Korean J. Math. **16** (2008), No. 4, pp. 565–571

ON SOME MATRIX INEQUALITIES

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Abstract. In this paper we present some trace inequalities for positive definite matrices in statistical mechanics. In order to prove the method of the uniform bound on the generating functional for the semi-classical model, we use some trace inequalities and matrix norms and properties of trace for positive definite matrices.

1. Introduction

Matrix inequalities play an important role in statistical mechanics([1,3,6,7]). We study quantum statistical mechanics for the semi-classical model in the lattice space. In order to investigate the uniform bound on the generating functional, the semi-classical model has been studied intensively by many authors([1,3,6,7,10]). The purpose of this paper is to establish the trace inequality for multiple product of powers of arbitrary finite positive definite matrices which is a tool to find the uniform bound on the generating functional for the semi-classical model in quantum statistical mechanics.

This paper is organized as follows. In section II, We introduce some definitions and theorems which are necessary to prove our main result and present some properties of trace for positive definite matrices on the finite dimensional Hilbert space([2,4,11]). In section III, we first describe some trace inequalities, and then we will prove our main result by applying these trace inequalities and properties of matrix norms,and induction argument.

2. Preliminaries

Let H_n denote the complex vector space of all $n \times\ n$ Hermitian

Received November 20, 2008. Revised December 7, 2008.
2000 Mathematics Subject Classification: 82B10.
Key words and phrases: positive definite matrices, matrix norms, trace.
This work was supported by the Cheongju University Scholar Overseas Funds.

matrices, endowed with the inner product $< A, B >= Tr(B^*A)$, where $Tr(\cdot)$ is the trace on the positive matrices and B^* is the adjoint of B, Then,this makes $(H_n, < \cdot, \cdot >)$ into a Hilbert space H([11,12]). We first define an $n \times\ n$ Hermitian matrix A is said to be positive definite, denoted by, $A > 0$, if $x^*Ax > 0$ for all nonzero x in $\mathbf{C}^n$. If $x^*Ax \geq 0$,then A is said to be positive semidefinite, denoted by, $A \geq 0$.

Definition 2.1. ([10,12]) Let A in H_n, Then, (1)The trace norm of A, defined by, $\|A\|_1 = \sum_i s_i(A)$. (2) The spectral norm of A,also denoted by, $\|A\|_2 = \max\{s_i(A)\}$, where $s_i(A)$ are the singular values of A, i.e., the eigenvalues of $|A| = (A^*A)^{\frac{1}{2}}$.

Theorem 2.2. ([6])*Let A be a positive definite matrix in H_n with finite trace norm and B be a positive matrix with finite trace norm, Then,*

$$(1) Tr(UAU^{-1}) = Tr(A)$$

for any unitary matrix U,

$$(2) Tr(AB) = Tr(BA).$$

We now restrict to the case the finite dimension of H, i.e., $\dim(H) < \infty$

Theorem 2.3. ([11])*[Cauchy-Schwartz inequality] Let A and B be positive definite matrices with finite trace norm,respectively, then,*

$$|Tr(A^*B)|^2 \leq Tr(A^*A)Tr(B^*B).$$

By the Cauchy-Schwartz inequality, we have

Theorem 2.4. ([11]) *Let A and B be positive definite matrices with finite trace norm, respectively, then,*

$$(1) Tr(AB) \leq |Tr(AB)| \leq \{Tr(A^*A)\}^{\frac{1}{2}}\{Tr(B^*B)\}^{\frac{1}{2}},$$

$$(2) Tr(AB) < Tr(A)Tr(B).$$

3. Trace Inequalities

In order to prove our main result, it is necessary to establish the following two lemmas.

LEMMA 3.1. ([4,5,9])*Let $A \geq 0$ and $B \geq 0$ in H_n, then,*

$$Tr(AB) \leq \ \|A\|_2 Tr(B),$$

where $\|A\|_2$ denotes the spectral norm or largest singular value of A.

LEMMA 3.2. ([9,11])*For any positive matrices C,D, and E in H_n, then,*

$$|Tr(CDE)| \leq \|D\|_2 \{Tr(C^*C)\}^{\frac{1}{2}} \{Tr(E^*E)\}^{\frac{1}{2}}.$$

Proof.

$$|Tr(CDE)| \leq \|D\|_2 Tr(CE)$$

By Theorem 2.2 and Theorem 2.4, combining the Cauchy-Schwartz inequality, we obtain the following :

$$|Tr(CDE)| \leq \|D\|_2 Tr(CE) \leq \|D\|_2 \{Tr(C^*C)\}^{\frac{1}{2}} \{Tr(E^*E)\}^{\frac{1}{2}}.$$

□

Now, we will prove our main result using above two lemmas.

THEOREM 3.3. *Let B_1, B_2, ... , B_n be positive definite matrices with finite matrix norm in H_n, and A be a positive definite matrix such that $A^{\frac{1}{n}}$ has a finite trace norm, then, for $p_i \geq 0$, $i = 1, 2, ..., n$, with $\sum_{i=1}^n p_i = 1$,*

$$|Tr(B_1 A^{p_1} B_2 A^{p_2} \cdots B_n A^{p_n})| \leq (\prod_{i=1}^{n} \|B_i\|_2) Tr(A).$$

Proof. We will prove the theorem by induction. For $n = 1$, it follows from Lemma 3.1. For $n = 2$, let $p_1 \leq \frac{1}{2}$,then,

$$|Tr(B_1 A^{p_1} B_2 A^{p_2})|$$

$$= |Tr(A^{\frac{p_1+p_2}{2}} B_1 A^{p_1} B_2 A^{\frac{p_1-p_2}{2}})|$$

$$\leq \|B_1\|_2 Tr(A)^{\frac{1}{2}} Tr(A^{\frac{p_1-p_2}{2}} B_2^* A^{2p_1} B_2 A^{\frac{p_1-p_2}{2}})^{\frac{1}{2}},$$

by Lemma 3.2.

Continuing in this process, after n-steps,we obtain

$$|Tr(B_1 A^{p_1} B_2 A^{p_2})|$$

$$\leq \|B_1\|_2 \|B_2\|_2{}^{\frac{1}{2}+\cdots+\frac{1}{2^{n-1}}} Tr(A)^{\frac{1}{2}+\cdots+\frac{1}{2^{n-1}}} |Tr(B_2^* A^{p_1{}'} B_2 A^{p_2{}'})|^{\frac{1}{2^n}}$$

for some $p_1{}' \geq 0$, $p_2{}' \geq 0$ with $p_1{}' + p_2{}' = 1$.

Since one of $p_i{}'$'s is less than $\frac{1}{2}$, we have

$$|Tr(B_2^* A^{p_1{}'} B_2 A^{p_2{}'})|^{\frac{1}{2^n}} \leq \|B_2\|_2{}^{\frac{1}{2^n}} \|A\|_2{}^{\frac{1}{2^n}} Tr(A^{\frac{1}{2}})^{\frac{1}{2^n}}.$$

By taking $n \longrightarrow \infty$, we proved the theorem for $n = 2$.

We now assume that the theorem holds for $n \leq m-1$. We will show that the theorem holds for $n = m$.

Since $p_1 + p_2 + ... + p_m = 1$, there exists j in $\mathbf{N}$ such that

$$p_j + p_{j+1 \pmod m} + ... + p_{j+[\frac{m}{2}] \pmod m} < \frac{1}{2}$$

$$p_j + p_{j+1 \pmod m} + ... + p_{j+1+[\frac{m}{2}] \pmod m} \geq \frac{1}{2},$$

where $[\frac{m}{2}]$ is the largest integer which is not greater than $\frac{m}{2}$.

Using the cyclic property of the trace : $Tr(AB) = Tr(BA)$ and rearranging B_j and p_j, we may assume that

$$p_1 + p_2 + ... + p_{[\frac{m}{2}]} < \frac{1}{2}$$

$$p_1 + p_2 + ... + p_{[\frac{m}{2}]+1} \geq \frac{1}{2} \cdots\cdots (1)$$

Let $p' = \frac{1}{2} - \sum_{i=1}^{[\frac{m}{2}]} p_i$ and let $m' = [\frac{m}{2}] + 1$, then,

$$2(\sum_{i=1}^{m'-1} p_i + p') = 1$$

$$2(\sum_{i=m'+1}^{m} p_i + (p_{m'} - p')) = 1 \cdots\cdots(2)$$

and

$$|Tr(B_1 A^{p_1} B_2 A^{p_2} \cdots B_m A^{p_m})|$$

$$\leq |Tr(A^{p'_m - p'} B_{m'+1} A^{p_{m'+1}} \cdots B_m A^{p_m} B_1 A^{p_1} \cdots B_{m'} A^{p'})|$$

$$\leq \|B_1\|_2 Tr(A^{p'} B^*_{m'} A^{p_{m'-1}} B_{m'-1} ... B^*_2 A^{2p_1} B_2 A^{p_2} ... B_{m'} A^{p_1'})^{\frac{1}{2}}$$

$$Tr(A^{p_m{}' - p'} B_{m'+1} ... B^*_m A^{2p_m} B_m A^{p_m - 1} ... B_{m'+1} A^{p_m{}' - p'})^{\frac{1}{2}} \cdots\cdots(3)$$

If m is odd, then $2(m' - 1) = m - 1$ and $2(m - (m' + 1)) = m - 1$. Thus, for odd integer m, the theorem follows from (2) and the induction hypothesis.

Next, let m be even, then $2(m' - 1) = m$ and $2(m - (m' + 1)) = m - 2$. So, by the induction hypothesis, (3) is bounded by

$$\|B_1\|_2 \prod_{j=m'+1}^{m} Tr(A)^{\frac{1}{2}} Tr(B^*_{m'} A^{p_{m'-1}} B^*_{m-1} ... B^*_2 A^{2p_1} B_2 ... B_{m'} A^{2p'})^{\frac{1}{2}} ......(4)$$

Notice that by (1) either

$$\sum_{j=2}^{m'-1} p_j < \frac{1}{2}$$

and

$$\sum_{j=2}^{m'-1} p_j + 2p' \geq \frac{1}{2} \cdots\cdots(5)$$

or else

$$\sum_{j=2}^{m'-1} p_j < \frac{1}{2}$$

and

$$\sum_{j=2}^{m'-1} p_j + 2p_1 \geq \frac{1}{2} \cdots\cdots (6)$$

In either case, we use the method to obtain (3) and (4).

$$Tr(B_{m'}^* A^{p_{m'-1}} B_{m'-1}^* \cdots B_2{}^* A^{2p_1} B_2 A^{p_2} ... B_{m'} A^{2p'})$$

$$\leq (\prod_{j=2}^{m'} \|B_j\|_2 Tr(A)^{\frac{1}{2}} Tr(A^{p_{m'}} B_{m'}^* \cdots B_2^* A^{2p_1'} B_2 A^{p_2'} \cdots B_{m'} A^{p_m'})$$

for some $p_1{}', ..., p_m{}'$ with $2(p_1{}' + ... + p_m{}') = 1$ and such that one of (5) and (6) holds for $p_1{}', ..., p_m{}'$

After n-steps of the above continuing process, we conclude that

$$|Tr(B_1 A^{p_1} ... B_m A^{p_m})|$$

$$\leq \|B_1\|_2 (\prod_{j=m'+1}^{m} \|B_j\|_2) Tr(A)^{\frac{1}{2}+\cdots+\frac{1}{2^{n-1}}} (\prod_{j=2}^{m'} \|B_j\|_2)^{\frac{1}{2}+\cdots+\frac{1}{2^{n-1}}}$$

$$|Tr(A^{q_{m'}} B_{m'}^* \cdots B_2^* A^{2q_1} B_2 \cdots B_{m'} A^{q_{m'}})|^{\frac{1}{2^n}} \cdots\cdots (7)$$

for some $q_1, ..., q_{m'}$ with $2(q_1 + ... + q_{m'}) = 1$

Since one of q_i's is less than $\frac{1}{m}$, the trace in (7) is bounded by $(\prod_{j=2}^{m'} \|B_j\|_2^{\frac{1}{2^{n-1}}}) \; \|A\|^{(1-\frac{1}{m})\frac{1}{2^n}} \; Tr(A^{\frac{1}{m}})^{\frac{1}{2^n}}$.

Hence, The theorem follows from (7). This completes the proof of the theorem. $\square$

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