Korean J. Math. **16** (2008), No. 4, pp. 545–551

SOLVABILITY FOR THE PARABOLIC PROBLEM WITH JUMPING NONLINEARITY CROSSING NO EIGENVALUES

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Abstract. We investigate the multiple solutions for a parabolic boundary value problem with jumping nonlinearity crossing no eigenvalues. We show the existence of the unique solution of the parabolic problem with Dirichlet boundary condition and periodic condition when jumping nonlinearity does not cross eigenvalues of the Laplace operator $-\Delta$. We prove this result by investigating the Lipschitz constant of the inverse compact operator of $D_t - \Delta$ and applying the contraction mapping principle.

1. Introduction

Ler Ω be a bounded region in R^n with smooth boundary $\partial\Omega$. Let $0 < \lambda_1 < \lambda_2 \le \cdots \le \lambda_k \to \infty$ be the eigenvalues of the eigenvalue problem $-\Delta u = \lambda u$ in Ω, $u|_{\partial\Omega} = 0$ and ϕ_k be the eigenfunction corresponding to the eigenvalue λ_k. We note that the first eigenfunction $\phi_1(x) > 0$.

In this paper we investigate the multiple solutions of the following parabolic boundary value problem

$$u_t = \Delta u + f(u) + h(x,t) \qquad \text{in } \ \Omega \times R, \tag{1.1}$$

$$u(x,t) = 0, \qquad x \in \partial\Omega,\ t \in R,$$

$$u(x,t) = u(x,t+T), \qquad \text{in } \ \Omega \times R,$$

where the period T is given.

Received November 13, 2008. Revised November 29, 2008.

2000 Mathematics Subject Classification: 35K20.

Key words and phrases: parabolic boundary value problem, eigenvalue problem, inverse compact operator, Lipschitz constant, contraction mapping principle, complex Hilbert space.

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In particular, we consider the case $T = 2\pi$, $f(u) = au^+ - bu^-$ and $h(x,t) = s\phi_1$, that is

$$u_t = \Delta u + au^+ - bu^- + s\phi_1 \qquad \text{in } \Omega \times R, \tag{1.2}$$
$$u(x,t) = 0, \qquad x \in \partial\Omega,\ t \in R,$$
$$u(x,t) = u(x,t+2\pi), \qquad \text{in } \Omega \times R,$$

The physical phenomena for this kind of parabolic problem occur in the heat flow dynamics with discontinuous nonlinearity.

The purpose of this paper is to show the existence of the unique periodic solution of problem (1.2) under the assumption that the jumping nonlinearity $au^+ - bu^-$ does not cross any eigenvalue of $-\Delta$.

The steady-state case of (1.1) is the elliptic problem

$$\Delta w + f(u) = h(x) \qquad \text{in } \Omega, \tag{1.3}$$
$$w = 0 \qquad \text{on } \partial\Omega.$$

In 1949, Dolph [7] shows that if the limits $f'(-\infty)$, $f'(+\infty)$ exist and, for some n, $\lambda_n < f'(-\infty), f'(+\infty) < \lambda_{n+1}$, then (1.3) has a solution for all $h \in L_2(\Omega)$. This extended an earlier result of Hammerstein which considerd the case $f'(-\infty), f'(+\infty) < \lambda_1$. For work in the general version of Hammerstein to unbounded regions or to more general h in $L_1(\Omega)$ or h a measurable function the reader may consult [4,8]. For work in generalization of these results to more general linear operators with discrete spectrum the reader may consult [5, 12, 13, 14]. For work in the general versions of Dolph's theorem under the assumption that the nonlinearity f is allowed to depend on x and u the reader may consult [12], and for work combining these last two topics the reader may consult the recent work of Mawhin and Ward [13].

The work of Ambrosetti and Prodi [1], and the work continued in [10,3,2,11] have a common feature in having the assumption that the nonlinearity depends only on u and the interval $[f'(-\infty), f'(+\infty)]$ contains some eigenvalues of the linear operator. Amann and Hess [2] and Dancer [6] showed that if $\lambda_1 \in [f'(-\infty), f'(+\infty)]$, then when the right-hand side of (1.3) is written

$$h(x) = s\phi_1 + h_1(x).$$

there exists $s_0 = s_0(h_1)$ such that if s is greater than, equal to, or less than s_0, then (1.3) has at least two, at least one, and zero solutions, respectively. In [11] it was shown that if $\lambda_1, \ldots, \lambda_n \in [f'(-\infty), f'(+\infty)]$

and n is even, then there exists $s_1(h_1)$ such that if $s > s_1$, then (1.3) has at least three (and generically four) solutions. More recently it has been shown by Hofer [9] that if $f'(-\infty) < \lambda_1, \lambda_2 < f'(+\infty) \neq \lambda_n$ for any n, then, for $h(x) = s\phi_1 + h_1(x)$ and s sufficiently large, (1.3) has at least four solutions. The restriction that $f'(+\infty) \neq \lambda_n$ for any n was subsequeently removed by Solimini [16]. McKenna and Walter in [15] generalize earlier work of [11] to more general operators, including both non selfadjoint operator, and operators whose resolvent is not compact. In this paper we improve these results to the parabolic problem with jumping nonlinearity. We investigate the existence of the solution of (1.2) when jumping nonlinearity $au^+ - bu^-$ does not cross any eigenvalues of $-\Delta$.

2. Main result

Let L be the parabolic operator in R^{n+1}

$$Lu = u_t - \Delta u.$$

Let Q be the space $\Omega \times (0, 2\pi)$ and H_0 the space defined by

$$H_0 = L_2(\Omega \times (0, 2\pi)).$$

Then H_0 is a Hilbert space equipped with the usual inner product

$$< v, w >= \int_0^{2\pi} \int_\Omega v(x,t)\bar{w}(x,t)dxdt$$

and a norm

$$\|v\|_{L^2(Q)} = \sqrt{< v, v >}.$$

First we shall work in the complex space H_0 but shall later switch to the real space.

The functions

$$\Phi_{mn}(x,t) = \phi_n \frac{e^{imt}}{\sqrt{2\pi}}, \qquad m = 0, \pm 1, \pm 2, \ldots, \; n = 1, 2, 3, \ldots$$

form a complete orthonormal basis in H_0. Every elements $v \in H_0$ has a Fourier expansion

$$v = \sum_{m\ n} v_{mn} \Phi_{mn}$$

with $\sum |v_{mn}|^2 < \infty$ and $v_{mn} = < v, \phi_{mn} >$. Let us define a subspace H of H_0 as

$$H = \{u \in H_0 | \sum_{m\ n} (m^2 + \lambda_n^2)^{\frac{1}{2}} u_{mn}^2 < \infty\}.$$

Then this is a complete normed space with a norm

$$\|u\| = [\sum_{m\ n} (m^2 + \lambda_n^2)^{\frac{1}{2}} u_{mn}^2]^{\frac{1}{2}}.$$

A weak solution of problem (1.2) is of the form $u = \sum u_{mn}\Phi_{mn}$ satisfying $\sum |u_{mn}|^2 (m^2 + \lambda_n^2)^{\frac{1}{2}} < \infty$, which implies $u \in H$. Thus we have that if u is a weak solution of (1.2), then $u_t = D_t u = \sum_{m\ n} imu_{mn}\Phi_{mn}$ belong to H and $-\Delta u = \sum \lambda_n u_{mn}\Phi_{mn}$ belong to H. We note that if $u \in H$, then $au^+ - bu^- + s\phi_1 \in H$. For simplicity of notation, a weak solution of (1.2) is characterized by

$$u_t - \Delta u = au^+ - bu^- + s\phi_1 \qquad \text{in} \ \ H. \tag{2.1}$$

Our main result is the following:

THEOREM 2.1. *Let $\lambda_k < a, b < \lambda_{k+1}$, $k \geq 1$. Then problem (2.1) has a unique solution in H.*

In section 3, we obtain some results on the operator $D_t - \Delta$ and prove Theorem 2.1 by investigating the Lipschitz constant of the inverse compact operator $D_t - \Delta$ and applying the contraction mapping principle.

3. Some results on the operator $D_t - \Delta$ and proof of Theorem 2.1

Since $|im + \lambda_n| \to \infty$ as m, $n \to \infty$, we have that:

LEMMA 3.1. *(i) $\|u\| \geq \|u(x, 0)\| \geq \|u(x, 0)\|_{L_2(\Omega)}$.*
(ii) $\|u\|_{L_2(Q)} = 0$ if and only if $\|u\| = 0$.
(iii) $u_t - \Delta u \in H$ implies $u \in H$.

Proof. (i) Let $u = \sum_{m\ n} u_{mn}\Phi_{mn}$. Then

$$\begin{aligned}\|u\|^2 &= \sum (m^2 + \lambda_n^2)^{\frac{1}{2}} u_{mn}^2 \\ &\geq \sum \lambda_n^2 u_{mn}^2(x, 0) \\ &= \|u(x, 0)\|^2 \geq \sum u_{mn}^2(x, 0) = \|u(x, 0)\|_{L_2(\Omega)}^2\end{aligned}$$

(ii) Let $u = \sum_{m\ n} u_{mn}\Phi_{mn}$.

$$\|u\| = 0 \Leftrightarrow \sum_{m\ n}(m^2+\lambda_n^2)^{\frac{1}{2}}u_{mn}^2 = 0 \Leftrightarrow \sum_{m\ n} u_{mn}^2 = 0 \Leftrightarrow \|u\|_{L_2(Q)} = 0.$$

(iii) Let $f = u_t - \Delta u \in H$. Then f can be expressed by

$$f = \sum f_{mn}\Phi_{mn}, \qquad \sum_{m\ n}(m^2+\lambda_n^2)^{\frac{1}{2}} f_{mn}^2 < \infty.$$

Then we have

$$\|(D_t-\Delta)^{-1}f\|^2 = \sum_{m\ n}\frac{(m^2+\lambda_n^2)^{\frac{1}{2}}}{m^2+\lambda_n^2} f_{mn}^2 < C\sum_{m\ n} f_{mn}^2 < \infty$$

for some $C > 0$. □

From Lemma 3.1, we obtain the following lemma:

Lemma 3.2. *Let $h(x,t) \in H_0 = L_2(\Omega\times(0,2\pi))$. Let a and b be not of the form $im+\lambda_n$, $m = 0, \pm1, \pm2, \ldots$, $n = 1, 2, \ldots$. Then all the solutions of*

$$u_t - \Delta u = au^+ - bu^- + h(x,t) \qquad \text{in } H_0$$

belong to H.

Lemma 3.3. *For any real $\alpha \neq \lambda_n$, the operator $(D_t-\Delta-\alpha)^{-1}$ is linear, self-adjoint, and a compact operator from H_0 to H with the operator norm $\frac{1}{|\alpha-\lambda_n|}$, where λ_n is an eigenvalue of $-\Delta$ closest to α.*

Proof. Suppose that $\alpha \neq \lambda_n$. Since $\lambda_n \to +\infty$, the number of elements in the set $\{\lambda_n | \ \lambda_n < \alpha\}$ is finite, where λ_n is an eigenvalue of $-\Delta$. Let $h = \sum_{m\ n} h_{mn}\Phi_{mn}$, where $\Phi_{mn} = \phi_n\frac{e^{imt}}{\sqrt{2\pi}}$. Then

$$(D_t-\Delta-\alpha)^{-1}h = \sum_{m\ n}\frac{1}{im+\lambda_n-\alpha}h_{mn}\Phi_{mn}.$$

Hence

$$\begin{aligned}\|(D_t-\Delta-\alpha)^{-1}\|^2 &= \sum_{m\ n}\frac{1}{m^2+(\lambda_n-\alpha)^2}(m^2+(\lambda_n-\alpha)^2)^{\frac{1}{2}}h_{mn}^2 \\ &\leq \sum_{m\ n} Ch_{mn}^2 < \infty\end{aligned}$$

for some $C > 0$. Thus $(D_t-\Delta-\alpha)^{-1}$ is a bounded operator from H_0 to H and it also sends bounded subset of H_0 to a compact subset of H, hence $(D_t-\Delta-\alpha)^{-1}$ is a compact operator. □

Next we will prove the main result.

4. Proof of theorem 2.1

Suppose that $\lambda_k < a, b < \lambda_{k+1}$, $k \geq 1$. Let us set

$$\alpha = \frac{1}{2}(\lambda_k + \lambda_{k+1}).$$

Then problem (2.1) can be rewritten as

$$u = (D_t - \Delta - \alpha)^{-1}[(a - \alpha)u^+ - (b + \alpha)u^- + s\phi_1]. \qquad (3.1)$$

From Lemma 3.3, $(D_t - \Delta - \alpha)^{-1}$ is a compact, self-adjoint, linear operator from H_0 to H with norm $\frac{1}{|\alpha - \lambda_n|}$. We note that

$$\|(a-\alpha)(u_2^+ - u_1^-) - (b-\alpha)(u_2^- - u_1^-)\|_{L_2} \leq \max\{|a-\alpha|, |b-\alpha|\}\|u_2 - u_1\|_{L_2}.$$

Thus the right side of (3.1) is a Lipschitz mapping from H_0 to H with Lipschitz constant $\gamma < 1$. By the contraction mapping principle, there exists a unique solution $u \in H_0$ of (3.1). By Lemma 3.2, the solution of (3.1) belongs to H. Thus we prove the theorem.

References

[1] A. Ambrosetti and ProG. Prodi, *On the inversion of some differentiable mappings with singularities between Banach spaces,* Annali Mat.pura appl. 93, 231-247 (1973).

[2] H. Amann and HesP. Hess, *A multiplicity result for a class of elliptic boundary value problems,* Proc. R. Soc. Edinb. 84-A, 145-151 (1979).

[3] M. Berger and PodE. Podolak, *On the solutions of a nonlinear Dirichlet problem,* Indiana Univ. Math. J. 24, 837-846 (1975).

[4] H. Brezis, *Semilinear equations in R^n without conditions at infinity,* IMAA preprint (1984).

[5] H. Brezis and NirL. Nirenberg, *Characterization of the ranges of some nonlinear operators and applications to boundary value problems,* Annali Scu. norm. sup. Pisa 5, 225-326 (1978).

[6] E. N. Dancer *On the ranges of certain weakly nonlinear elliptic partial differential equations,* J. Math. pures appl. 57, 351-366 (1978).

[7] C. L. Dolph, *Nonlinear integral equations of the Hammerstein type,* Trans. Am. math.Soc. 60-289-307 (1949).

[8] J. Glover and **P. J. McKenna**, *Solving semilinear partial differential equations with probabilistic potential theory,* Trans. Am. math. Soc.(to appear).

[9] H. Hofer, *Variational and topological methods in partially ordered Hilbert spaces,* Math. Annln 261, 493-514.

[10] J. Kazdan and WarF. Warner, *Remarks on some quasilinear elliptic equations,* Communs pure appl. Math. 28, 567-597 (1975).
[11] A. C. Lazer and McKP. J. McKenna, *On the number of solutions of a nonlinear Dirichlet problem,* J. Math. Analysis Applic 84, 282-294 (1981).
[12] J. Mawhin, *Nonresonance conditions of non-uniform type in nonlinear boundary value problems,* Dynamical systems II, (Edited by Bednarek and L. Cesari), 255-276, Academic Press, New York (1982).
[13] J. Mawhin and WarJ. Ward, *Nonuniform non-resonance conditions in the periodic Dirichlet problem for semilinear wave equations with jumping nonlinearities,* J. Nigerian Math, Soc. (to appear).
[14] P. J. McKenna, *On some semilinear problems without compactness,* in Nonlinear Phenomena in Mathematical Sciences, (Edited by V. Lakshmikantham), 683-692, Academic Press, New York (1982).
[15] P.J. McKenna and WalW. Walter, *On the multiplicity of the solution set of some nonlinear boundary value problems,* Nonlinear Analysis 8, 893-907 (1984).
[16] S. Solimini, *Remarks on the number of solutions of some nonlinear elliptic problems,* J. d'Analyse Nonlinéaire (to appear).

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