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ON DENJOY$_*$-STIELTJES INTEGRAL

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Abstract. In this paper we introduce the concepts of the generalized bounded variation in the restricted sense with respect to a strictly increasing function and Denjoy$_*$-Stieltjes integral of real-valued functions and investigate their properties.

1.Introduction

The Riemann integral is fundamental in elementary calculus. However, the Riemann integral has its limitations. The Lebesgue integral is the generalization of the Riemann integral and offers some advantages over the Riemann integral. Also generalizations of the Lebesgue integral were studied in many directions. Some authors([1],[3],[4],[6]) studied the generalized bounded variation and the Denjoy, Denjoy$_*$ integrals of real-valued functions which are extensions of the Lebesgue integral.

In this paper we define the generalized bounded variation in the restricted sense with respect to a strictly increasing function and the Denjoy$_*$-Stieltjes integral of real-valued functions which is the extension of the Denjoy$_*$ integral and then obtain their properties.

2. Preliminaries

Throughout this paper X will denote a real Banach space. Let $\omega(F, [c, d]) = \sup\{\|F(y) - F(x)\| : c \leq x < y \leq d\}$ denote the oscillation of a function $F : [a, b] \to X$ on an interval $[c, d]$.

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DEFINITION 2.1 [4]. Let $F : [a, b] \to X$ and let $E \subset [a, b]$.

(a) The function F is BV (resp. BV_*) on E if $V(F, E) = \sup\{\sum_{i=1}^{n} \|F(d_i) - F(c_i)\|\}$ (resp. $V_*(F, E) = \sup\{\sum_{i=1}^{n} \omega(F, [c_i, d_i])\}$) is finite where the supremum is taken over all finite collections $\{[c_i, d_i] : 1 \leq i \leq n\}$ of non-overlapping intervals that have endpoints in E.

(b) The function F is AC (resp. AC_*) on E if (resp. F is bounded on an interval that contains E and) for each $\epsilon > 0$ there exists $\delta > 0$ such that $\sum_{i=1}^{n} \|F(d_i) - F(c_i)\| < \epsilon$ (resp. $\sum_{i=1}^{n} \omega(F, [c_i, d_i]) < \epsilon$) whenever $\{[c_i, d_i] : 1 \leq i \leq n\}$ is a finite collection of non-overlapping intervals that have endpoints in E and satisfy $\sum_{i=1}^{n} (d_i - c_i) < \delta$.

(c) The function F is BVG (resp. BVG_*) on E if E can be expressed as a countable union of sets on each of which F is BV (resp. BV_*).

(d) The function F is ACG (resp. ACG_*) on E if F is continuous on E and E can be expressed as a countable union of sets on each of which F is AC (resp. AC_*).

DEFINITION 2.2 [6]. Let $F : [a, b] \to X$ and let $t \in (a, b)$. A vector z in X is the approximate derivative of F at t if there exists a measurable set $E \subset [a, b]$ that has t as a point of density such that $\lim_{\substack{s \to t \\ s \in E}} \frac{F(s) - F(t)}{s - t} = z$. We will write $F'_{ap}(t) = z$.

A function $f : [a, b] \to \mathbb{R}$ is Denjoy (resp. Denjoy_*) integrable on $[a, b]$ if there exists an ACG (resp. ACG_*) function $F : [a, b] \to \mathbb{R}$ such that $F'_{ap} = f$ (resp. $F' = f$) almost everywhere on $[a, b]$. The function f is Denjoy (Denjoy_*) integrable on a set $E \subset [a, b]$ if $f\chi_E$ is Denjoy (Denjoy_*) integrable on $[a, b]$.

THEOREM 2.3 [4]. *Let $f : [a, b] \to \mathbb{R}$.*

(a) *If f is Denjoy (resp. Denjoy_*) integrable on $[a, b]$, then f is measurable.*

(b) *If f is nonnegative and Denjoy (resp. Denjoy_*) integrable on $[a, b]$, then f is Lebesgue integrable on $[a, b]$.*

(c) *If f is Denjoy (resp. Denjoy$_*$) integrable on $[a,b]$, then every perfect set in $[a,b]$ contains a portion on which f is Lebesgue integrable.*

3. Generalized bounded variation with respect to α

In this section, we introduce the concepts of BV, BV$_*$, AC, AC$_*$, BVG, BVG$_*$, ACG and ACG$_*$ with respect to a strictly increasing function and obtain their properties.

DEFINITION 3.1. Let $F : [a,b] \to X$ and let $\alpha : [a,b] \to \mathbb{R}$ be a strictly increasing function and let $E \subset [a,b]$.

(a) The function F is BV (resp. BV$_*$) with respect to α on E if $V(F,\alpha,E) = \sup\{\sum_{i=1}^{n} \|F(d_i)-F(c_i)\| \frac{\alpha(d_i)-\alpha(c_i)}{d_i - c_i}\}$ (resp. $V_*(F,\alpha,E) = \sup\{\sum_{i=1}^{n} \omega(F,[c_i,d_i]) \frac{\alpha(d_i)-\alpha(c_i)}{d_i - c_i}\}$) is finite where the supremum is taken over all finite collections $\{[c_i,d_i] : 1 \le i \le n\}$ of non-overlapping intervals that have endpoints in E.

(b) The function F is AC (resp. AC$_*$) with respect to α on E if (resp. F is bounded on an interval that contains E and) for each $\epsilon > 0$ there exists $\delta > 0$ such that $\sum_{i=1}^{n} \|F(d_i)-F(c_i)\| < \epsilon$ (resp. $\sum_{i=1}^{n} \omega(F,[c_i,d_i]) < \epsilon$) whenever $\{[c_i,d_i] : 1 \le i \le n\}$ is a finite collection of non-overlapping intervals that have endpoints in E and satisfy $\sum_{i=1}^{n}[\alpha(d_i)-\alpha(c_i)] < \delta$.

(c) The function F is BVG (resp. BVG$_*$) with respect to α on E if E can be expressed as a countable union of sets on each of which F is BV (resp. BV$_*$) with respect to α.

(d) The function F is ACG (resp. ACG$_*$) with respect to α on E if F is continuous on E and E can be expressed as a countable union of sets on each of which F is AC (resp. AC$_*$) with respect to α.

THEOREM 3.2. *Let $F : [a,b] \to X$ and let $\alpha : [a,b] \to \mathbb{R}$ be a strictly increasing function.*

(a) *If F is BV_* with respect to α on $[a,b]$, then F is BV_* with respect to α on every subinterval of $[a,b]$ and $V_*(F,\alpha,[a,b]) = V_*(F,\alpha,[a,c]) + V_*(F,\alpha,[c,b])$ for each $c \in (a,b)$.*

(b) *If F is BV_* with respect to α on $[a, c]$ and $[c, b]$, then F is BV_* with respect to α on $[a, b]$.*

Proof. Suppose that F is BV_* with respect to α on $[a, b]$. Since $V_*(F, \alpha, [c, d]) \leq V_*(F, \alpha, [a, b])$ for each interval $[c, d] \subset [a, b]$, F is BV_* with respect to α on every subinterval of $[a, b]$. Now let $c \in (a, b)$ and let $\{[c_i, d_i] : 1 \leq i \leq n\}$ be any collection of non-overlapping intervals in $[a, b]$. By splitting an interval if necessary, we may assume that either $[c_i, d_i] \subset [a, c]$ or $[c_i, d_i] \subset [c, b]$ for each i. Then

$$\begin{aligned}\sum_{i=1}^{n} \omega(F, [c_i, d_i]) \frac{\alpha(d_i) - \alpha(c_i)}{d_i - c_i} &= \sum_{d_i \leq c} \omega(F, [c_i, d_i]) \frac{\alpha(d_i) - \alpha(c_i)}{d_i - c_i} \\ &\quad + \sum_{c_i \geq c} \omega(F, [c_i, d_i]) \frac{\alpha(d_i) - \alpha(c_i)}{d_i - c_i} \\ &\leq V_*(F, \alpha, [a, c]) + V_*(F, \alpha, [c, b]).\end{aligned}$$

Hence $V_*(F, \alpha, [a, b]) \leq V_*(F, \alpha, [a, c]) + V_*(F, \alpha, [c, d])$. Thus (b) is proved.

Now let $\epsilon > 0$ and choose non-overlapping collections $\{[s_i, t_i] : 1 \leq i \leq m\}$ in $[a, c]$ and $\{[u_j, v_j] : 1 \leq j \leq n\}$ in $[c, b]$ such that

$$\sum_{i=1}^{m} \omega(F, [s_i, t_i]) \frac{\alpha(t_i) - \alpha(s_i)}{t_i - s_i} > V_*(F, \alpha, [a, c]) - \frac{\epsilon}{2} \ ;$$

$$\sum_{j=1}^{n} \omega(F, [u_j, v_j]) \frac{\alpha(v_j) - \alpha(u_j)}{v_j - u_j} > V_*(F, \alpha, [c, b]) - \frac{\epsilon}{2} \ .$$

Then we have

$$\begin{aligned}V_*(F, \alpha, [a, b]) &\geq \sum_{i=1}^{m} \omega(F, [s_i, t_i]) \frac{\alpha(t_i) - \alpha(s_i)}{t_i - s_i} \\ &\quad + \sum_{j=1}^{n} \omega(F, [u_j, v_j]) \frac{\alpha(v_j) - \alpha(u_j)}{v_j - u_j} \\ &> V_*(F, \alpha, [a, c]) + V_*(F, \alpha, [c, b]) - \epsilon.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $V_*(F, \alpha, [a, b]) \geq V_*(F, \alpha, [a, c]) + V_*(F, \alpha, [c, b])$. Hence $V_*(F, \alpha, [a, b]) = V_*(F, \alpha, [a, c]) + V_*(F, \alpha, [c, b])$. Thus (a) is also proved. $\square$

THEOREM 3.3. *Let $F : [a, b] \to X$ and let $\alpha : [a, b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a, b])$ and let $E \subset [a, b]$. If F is AC_* with respect to α on E, then F is BV_* with respect to α on E.*

Proof. If F is AC$_*$ with respect to α on E, then F is bounded on an interval that contains E and for given $\epsilon = 1$ there exists $\delta > 0$ such that $\sum_{i=1}^{n} \omega(F, [c_i, d_i]) < 1$ whenever $\{[c_i, d_i] : 1 \leq i \leq n\}$ is any collection of non-overlapping intervals that have endpoints in E and satisfy $\sum_{i=1}^{n} [\alpha(d_i) - \alpha(c_i)] < \delta$. Since α' is bounded on $[a, b]$, there exists $M > 0$ such that $|\alpha'(t)| = \alpha'(t) \leq M$ for all $t \in [a, b]$. Let $[c, d]$ be any subinterval of $[a, b]$ of length $< \delta/M$ and let $\{[c_i, d_i] : 1 \leq i \leq n\}$ be any collection of nonoverlapping intervals that have endpoints in $E \cap [c, d]$. Then by the Mean Value Theorem there exists $t \in (c, d)$ such that $\dfrac{\alpha(d) - \alpha(c)}{d - c} = \alpha'(t)$. So $\alpha(d) - \alpha(c) = \alpha'(t)(d - c) < M \cdot \delta/M = \delta$. Hence $\sum_{i=1}^{n} [\alpha(d_i) - \alpha(c_i)] \leq \alpha(d) - \alpha(c) < \delta$ since α is strictly increasing. Thus $\sum_{i=1}^{n} \omega(F, [c_i, d_i]) < 1$. Also we have

$$\sum_{i=1}^{n} \omega(F, [c_i, d_i]) \frac{\alpha(d_i) - \alpha(c_i)}{d_i - c_i} \leq \sum_{i=1}^{n} \omega(F, [c_i, d_i]) M \leq M.$$

Since F is bounded on an interval that contains E, there exists $K > 0$ such that $\|F(x)\| \leq K$ for all x in an interval that contains E. Hence $\omega(F, [c, d]) \leq 2K$ for any subinterval $[c, d]$ of $[a, b]$ that has endpoints in E.

Now let $[u, v] \subset [a, b]$ be an interval containing E. Then $[u, v]$ is the union of a finite number of non-overlapping intervals $[u_1, v_1], [u_2, v_2], \ldots, [u_p, v_p]$ each of which is of length $< \delta/M$. Let $\{[c_i, d_i] : 1 \leq i \leq n\}$

be any collection of non-overlapping intervals that have endpoints in E. Then there exist at most p number of intervals $[c_i, d_i]$ such that both of endpoints of $[c_i, d_i]$ cannot be contained in any $E \cap [u_j, v_j]$, where $j = 1, 2, \ldots, p$. For such intervals $[c_i, d_i]$, we have

$$\omega(F, [c_i, d_i]) \frac{\alpha(d_i) - \alpha(c_i)}{d_i - c_i} \leq 2KM.$$

Hence we have

$$\begin{aligned} &\sum_{i=1}^{n} \omega(F, [c_i, d_i]) \frac{\alpha(d_i) - \alpha(c_i)}{d_i - c_i} \\ &\leq \sum_{j=1}^{p} \sum_{c_i, d_i \in E \cap [u_j, v_j]} \omega(F, [c_i, d_i]) \frac{\alpha(d_i) - \alpha(c_i)}{d_i - c_i} + p(2KM) \\ &\leq pM + 2pKM. \end{aligned}$$

Hence F is BV_* with respect to α on E. □

COROLLARY 3.4. *Let $F : [a, b] \to X$ and let $\alpha : [a, b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a, b])$ and let $E \subset [a, b]$. If F is ACG_* with respect to α on E, then F is BVG_* with respect to α on E.*

THEOREM 3.5. *Let $F : [a, b] \to X$ and let $\alpha : [a, b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a, b])$ and let $E \subset [a, b]$. Then F is BV_* on E if and only if F is BV_* with respect to α on E.*

Proof. If F is BV_* on E, then $V_*(F, E) = \sup \left\{ \sum_{i=1}^{n} \omega(F, [c_i, d_i]) \right\}$ is finite where the supremum is taken over all finite collections $\{[c_i, d_i] : 1 \leq i \leq n\}$ of non-overlapping intervals that have endpoints in E. Let $\{[c_i, d_i] : 1 \leq i \leq n\}$ be any finite collection of non-overlapping intervals that have endpoints in E. Since $\alpha \in C^1([a, b])$, there exists $M > 0$ such that $|\alpha'(t)| \leq M$ for all $t \in [a, b]$. By the Mean Value Theorem there exists $t_i \in (c_i, d_i)$ such that $\dfrac{\alpha(d_i) - \alpha(c_i)}{d_i - c_i} = \alpha'(t_i),\ 1 \leq i \leq n$. Hence we have

$$\sum_{i=1}^{n} \omega(F,[c_i,d_i])\frac{\alpha(d_i)-\alpha(c_i)}{d_i-c_i} \leq M\sum_{i=1}^{n} \omega(F,[c_i,d_i]) \leq MV_*(F,E).$$

Therefore F is BV$_*$ with respect to α on E.

Conversely, if F is BV$_*$ with respect to α on E, then $V_*(F,\alpha,E) = \sup\left\{\sum_{i=1}^{n} \omega(F,[c_i,d_i])\frac{\alpha(d_i)-\alpha(c_i)}{d_i-c_i}\right\}$ is finite where the supremum is taken over all finite collections $\{[c_i,d_i] : 1 \leq i \leq n\}$ of non-overlapping intervals that have endpoints in E. Let $\{[c_i,d_i] : 1 \leq i \leq n\}$ be any finite collection of non-overlapping intervals that have endpoints in E. Since α is a strictly increasing function such that $\alpha \in C^1([a,b])$, there exists $m > 0$ such that $|\alpha'(t)| = \alpha'(t) \geq m$ for all $t \in [a,b]$. By the Mean Value Theorem there exists $t_i \in (c_i,d_i)$ such that $\frac{\alpha(d_i)-\alpha(c_i)}{d_i-c_i} = \alpha'(t_i),\ 1 \leq i \leq n$. Hence we have

$$V_*(F,\alpha,E) \geq \sum_{i=1}^{n} \omega(F,[c_i,d_i])\frac{\alpha(d_i)-\alpha(c_i)}{d_i-c_i} \geq m\sum_{i=1}^{n} \omega(F,[c_i,d_i]).$$

Therefore $\sum_{i=1}^{n} \omega(F,[c_i,d_i]) \leq \frac{1}{m}V_*(F,\alpha,E)$. Thus F is BV$_*$ on E. □

THEOREM 3.6. *Let $F : [a,b] \to X$ and let $\alpha : [a,b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a,b])$ and let $E \subset [a,b]$. Then F is AC$_*$ on E if and only if F is AC$_*$ with respect to α on E.*

Proof. Suppose that F is AC$_*$ on E. Let $\epsilon > 0$ be given. Then there exists $\eta > 0$ such that $\sum_{i=1}^{n} \omega(F,[c_i,d_i]) < \epsilon$ whenever $\{[c_i,d_i] : 1 \leq i \leq n\}$ is any finite collection of non-overlapping intervals that have endpoints in E and satisfy $\sum_{i=1}^{n}(d_i - c_i) < \eta$. Since α is a strictly increasing function such that $\alpha \in C^1([a,b])$, there exists $m > 0$ such that $|\alpha'(t)| = \alpha'(t) \geq m$ for all $t \in [a,b]$. Take $\delta = m\eta$. Let $\{[c_i, d_i] : 1 \leq i \leq n\}$ be any finite collection of non-overlapping intervals that have endpoints in E and satisfy $\sum_{i=1}^{n}[\alpha(d_i) - \alpha(c_i)] < \delta$. Then by the Mean Value Theorem there exists $t_i \in (c_i, d_i)$ such that $\alpha(d_i) - \alpha(c_i) = \alpha'(t_i)(d_i - c_i),\ 1 \leq i \leq n$. So $\alpha(d_i) - \alpha(c_i) \geq m(d_i - c_i),\ 1 \leq i \leq n$. Hence $\sum_{i=1}^{n}(d_i - c_i) \leq \frac{1}{m}\sum_{i=1}^{n}[\alpha(d_i) - \alpha(c_i)] \leq \frac{1}{m} \cdot \delta = \eta$. So $\sum_{i=1}^{n}\omega(F, [c_i, d_i]) < \epsilon$. Thus F is AC$_*$ with respect to α on E.

Conversely, suppose that F is AC$_*$ with respect to α on E. Let $\epsilon > 0$ be given. Then there exists $\eta > 0$ such that $\sum_{i=1}^{n}\omega(F, [c_i, d_i]) < \epsilon$ whenever $\{[c_i, d_i] : 1 \leq i \leq n\}$ is any finite collection of non-overlapping intervals that have endpoints in E and satisfy $\sum_{i=1}^{n}[\alpha(d_i) - \alpha(c_i)] < \eta$. Since $\alpha \in C^1([a,b])$, there exists $M > 0$ such that $|\alpha'(t)| \leq M$ for all $t \in [a,b]$. Take $\delta = \frac{\eta}{M}$. Let $\{[c_i, d_i] : 1 \leq i \leq n\}$ be any finite collection of non-overlapping intervals that have endpoints in E and satisfy $\sum_{i=1}^{n}(d_i - c_i) < \delta$. Then by the Mean Value Theorem there exists $t_i \in (c_i, d_i)$ such that $\alpha(d_i) - \alpha(c_i) = \alpha'(t_i)(d_i - c_l),\ 1 \leq i \leq n$. So $\alpha(d_i) - \alpha(c_i) \leq M(d_i - c_i),\ 1 \leq i \leq n$. Hence $\sum_{i=1}^{n}[\alpha(d_i) - \alpha(c_i)] \leq M\sum_{i=1}^{n}(d_i - c_i) < M\delta = \eta$. So $\sum_{i=1}^{n}\omega(F, [c_i, d_i]) < \epsilon$. Thus F is AC$_*$ on E. □

4. Denjoy$_*$-Stieltjes integral

In this section, we introduce the concepts of the Denjoy$_*$-Stieltjes integral with respect to a strictly increasing function which belongs to $C^1([a,b])$ and investigate some properties of this integral.

Definition 4.1. Let $F : [a,b] \to X$ and let $t \in (a,b)$ and let $\alpha : [a,b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a,b])$. A vector $z \in X$ is the derivative of F with respect to α at t if $\lim\limits_{\substack{s\to t\\ s\in[a,b]}} \dfrac{F(s)-F(t)}{\alpha(s)-\alpha(t)} = z$. We will write $F'_\alpha(t) = z$.

Note that $F'(t) = F'_\alpha(t)\ \alpha'(t)$ for each $t \in (a,b)$.

Definition 4.2. Let $\alpha : [a,b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a,b])$. A function $f : [a,b] \to \mathbb{R}$ is Denjoy$_*$-Stieltjes integrable with respect to α on $[a,b]$ if there exists an ACG$_*$ function $F : [a,b] \to \mathbb{R}$ with respect to α such that $F'_\alpha = f$ almost everywhere on $[a,b]$. The function f is Denjoy$_*$-Stieltjes integrable with respect to α on a set $E \subset [a,b]$ if $f\chi_E$ is Denjoy$_*$-Stieltjes integrable with respect to α on $[a,b]$.

Theorem 4.3. *Let $f : [a,b] \to \mathbb{R}$ and let $\alpha : [a,b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a,b])$ and let $E \subset [a,b]$. Then f is Denjoy$_*$-Stieltjes integrable with respect to α on E if and only if $\alpha' f$ is Denjoy$_*$ integrable on E.*

Proof. If f is Denjoy$_*$-Stieltjes integrable with respect to α on E, then there exists an ACG$_*$ function $F : [a,b] \to \mathbb{R}$ with respect to α such that $F'_\alpha = f\chi_E$ almost everywhere on $[a,b]$. By Theorem 3.6, F is an ACG$_*$ function on $[a,b]$ such that $F' = \alpha' f\chi_E$ almost everywhere on $[a,b]$. Hence $\alpha' f\chi_E$ is Denjoy$_*$ integrable on $[a,b]$. Thus $\alpha' f$ is Denjoy$_*$ integrable on E.

Conversely, if $\alpha' f$ is Denjoy$_*$ integrable on E, then there exists an ACG$_*$ function $F : [a,b] \to \mathbb{R}$ on $[a,b]$ such that $F' = \alpha' f\chi_E$ almost everywhere on $[a,b]$. By Theorem 3.6, F is an ACG$_*$ function with respect to α on $[a,b]$ such that $F'_\alpha = f\chi_E$ almost everywhere on $[a,b]$. Hence $f\chi_E$ is Denjoy$_*$-Stieltjes integrable with respect to α on $[a,b]$. Thus f is Denjoy$_*$-Stieltjes integrable with respect to α on E.

□

From Theorem 4.3, we can deal with the Denjoy$_*$-Stieltjes integrability by means of the Denjoy$_*$ integrability.

THEOREM 4.4. *Let* $f : [a, b] \to \mathbb{R}$ *and let* $\alpha : [a, b] \to \mathbb{R}$ *be a strictly increasing function such that* $\alpha \in C^1([a, b])$.

(a) *If* f *is Denjoy*$_*$*-Stieltjes integrable with respect to* α *on* $[a, b]$, *then* f *is measurable.*

(b) *If* f *is nonnegative and Denjoy*$_*$*-Stieltjes integrable with respect to* α *on* $[a, b]$, *then* $\alpha' f$ *is Lebesgue integrable on* $[a, b]$.

(c) *If* f *is Denjoy*$_*$*-Stieltjes integrable with respect to* α *on* $[a, b]$, *then every perfect set in* $[a, b]$ *contains a portion on which* $\alpha' f$ *is Lebesgue integrable.*

Proof. (a) If f is Denjoy$_*$-Stieltjes integrable with respect to α on $[a, b]$, then $\alpha' f$ is Denjoy$_*$ integrable on $[a, b]$ by Theorem 4.3. Hence $\alpha' f$ is measurable by Theorem 2.3. Since $\dfrac{1}{\alpha'}$ is continuous on $[a, b]$, $\dfrac{1}{\alpha'}$ is measurable. Hence f is also measurable.

(b) and (c) follow easily from Theorem 2.3 and Theorem 4.3. □

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