+\alpha}_{2n-1-k-\alpha} = \delta_{\alpha,k-\beta}\frac{1}{\sqrt3}(\hat\theta^1 - \sqrt2\hat\theta^2)\\
&\hat\theta^{k+1+\alpha}_{k+1+\beta} = \hat\theta^{k+1+2p}_{2n-1-k-\beta} = \hat\theta^{k+2p}_{2n-1-k-\beta} = 0,\\
&\hat\theta^{k+1+2p}_{2n-1} = \sqrt{\frac{2}{3}}\hat\theta^{k+1+2p} \quad \hat\theta^{k++2p}_{2n-1} = \sqrt{\frac{2}{3}}\hat\theta^{k+2p}\\
&\hat\theta^{2n-1-k-\alpha}_{2n-1-k-\beta} = \theta^{2k+2-\alpha}_{2k+2-\beta} = \theta^{u'-\alpha}_{u'-\beta}, \quad \hat\theta^{2n-1-k-\alpha}_{2n-1} = -\sqrt{\frac{2}{3}}\theta^{2n-1-k-\alpha}
\end{aligned} \tag{3.19}
$$

and

$$
\begin{aligned}
&\hat\theta^1_2 = -\frac{1}{\sqrt3}\hat\theta^{2n-1}, \quad \hat\theta^1_{2+\alpha} = -\frac{1}{\sqrt3}\hat\theta^{2n-1-\alpha}, \quad \hat\theta^1_{n+\alpha} = \sqrt3\hat\theta^{n+1-\alpha},\\
&\hat\theta^2_{2+\alpha} = -\sqrt{\frac{2}{3}}\hat\theta^{2n-1-\alpha}, \quad \hat\theta^2_{n+\alpha} = 0, \quad \hat\theta^{2+\alpha}_{2+\beta} = \theta^{2+\alpha}_{2+\beta},\\
&\hat\theta^{2+\alpha}_{n+\beta} = \frac{1}{\sqrt3}\delta_{\alpha,k-\beta}(\hat\theta^1 + \sqrt2\hat\theta^2), \quad \hat\theta^{n+\alpha}_{n+\beta} = \hat\theta^{n+1-\alpha}_{n+1-\beta},\\
&\hat\theta^1_{2n-1} = \frac{1}{3\sqrt3}(2\sqrt2\hat\theta^1 + \hat\theta^2) \quad \hat\theta^2_{2n-1} = \frac{1}{3\sqrt3}\hat\theta^1 + \frac{7\sqrt2}{3\sqrt3}\hat\theta^2\\
&\hat\theta^{2+\alpha}_{2n-1} = 0, \quad \hat\theta^{n+\alpha}_{2n-1} = \sqrt{\frac{2}{3}}\hat\theta^{n+\alpha}, \quad \text{for } 1 \le \alpha,\beta \le k-1
\end{aligned} \tag{3.20}
$$

for $m(z) > m(y)$ and $m(z) = m(y)$, respectively.

4. Reconstructions of extrinsically homogeneous real hypersurfaces

In this section we shall construct extrinsically homogeneous real hypersurfaces with four distinct principal curvatures and their structure vector fields are not principal. In the following we construct this for $m(y) = m(z)$ and abbreviate the construction for $m(z) > m(y)$ since we have only to apply the same method.

Basically we shall adopt the notations in S. Helgason([3]).

Let $GL(n+1,\mathbb{C})$ be the general linear group of degree $n+1$ over $\mathbb{C}$. For $I = E_{11} - E_{22} - \cdots - E_{n+1,n+1}$, we put

$$G = \{\sigma \in GL(n+1,\mathbb{C}) \,|\, \sigma^t I \bar{\sigma} = I,\, \det\sigma = 1\}$$

and

$$K = \{\begin{pmatrix} \sigma & 0 \\ 0 & \tau \end{pmatrix} \,|\, \sigma \in U(1),\, \det\sigma \det\tau = 1\}.$$

Then K is a closed subgroup of G, and the homogeneous space G/K is just the hyperbolic complex space form of complex dimension n, which is denoted by H_n. The Riemannian metric and the complex structure on H_n will be stated later.

In the following, given a Lie group (e.g. G), we denote the associated Lie algebra of G by the corresponding German character $\mathfrak{g}$. Conversely, given a subalgebra (e.g. $\mathfrak{l}$) of $\mathfrak{g}$, we denote by the corresponding Roman character L the connected Lie subgroup of G whose Lie algebra is $\mathfrak{l}$.

We put

$$\begin{aligned} &A_\alpha = iE_{11} - iE_{\alpha+1,\alpha+1}, \tilde{Y}_{jk} = iE_{jk} + iE_{kj}, \\ &Y_{jk} = E_{jk} - E_{kj}, \quad X_\alpha = iE_{1,\alpha+1} - iE_{\alpha+1,1}, \\ &X_{n+\alpha} = E_{1,n+2-\alpha} + E_{n+2-\alpha,1}, \end{aligned} \tag{4.1}$$

where $1 \leq \alpha \leq n$ and $2 \leq j < k \leq n+1$. Then the set $\{A_\alpha, \tilde{Y}_{jk}, Y_{jk}, X_\alpha, X_{n+\alpha}\}_{1\leq\alpha\leq n,\, 2\leq j<k\leq n+1}$ (resp. the set $\{A_\alpha, \tilde{Y}_{jk}, Y_{jk}\}_{1\leq\alpha\leq n,\, 2\leq j<k\leq n+1}$) forms a basis for $\mathfrak{g}$ (resp. $\mathfrak{k}$). The bracket product of vectors given in (4.1) can be obtained by

$$[E_{jk}, E_{lm}] = \delta_{kl}E_{jm} - \delta_{mj}E_{lk} \tag{4.2}$$

for $1 \leq j, k, l, m \leq n+1$.

We put $\mathfrak{p} = \sum_{\alpha=1}^{n} \mathbb{R}X_\alpha + \sum_{\alpha=1}^{n} \mathbb{R}X_{n+\alpha}$. Then we have a Cartan decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}, \quad [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}.$$

We can identify $\mathfrak{p}$ with the tangent space $T_o(H_n)$ of H_n at the origin o. We give on H_n, regarded as a symmetric space, a Riemannian metric $<,>$ in such a way that

$$\begin{aligned} &< X_\alpha, X_\beta >= \delta_{\alpha\beta}, \quad < X_\alpha, X_{n+\beta} >= 0 \\ &< X_{n+\alpha}, X_{n+\beta} >= \delta_{\alpha\beta} \quad \text{for} \quad 1 \leq \alpha, \beta \leq n \end{aligned} \tag{4.3}$$

at o. Such a H_n is the hyperbolic complex space form of constant holomorphic sectional curvature -4 of complex dimension n, which is denoted by $H_n(\mathbb{C})$. Then G acts on $H_n(\mathbb{C})$ as a group of isometries. The complex structure $\hat{J}$ on $H_n(\mathbb{C})$ is given by at o

$$\hat{J}(X_\alpha) = X_{2n+1-\alpha}, \quad \text{for} \quad 1 \leq \alpha \leq n.$$

For any element Z of $\mathfrak{g}$, we denote the $\mathfrak{k}$ (resp. $\mathfrak{p}$)-component of Z by $Z_\mathfrak{k}$ (resp. $Z_\mathfrak{p}$).

Here we shall construct the Lie algebra $\mathfrak{l}_n = \sum_{i=1}^{2n-1} \mathbb{R}Z_i$ such that Z_i satisfy and (3.18), (3.19) and $(Z_i)_\mathfrak{p} = X_i = \hat{e}_i$.

Let ∇ be the Riemannian connection of $H_n(\mathbb{C})$ with respect to the Riemannian metric $<,>$ given in (4.3). Then it is also known that

$$\nabla_{(Z_i)_\mathfrak{p}} Z_j = [(Z_i)_\mathfrak{k}, (Z_j)_\mathfrak{p}]. \tag{4.4}$$

From now on we assume that $m(y) = m(z)$. Let $z = \frac{1}{\sqrt{3}}$, and let $C^i_{j,k}$ $(1 \leq i, j, k \leq 2n-1)$ be scalar functions on $H_n(\mathbb{C})$ such that $C^i_{j,k} = -C^j_{i,k}$. Using

(4.4), (3.18) and (3.20) we can put

$$
\begin{aligned}
Z_1 &= X_1 + \frac{5}{3}zA_1 - \frac{2}{3}zA_2 - \frac{2\sqrt{2}}{3}z\tilde{Y}_{23},\\
Z_2 &= X_2 - \frac{\sqrt{2}}{3}zA_1 + \frac{4\sqrt{2}}{3}zA_2 - \frac{z}{3}\tilde{Y}_{23},\\
Z_{2+\alpha} &= X_{2+\alpha} - 3z\tilde{Y}_{2,3+\alpha} + \sum_{1\le\mu<\gamma\le n-2} C^{2+\alpha}_{3+\mu,3+\gamma}Y_{3+\mu,3+\gamma},\\
Z_{n+\alpha} &= X_{n+\alpha} - zY_{2,n+2-\alpha} - \sqrt{2}zY_{3,n+2-\alpha},\\
Z_{2n-1} &= X_{2n-1} - zY_{23} + \sum_{1\le\mu<\gamma\le n-2} C^{2n-1}_{3+\mu,3+\gamma}Y_{3+\mu,3+\gamma},
\end{aligned}
\tag{4.5}
$$

where $\hat{\theta}^{2+\alpha}_{2+\beta}(X_{2+\gamma}) = C^{2+\gamma}_{3+\alpha,3+\beta}$, $\hat{\theta}^{2+\alpha}_{2+\beta}(X_{2n-1}) = C^{2n-1}_{3+\alpha,3+\beta}$ and $1 \le \alpha, \beta, \gamma \le n-2$.

Then it follows that

$$
\begin{aligned}
&[Z_1, Z_2] = 0, \quad [Z_1, Z_{2+\alpha}] = 2zZ_{2n-1-\alpha}, \quad [Z_1, Z_{n+\alpha}] = 0,\\
&[Z_1, Z_{2n-1}] = \frac{2\sqrt{2}}{3}zZ_1 - \frac{2}{3}zZ_2, \quad [Z_2, Z_{2+\alpha}] = -\sqrt{2}zZ_{2n-1-\alpha},\\
&[Z_2, Z_{n+\alpha}] = 0, \quad [Z_2, Z_{2n-1}] = \frac{4}{3}zZ_1 + \frac{7\sqrt{2}}{3}Z_2,\\
&[Z_{2+\alpha}, Z_{2+\beta}] = \sum_{1\le\lambda\le n-2}(C^{2+\beta}_{3+\alpha,3+\lambda} - C^{2+\alpha}_{3+\beta,3+\lambda})Z_{2+\lambda}\\
&\qquad + \sum_{1\le\mu<\gamma\le n-2}((dC^{2+\alpha}_{3+\mu,3+\gamma})(X_{2+\beta})\\
&\qquad\qquad - (dC^{2+\beta}_{3+\mu,3+\gamma})(X_{2+\alpha}))Y_{3+\mu,3+\gamma},\\
&[Z_{2+\alpha}, Z_{n+\beta}] = \sum_{\gamma=1}^{n-2} C^{2+\alpha}_{3+\gamma,n+2-\beta}Z_{2n-1-\gamma}\\
&\qquad + \begin{cases} 4zZ_1 + \sqrt{2}zZ_2 & \text{if } \alpha+\beta = n-1\\ 0 & \text{otherwise,}\end{cases}\\
&[Z_{2+\alpha}, Z_{2n-1}] = \sum_{1\le\lambda\le n-2} C^{2n-1}_{3+\alpha,3+\lambda_i}Z_{2+\lambda}\\
&\qquad + \sum_{1\le\mu<\gamma\le n-2}((dC^{2+\alpha}_{3+\mu,3+\gamma})(X_{2n-1}) - (dC^{2n-1}_{3+\mu,3+\gamma})(X_{2+\alpha})),\\
&[Z_{n+\alpha}, Z_{n+\beta}] = 0,\\
&[Z_{n+\alpha}, Z_{2n-1}] = \sqrt{2}z\, Z_{n+\alpha} + \sum_{1\le\lambda\le n-2} C^{2n-1}_{n+2-\alpha,3+\lambda}Z_{2n-1-\lambda}.
\end{aligned}
\tag{4.6}
$$

For any Greek letters μ, γ and λ, we see that the coefficients $C^{2+\lambda}_{3+\mu,3+\gamma}$ and $C^{2n-1}_{3+\mu,3+\gamma}$ of $Z_{2+\lambda}$ and Z_{2n-1} satisfy

$$
\begin{aligned}
&(dC^{2+\alpha}_{3+\alpha,3+\beta})(X_{2+\beta}) - (dC^{2+\beta}_{3+\alpha,3+\beta})(X_{2+\alpha}) + (9z^2-1)\\
&= \sum_{\lambda=1}^{n-2}(C^{2+\alpha}_{3+\alpha,3+\lambda}C^{2+\beta}_{3+\lambda,3+\beta} - C^{2+\beta}_{3+\alpha,3+\lambda}C^{2+\alpha}_{3+\lambda,3+\beta})\\
&+ \sum_{\lambda=1}^{n-2}(C^{2+\alpha}_{3+\beta,3+\lambda}C^{2+\lambda}_{3+\alpha,3+\beta} - C^{2+\beta}_{3+\alpha,3+\lambda}C^{2+\lambda}_{3+\alpha,3+\beta})\\
&\text{and}\\
&(dC^{2n-1}_{3+\alpha,3+\beta})(X_{2+\mu}) - (dC^{2+\beta}_{3+\alpha,3+\beta})(X_{2n-1})\\
&= \sum_{\lambda=1}^{n-2}(C^{2n-1}_{3+\alpha,3+\lambda}C^{2+\mu}_{3+\lambda,3+\beta} - C^{2+\mu}_{3+\alpha,3+\lambda}C^{2n-1}_{3+\lambda,3+\beta}\\
&\qquad\qquad\qquad - C^{2+\lambda}_{3+\alpha,3+\beta}C^{2n-1}_{3+\lambda,3+\mu})
\end{aligned}
\tag{4.7}
$$

respectively.

If we define a subspace $\mathfrak{l}_n$ of $\mathfrak{g}$ by

$$\mathfrak{l}_n = \sum_{i=1}^{2n-1} \mathbb{R}Z_i,$$

then we see from (4.6) and (4.7) that $\mathfrak{l}_n$ is a Lie subalgebra of $\mathfrak{g}$. For this Lie subalgebra $\mathfrak{l}_n$ we know the following:

PROPOSITION 2. *If we put $n = 3$ in (4.6) then it is easy to see that the above Lie algebra $\mathfrak{l}_3$ is solvable. In the case where $n \geq 4$ we see from (4.6) and (4.7) that the Lie algebra $\mathfrak{l}_n$(i.e. $n \geq 4$) is not solvable.*

Now, we shall investigate the principal curvatures of each orbit in $H_n(\mathbb{C})$ under the Lie subgroup L_n. We put

$$\sigma_t = \exp tX_{2n} \quad \text{for} \quad t \in \mathbb{R},$$

which is a 1-parameter subgroup of G.

Since the orbit $L_n(\sigma_t(o))$ is congruent to the orbit $(Ad(\sigma_t^{-1})L_n)(o)$ under $Ad(\sigma_t^{-1})L_n$ in $H_n(\mathbb{C})$, we shall compute the shape operator and the structure vector on the latter. For simplicity, we put

$$c_t = \cosh t \quad \text{and} \quad s_t = \sinh t.$$

Then we see that $\sigma_t = c_t E_{11} + c_t E_{22} + s_t E_{12} + s_t E_{21} + \sum_{k=3}^{n+1} E_{kk}$.

By a simple calculation, it follows from (4.5) that

$$
\begin{aligned}
Ad(\sigma_t)Z_1 &= (c_t^2 + s_t^2 - \frac{8}{3}c_t s_t z)X_1 - \frac{2\sqrt{2}}{3}s_t z X_2 \\
&\quad + (\frac{5}{3}c_t^2 z + s_t^2 z - 2c_t s_t)A_1 - \frac{2}{3}zA_2 - \frac{2\sqrt{2}}{3}c_t z\tilde{Y}_{23} \\
Ad(\sigma_t)Z_2 &= -\frac{2\sqrt{2}}{3}c_t s_t z X_1 + (c_t - \frac{s_t}{3}z)X_2 \\
&\quad + (s - \frac{c_t}{3}z)\tilde{Y}_{23} + (\sqrt{2}s_t^2 z - \frac{\sqrt{2}c_t^2}{3}z)A_1 + \frac{4\sqrt{2}}{3}zA_2 \\
Ad(\sigma_t)Z_{2+\alpha} &= (c_t - 3s_t z)X_{2+\alpha} + (s_t - 3c_t z)\tilde{Y}_{2,3+\alpha} \\
&\quad + \sum_{1\le\mu<\gamma\le n-2} C^{2+\alpha}_{3+\mu,3+\gamma}Y_{3+\mu,3+\gamma} \\
Ad(\sigma_t)Z_{n+\alpha} &= (c_t - s_t z)X_{n+\alpha} + (s_t - c_t z)Y_{2,n+2-\alpha} - \sqrt{2}zY_{3,n+2-\alpha} \\
Ad(\sigma_t)Z_{2n-1} &= (c_t - s_t z)X_{2n-1} + (s_t - c_t z)Y_{23} \\
&\quad + \sum_{1\le\mu<\gamma\le n-2} C^{2n-1}_{3+\mu,3+\gamma}Y_{3+\mu,3+\gamma},
\end{aligned}
\tag{4.8}
$$

for $1 \le \alpha, \mu, \gamma \le n-2$.

The vector $\nu = X_{2n}$ is the normal vector of the orbit $(Ad(\sigma_t^{-1})L_n)(o)$. Then the shape operator T_ν of $(Ad(\sigma_t^{-1})L_n)(o)$ in the direction ν is given by

$$T_\nu((Z_j)_{\mathfrak{p}}) = -[(Z_j)_{\mathfrak{k}},\, \nu]_{(Ad(\sigma_t^{-1})L_n)(o)}$$

for $1 \le j \le 2n-1$ (see also [9]).

Then it follows from (4.6) that

$$(c_t^2 + s_t^2 - \frac{8}{3}c_t s_t z)T_\nu(X_1) - \frac{2\sqrt{2}}{3}s_t z T_\nu(X_2)$$
$$= (4c_t s_t - \frac{8}{3}(c_t^2 + s_t^2)z)X_1 - \frac{2\sqrt{2}}{3}c_t z X_2,$$
$$-\frac{2\sqrt{2}}{3}c_t s_t z T_\nu(X_1) + (c_t - \frac{s_t}{3}z)T_\nu(X_2)$$
$$= -\frac{2\sqrt{2}}{3}z(c_t^2 + s_t^2)X_1 + (s_t - \frac{c_t}{3}z)X_2,$$
$$(c_t - 3s_t z)T_\nu(X_{2+\alpha}) = (s_t - 3c_t z)X_{2+\alpha}$$
$$(c_t - s_t z)T_\nu(X_{n+\alpha}) = (s_t - cz)X_{n+\alpha},$$
$$(c_t - s_t z)T_\nu(X_{2n-1}) = (s_t - cz)X_{2n-1}.$$

We see that $\tanh t - 3z \neq 0$ because $z^2 = \frac{1}{3}$. Therefore, from the above equations, the shape operator T_ν is given by

$$\begin{aligned}
T_\nu(X_1) &= \frac{12(\tanh t - z)(\tanh t - 2z)}{(\tanh t - 3z)^3}X_1 + \frac{2\sqrt{2}\mathrm{sech}^3 t}{(\tanh t - 3z)^3}X_2,\\
T_\nu(X_2) &= \frac{2\sqrt{2}\mathrm{sech}^3 t}{(\tanh t - 3z)^3}X_1 - \frac{9z(\tanh t - z)^3}{(\tanh t - 3z)^3}X_2,\\
T_\nu(X_{2+\alpha}) &= -\frac{z(\tanh t - 3z)}{\tanh t - z}X_{2+\alpha},\\
T_\nu(X_{n+\alpha}) &= -\frac{3z(\tanh t - z)}{\tanh t - 3z}X_{n+\alpha},\\
T_\nu(X_{2n-1}) &= -\frac{3z(\tanh t - z)}{\tanh t - 3z}X_{2n-1}, \quad (1 \leq \alpha \leq n-2)
\end{aligned} \tag{4.9}$$

for any $t \in \mathbb{R} \setminus \{t \mid \tanh t = z\}$.

It is easy to see from (4.9) that the orbit $L_n(\sigma_t(o))$ has four distinct principal curvatures

$$\frac{-9z(\tanh t - z) \pm \sqrt{9 - 6z\tanh t - 5\tanh^2 t}}{2(\tanh t - 3z)},$$

$$\frac{-z(\tanh t - 3z)}{\tanh t - z}, \frac{-3z(\tanh t - z)}{\tanh t - 3z}$$

with multiplicities 1, 1, $n-2$, $n-1$ respectively, provided that $t \neq 0, \tanh^{-1} z$. When $t = 0$, the orbit $L_n(\sigma_0(o))$ has three distinct principal curvatures

$$0, \ -3z, \ -z$$

with multiplicities 1, $n-1$, $n-1$ respectively.

In the case where $m(y) < m(z)$, by a similar computation as in the above, we see that the orbit $L_n(\sigma_t(o))$ has four distinct principal curvatures

$$\frac{-9z(\tanh t - z) \pm \sqrt{9 - 6z \tanh t - 5 \tanh^2 t}}{2(\tanh t - 3z)},$$
$$\frac{-z(\tanh t - 3z)}{\tanh t - z}, \ \frac{-3z(\tanh t - z)}{\tanh t - 3z}$$

with multiplicities 1, 1, k, $2n-k-3$ respectively, provided that $t \neq 0, \tanh^{-1} z$. When $t = 0$, the orbit $L_n(\sigma_0(o))$ has three distinct principal curvatures

$$0, \ -3z, \ -z$$

with multiplicities 1, k, $2n-k-2$ respectively.

From (4.9) we see that the structure vector field X_1 of $L_n(\sigma_t(o))$ is not principal.

Summing up the above results, we have

Theorem 3. *There is an extrinsically homogeneous real hypersurface M such that M has four distinct principal curvatures*

$$\frac{-9z(\tanh t - z) \pm \sqrt{9 - 6z \tanh t - 5 \tanh^2 t}}{2(\tanh t - 3z)},$$
$$\frac{-z(\tanh t - 3z)}{\tanh t - z}, \ \frac{-3z(\tanh t - z)}{\tanh t - 3z}$$

with multiplicities 1, 1, k, $(\geq 2)\, 2n-k-3$ *and the structure vector field on M is not principal.*

Moreover if M is an extrinsically homogeneous real hypersurface with case (3), then M must obtain the connection forms satisfying (3.15). Therefore if such M is existed, then M is congruent to one of new model spaces Be type type of second kind. From Theorem C, Theorem 1 and the above results, we have

Theorem 4. *Let M be an extrinsically homogeneous real hypersurface of 3-type whose structure vector field is not principal. Then M is holomorphic congruent to an open part of one of the new model spaces Be type of first kind and Be type of second kind.*

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