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SLLN FOR INDEPENDENT FUZZY RANDOM VARIABLES

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Abstract. We obtain an improvement of strong laws of large numbers for independent fuzzy random variables.

1. Introduction

Limit theorems for sums of fuzzy random variables have received much attentions because of its usefulness and in several applied fields. Hence many scholars have studied strong laws of large numbers(SLLN) for sums of independent fuzzy random variables. SLLN in the sense of L^1-metric d_1 have been studied by Klement et al. [13], Inoue [7], and SLLN with respect to uniform metric d_∞ have been studied by Colubi et al. [1], Molchanov [14], Joo et al. [9], Joo and Kim [11], and so on. Moreover, Joo [8] obtained a SLLN with respect to the Skorokhod metric d_s which was introduced by Joo and Kim [10]. Beside that, there are many other things such as Colubi et al. [2], Feng [3], Proske and Puri [15], Uemura [16] which studied SLLN for Banach space-valued fuzzy random variables.

Recently, Guan and Li [4], Hyun et al. [6] generalized SLLN for sums of fuzzy random variables to the case of weighted sums under restrictive conditions.

In this paper, we establish much better SLLN for independent fuzzy random variables.

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2. Preliminaries

Let R denote the real line. A fuzzy number is a fuzzy set $\tilde{u} : R \to [0,1]$ with the following properties ;

(1) $\tilde{u}$ is normal, i.e., there exists $x \in R$ such that $\tilde{u}(x) = 1$.
(2) $\tilde{u}$ is upper semicontinuous.
(3) supp $\tilde{u} = cl\{x \in R : \tilde{u}(x) > 0\}$ is compact.
(4) $\tilde{u}$ is convex, i.e. $\tilde{u}(\lambda x + (1-\lambda)y) \geq \min(\tilde{u}(x), \tilde{u}(y))$ for $x, y \in R$ and $\lambda \in [0,1]$.

Let $F(R)$ be the family of all fuzzy numbers. For a fuzzy set $\tilde{u}$, if we define

$$L_\alpha \tilde{u} = \begin{cases} \{x : \tilde{u}(x) \geq \alpha\}, 0 < \alpha \leq 1, \\ supp\ \tilde{u}, \alpha = 0, \end{cases}$$

then, it follows that $\tilde{u}$ is a fuzzy number if and only if $L_1\tilde{u} \neq \phi$ and $L_\alpha \tilde{u}$ is a closed bounded interval for each $\alpha \in [0,1]$. From this characterization of fuzzy numbers, a fuzzy number $\tilde{u}$ is completely determined by the closed intervals $L_\alpha \tilde{u} = [u^l_\alpha, u^r_\alpha]$.

Theorem 2.1. *For $\tilde{u} \in F(R)$, the followings hold;*

(1) u^l *is a bounded increasing function on* $[0,1]$.
(2) u^r *is a bounded decreasing function on* $[0,1]$.
(3) $u^l_1 \leq u^r_1$.
(4) u^l *and* u^r *are left continuous on* $[0,1]$ *and right continuous at 0.*

Furthermore, if v^l and v^r satisfy above $(1)-(4)$, then there exists a unique $\tilde{v} \in F(R)$ such that $L_\alpha \tilde{v} = [v^l_\alpha, v^r_\alpha]$.

Proof :. see Goetschel and Voxman [5]. □

The above theorem implies that we can identify a fuzzy number $\tilde{u}$ with the family of closed intervals $\{[u^l_\alpha, u^r_\alpha] : 0 \leq \alpha \leq 1\}$, where u^l and u^r satisfy (1)-(4) of Theorem 2.1.

We can define uniform metric d_∞ on $F(R)$ as follows;

$$\begin{aligned} d_\infty(\tilde{u}, \tilde{v}) &= \sup_{0 \leq \alpha \leq 1} \max(|u^l_\alpha - v^l_\alpha|, |u^r_\alpha - v^r_\alpha|) \\ &= \max(\sup_{0 \leq \alpha \leq 1} |u^l_\alpha - v^l_\alpha|, \sup_{0 \leq \alpha \leq 1} |u^r_\alpha - v^r_\alpha|). \end{aligned}$$

The norm of $\tilde{u} \in F(R)$ is defined by

$$\|\tilde{u}\| = d_\infty(\tilde{u}, \tilde{0}) = \max(|u_0^l|, |u_0^r|).$$

Unfortunately, $(F(R), d_\infty)$ is complete but is not separable (For details, see Klement et al. [13]). Joo and Kim [10] introduced the Skorokod metric d_s on $F(R)$ which makes it a separable and topologically complete metric space(For details, see Joo and Kim [10]).

3. Strong laws of large numbers

In this section, let $(\Omega, \mathcal{A}, P)$ be a probability space. A fuzzy number valued function

$$\tilde{X} : \Omega \to F(R), \tilde{X} = \{[X_\alpha^l, X_\alpha^r] : 0 \le \alpha \le 1\}$$

is called a fuzzy random variable if for each $\alpha \in [0,1]$, X_α^l and X_α^r are random variable in the usual sense. It is well-known that $\tilde{X}$ is a fuzzy random variable if and only if $\tilde{X} : \Omega \to (F(R), d_s)$ is measurable (See Kim [12]). So we assume that the space $F(R)$ is considered as the metric space endowed with the metric d_s, unless otherwise stated.

A fuzzy random variable $\tilde{X}$ is called integrable if $E\|\tilde{X}\| < \infty$. The expectation of integrable fuzzy random variable $\tilde{X}$ is a fuzzy number defined by

$$E(\tilde{X}) = \{[EX_\alpha^l, EX_\alpha^r] \ : \ 0 \le \alpha \le 1\}.$$

Now we need the concept of independence for fuzzy random variables. In this paper, we adopt the following notion of independence:

Definition 3.1. *A sequence of fuzzy random variables $\{\tilde{X}_n\}$ is called independent if a sequence of σ-fields $\{\sigma(\tilde{X}_n)\}$ is independent, where $\sigma(\tilde{X})$ is the smallest σ-field of subsets of Ω such that $\tilde{X} : \Omega \to (F(R^p), d_s)$ is measurable.*

Note that $\sigma(\tilde{X}) = \sigma(\{X_\alpha^l, X_\alpha^r : 0 \le \alpha \le 1\})$ because $\tilde{X} : \Omega \to (F(R), d_s)$ is measurable if and only if for each $\alpha \in [0,1], X_\alpha^l$ and X_α^r are measurable.

Let $\{\tilde{X}_n\}$ be a sequence of integrable fuzzy random variables and $\{\lambda_{ni}\}$ be a Toeplitz sequence, i.e., $\{\lambda_{ni}\}$ is a double array of real numbers satisfying

(1) For each i, $\lim_{n\to\infty} \lambda_{ni} = 0$;

(2) There exists $C > 0$ such that $\sum_{i=1}^{\infty} |\lambda_{ni}| \le C$ for each n.

Now, we write $\tilde{X}_n = \{[X^l_{n,\alpha}, X^r_{n,\alpha}] : 0 \le \alpha \le 1\}$ and assume the following condition:

(3.1): For each $\epsilon > 0$, there exists a partition $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_m = 1$ of $[0, 1]$ such that for all n,

$$\max(\max_{1\le k\le m} E|X^l_{n,\alpha^+_{k-1}} - X^l_{n,\alpha_k}|, \max_{1\le k\le m} E|X^r_{n,\alpha^+_{k-1}} - X^r_{n,\alpha_k}|) < \epsilon.$$

Theorem 3.2. *Let $\{\tilde{X}_n\}$ be a sequence of independent fuzzy random variables satisfying* (3.1)*. Suppose that there exists a nonnegative random variable ξ with $E\xi^{1+\frac{1}{\gamma}} < \infty$ for some $\gamma > 0$ such that for each n,*

$$P(\|\tilde{X}_n\| \ge \lambda) \le P(\xi \ge \lambda) \quad \textit{for all } \lambda > 0.$$

If $\{\lambda_{ni}\}$ is a Toeplitz sequence satisfying $\max_{1\le i\le n} |\lambda_{ni}| = O(n^{-\gamma})$, then

$$\lim_{n\to\infty} d_\infty(\oplus_{i=1}^n \lambda_{ni}\tilde{X}_i, \oplus_{i=1}^n \lambda_{ni}E\tilde{X}_i) = 0 \quad a.s.$$

Proof. See Hyun et al. [6]. □

Theorem 3.3. *Let $\{\tilde{X}_n\}$ be a sequence of independent fuzzy random variables satisfying* (3.1)*. If*

$$\sup_n E\|\tilde{X}_n\|^p = M < \infty \quad \textit{for some } p > 1, \tag{3.2}$$

then for any Toeplitz sequence $\{\lambda_{ni}\}$ satisfying $\max_{1\le i\le n} |\lambda_{ni}| = O(n^{-\gamma})$ for $\gamma > \frac{1}{p-1}$,

$$\lim_{n\to\infty} d_\infty(\oplus_{i=1}^n \lambda_{ni}\tilde{X}_i, \oplus_{i=1}^n \lambda_{ni}E\tilde{X}_i) = 0 \quad a.s.$$

Proof. By Lemma 3 of Wei and Taylor [17], (3.2) implies that there exists a nonnegative random variable ξ with $E\xi^{1+\frac{1}{\gamma}} < \infty$ for $0 < \frac{1}{\gamma} < p - 1$ such that for each n,

$$P(\|\tilde{X}_n\| \ge \lambda) \le P(\xi \ge \lambda) \quad \text{for all } \lambda > 0.$$

Thus the result follows immediately from Theorem 3.2. □

If we apply Theorem 3.2 to

$$\lambda_{ni} = \begin{cases} 1/n, & \text{if } 1 \le i \le n, \\ 0, & \text{if } i > n, \end{cases}$$

then we can obtain a SLLN for independent fuzzy random variables. But in this case, we need the restrictive condition $E|\xi|^2 < \infty$. Similarly, we need the restrictive condition $\sup_n E\|\tilde{X}_n\|^p < \infty$ for some $p > 2$ in order to provide a SLLN by applying Theorem 3.3. However, we can obtain much better SLLN by similar arguments as in the proof of Theorem 3.2.

Theorem 3.4. *Let $\{\tilde{X}_n\}$ be a sequence of independent fuzzy random variables satisfying* (3.1)*. If*

$$\sum_{n=1}^{\infty} \frac{1}{n^p} E\|\tilde{X}_n\|^p < \infty \quad \text{for some } 1 \le p \le 2,$$

then

$$\lim_{n\to\infty} \frac{1}{n} d_\infty(\oplus_{i=1}^n \tilde{X}_i, \oplus_{i=1}^n E\tilde{X}_i) = 0 \quad a.s.$$

Proof. Let $\epsilon > 0$ be given. By assumption (3.1), we choose $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_m = 1$ of $[0,1]$ such that for all n,

$$\max_{1\le k\le m} E|X^l_{n,\alpha^+_{k-1}} - X^l_{n,\alpha_k}| < \epsilon. \tag{3.3}$$

We note that

$$\begin{aligned} &\frac{1}{n} \sup_{\alpha_{k-1}<\alpha\le\alpha_k} |\sum_{i=1}^n (X^l_{i,\alpha} - EX^l_{i,\alpha})| \\ &\le \frac{1}{n} \sup_{\alpha_{k-1}<\alpha\le\alpha_k} |\sum_{i=1}^n (X^l_{i,\alpha} - X^l_{i,\alpha_k})| + \frac{1}{n}|\sum_{i=1}^n (X^l_{i,\alpha_k} - EX^l_{i,\alpha_k})| \\ &\qquad + \frac{1}{n} \sup_{\alpha_{k-1}<\alpha\le\alpha_k} \sum_{i=1}^n |EX^l_{i,\alpha_k} - EX^l_{i,\alpha}| \end{aligned}$$

$$\leq \frac{1}{n}\sum_{i=1}^{n}|X^l_{i,\alpha^+_{k-1}} - X^l_{i,\alpha_k}| + \frac{1}{n}|\sum_{i=1}^{n}(X^l_{i,\alpha_k} - EX^l_{i,\alpha_k})|$$
$$+ \frac{1}{n}\sum_{i=1}^{n}|EX^l_{i,\alpha_k} - EX^l_{i,\alpha^+_{k-1}}|$$
$$= \text{(I)} + \text{(II)} + \text{(III)}.$$

For (I), we first note that for all n,

$$E|X^l_{n,\alpha^+_{k-1}} - X^l_{n,\alpha_k}|^p \leq 2^p E\|\tilde{X}_n\|^p,$$

and so

$$\sum_{n=1}^{\infty}\frac{1}{n^p}E|X^l_{n,\alpha^+_{k-1}} - X^l_{n,\alpha_k}|^p < \infty.$$

By Chung's strong law of large numbers for real-valued random variables,

$$\sum_{i=1}^{n}\frac{1}{n}(|X^l_{i,\alpha^+_{k-1}} - X^l_{i,\alpha_k}| - E|X^l_{i,\alpha^+_{k-1}} - X^l_{i,\alpha_k}|) \to 0 \quad a.s.$$

Thus, by (3.3),

$$\text{(I)} = \frac{1}{n}\sum_{i=1}^{n}(|X^l_{i,\alpha^+_{k-1}} - X^l_{i,\alpha_k}| - E|X^l_{i,\alpha^+_{k-1}} - X^l_{i,\alpha_k}|)$$
$$+ \frac{1}{n}\sum_{i=1}^{n}E|X^l_{i,\alpha^+_{k-1}} - X^l_{i,\alpha_k}|$$
$$< \epsilon \quad a.s. \quad \text{for large} \quad n.$$

For (II), since $E|X^l_{n,\alpha_k}|^p \leq E\|\tilde{X}_n\|^p$, we have that (II) $\to 0$ $a.s.$ by Chung's strong law of large numbers.

Finally, it is trivial that (III) $\leq \epsilon$.

Hence we obtain

$$\begin{aligned}
&\frac{1}{n}\sup_{0\le\alpha\le 1}|\sum_{i=1}^{n}(X_{i,\alpha}^{l}-EX_{i,\alpha}^{l})| \\
=&\frac{1}{n}\max_{0\le k\le m}\sup_{\alpha_{k-1}<\alpha\le\alpha_k}|\sum_{i=1}^{n}(X_{i,\alpha}^{l}-EX_{i,\alpha}^{l})| \\
<& \ 2\epsilon \ \ a.s. \text{ for large } n.
\end{aligned}$$

Similarly, it can be proved that

$$\frac{1}{n}\sup_{0\le\alpha\le 1}|\sum_{i=1}^{n}(X_{i,\alpha}^{r}-EX_{i,\alpha}^{r})| < \ 2\epsilon \ \ a.s. \text{ for large } n.$$

Therefore,

$$\begin{aligned}
&\frac{1}{n}d_\infty(\oplus_{i=1}^{n}\tilde{X}_i,\oplus_{i=1}^{n}E\tilde{X}_i) \\
=&\frac{1}{n}\max(\sup_{0\le\alpha\le 1}|\sum_{i=1}^{n}(X_{i,\alpha}^{l}-EX_{i,\alpha}^{l})|,\sup_{0\le\alpha\le 1}|\sum_{i=1}^{n}(X_{i,\alpha}^{r}-EX_{i,\alpha}^{r})|) \\
<& \ 2\epsilon \ \ a.s. \text{ for large } n.
\end{aligned}$$

Since ϵ is arbitrary, this completes the proof. □

Corollary 3.5. *Let $\{\tilde{X}_n\}$ be a sequence of independent fuzzy random variables satisfying* (3.1). *If*

$$\sup_n E\|\tilde{X}_n\|^p<\infty \ \text{ for some } p>1,$$

then

$$\lim_{n\to\infty}\frac{1}{n}d_\infty(\oplus_{i=1}^{n}\tilde{X}_i,\oplus_{i=1}^{n}co(E\tilde{X}_i))=0 \ \ a.s.$$

Corollary 3.6. *Let $\{\tilde{X}_n\}$ be a sequence of independent fuzzy random variables. If $\{\tilde{X}_n\}$ is convex-compactly uniformly integrable or convexly tight and*

$$\sup_n E\|\tilde{X}_n\|^p=M<\infty \ \text{ for some } p>1,$$

then

$$\lim_{n\to\infty}\frac{1}{n}d_\infty(\oplus_{i=1}^{n}\tilde{X}_i,\oplus_{i=1}^{n}E\tilde{X}_i)=0 \ \ a.s.$$

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