Korean J. Math. **16** (2008), No. 4, pp. 481–494

A GENERATION OF A DETERMINANTAL FAMILY OF ITERATION FUNCTIONS AND ITS CHARACTERIZATIONS

YoonMee Ham, Sang-Gu Lee* and Jerry Ridenhour

Abstract. Iteration functions $K_m(z)$ and $U_m(z)$, $m \geq 2$ are defined recursively using the determinant of a matrix. We show that the fixed-iterations of $K_m(z)$ and $U_m(z)$ converge to a simple zero with order of convergence m and give closed form expansions of $K_m(z)$ and $U_m(z)$. To show the convergence, we derive a recursion formula for L_m and then apply the idea of Ford or Pomentale. We also find a Toeplitz matrix whose determinant is $L_m(z)/(f')^m$, and then we adapt the well-known results of Gerlach and Kalantari et.al. to give closed form expansions.

1. Introduction

Suppose that $f(z)$ is analytic with a simple zero at α in either the reals or the complex numbers. Let $L_0(z) = 1$ and

$$L_m(z) = \det \begin{pmatrix} f'(z) & f(z) & 0 & \cdots & 0 \\ f''(z) & f'(z) & f(z) & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ \frac{f^{(m-1)}(z)}{(m-2)!} & \frac{f^{(m-2)}(z)}{(m-2)!} & \frac{f^{(m-3)}(z)}{(m-3)!} & \ldots & f(z) \\ \frac{f^{(m)}(z)}{(m-1)!} & \frac{f^{(m-1)}(z)}{(m-1)!} & \frac{f^{(m-2)}(z)}{(m-2)!} & \ldots & f'(z) \end{pmatrix} \tag{1}$$

where $\det(\cdot)$ denotes determinant. The matrix of $L_m(z)$ is the determinant of a kind of Toeplitz matrix.

Received September 19, 2008. Revised October 10, 2008.

2000 Mathematics Subject Classification: 65-01, 65B99, 65H05.

Key words and phrases: Laguerre's iteration function; Toeplitz; iterative methods; basic family; order of convergence.

This paper was supported by the SRC/ERC program of MOST/KOSEF R11-1999-054 & BK21.

*Corresponding author.

In [9], L_m is introduced and is evaluated recursively,

(2)
$$L_m = f' L_{m-1} - \frac{1}{2} f f'' L_{m-2} + \ldots + \frac{(-1)^{m-2}}{(m-1)!} f^{m-2} f^{(m-1)} L_1 + \frac{(-1)^{m-1}}{(m-1)!} f^{m-1} f^{(m)},$$

where the second term will be $-f\,f''$ when $m = 2$. This formula becomes apparent once the determinant of a matrix is expanded along its last column.

French mathematician E. N. Laguerre [10] gave a proposition which says that any two numbers u and v satisfying the relation

$$(u - x)(v - x)(f'^2 - f f'') + (u + v - 2x) f f'' + N f^2 = 0 \tag{3}$$

where $f = f(x)$, $f(x) = 0$ is an algebraic equation of degree N, separate the roots of the equation. Kulik [9] showed that

$$u = x - f \frac{(v - x) L_{m-1} + f L_{m-2}}{(v - x) L_m + f L_{m-1}}$$

where L_m is as in (1).

We define the following iteration schemes; for each $m \geq 2$, define

$$K_m(z) = z - f(z) \frac{L_{m-1}(z)}{L_m(z)} \tag{4}$$

and

$$U_m(v, z) = z - f(z) \frac{(v - z) L_{m-1}(z) + f(z) L_{m-2}(z)}{(v - z) L_m(z) + f(z) L_{m-1}(z)} \tag{5}$$

for a fixed complex constant v. The Laguerre case (3) can be obtained from (5) by taking a polynomial f with $m = 2$.

We state the well-known results of Gerlach in [2], Ford in [1] and Kalantari et.al. in [5, 7, 8].

THEOREM 1.1. *(Gerlach [2]). Set $F_1(x) = f(x)$, and for each $m > 2$, recursively define*

$$F_m(x) = \frac{F_{m-1}(x)}{F'_{m-1}(x)^{1/m}}.$$

Then, the function

$$\hat{F}_m(x) = x - \frac{F_{m-1}(x)}{F'_{m-1}(x)}$$

defines an iteration function whose order of convergence for simple roots is m.

No closed formula for $\hat{F}_m(x)$ was given previously. Indeed it is not even clear that $\hat{F}_m(x)$ would simplify into a rational function of $x, f(x)$, and its derivatives. Ford and Pennline [1], give a rational formulation of $\hat{F}_m(x)$. More precisely, they show:

THEOREM 1.2. *(Ford and Pennline [1]). The iteration function $G_m(x)$ can be written as*

$$G_m(x) = x - f(x)\frac{Q_m(x)}{Q_{m+1}(x)}$$

where $Q_2(x) = 1$ and $Q_{m+1}(x) = f'(x)Q_m(x) - \frac{1}{m-1}f(x)Q'_m(x)$.

In Kalantari et.al. in [5, 7, 8], they give a closed formula for $G_m(x)$ by proving the equivalence of the family $\{G_m(x)\}_{m=2}^{\infty}$ a family of iteration functions, $\{B_m(x)\}_{m=2}^{\infty}$, called the Basic Family. To define the Basic Family, let $D_0(x) = 1$ and define

$$D_m(x) = \det \begin{pmatrix} f(x) & 0 & 0 & \cdots & 0 \\ f'(x) & f(x) & 0 & \cdots & 0 \\ \frac{f''(x)}{2!} & f'(x) & f(x) & \cdots & 0 \\ \frac{f'''(x)}{3!} & \frac{f''(x)}{2} & f'(x) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{f^{(m-1)}(x)}{(m-1)!} & \frac{f^{(m-2)}(x)}{(m-2)!} & \cdots & \cdots & f(x) \end{pmatrix} \tag{6}$$

for $m \geq 1$. Also, for each $i = m+1, \ldots, n+m-1$, define

(7)

$$\hat{D}_{m,i}(x) = \det \begin{pmatrix} \frac{f''(x)}{2!} & f'(x) & f(x) & \cdots & 0 \\ \frac{f'''(x)}{3!} & \frac{f''(x)}{2!} & f'(x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \frac{f^{(m)}(x)}{(m)!} & \frac{f^{(m-1)}(x)}{(m-1)!} & \cdots & \frac{f''(x)}{2!} & f'(x) \\ \frac{f^{(i)}(x)}{i!} & \frac{f^{(i-1)}(x)}{(i-1)!} & \cdots & \frac{f^{(i-m+2)}(x)}{(i-m+2)!} & \frac{f^{(i-m+1)}(x)}{(i-m+1)!} \end{pmatrix}.$$

Note that $D_m(x)$ corresponds to the determinant of a Toeplitz matrix defined with respect to the normalized derivatives of $f(x)$.

THEOREM 1.3. *(Kalantari et al. [6], Kalantari [4]). For each $m \geq 2$, define*

$$B_m(x) = x - f(x)\frac{D_{m-2}(x)}{D_{m-1}(x)}.$$

Let θ be a simple root of $f(x)$. Then,

$$B_m(x) = \theta + \sum_{i=m}^{m+n-2} (-1)^m \frac{\hat{D}_{m-1,i}(x)}{D_{m-1}(x)} (x-\theta)^m.$$

In particular, there exists $r > 0$ such that given any $x_0 \in Nr(\theta) = \{x : |x-\theta| < r\}$, the fixed-point iteration

$$x_{k+1} = B_m(x_k), k = 1, 2, \ldots$$

is well-defined, it converges to θ having order m. Specifically,

$$\lim_{k\to\infty} \frac{(\theta - x_{k+1})}{(\theta - x_k)^m} = (-1)^{m-1} \frac{\hat{D}_{m-1,m}(\theta)}{D_{m-1}(\theta)} = (-1)^{m-1} \frac{\hat{D}_{m-1,m}(\theta)}{(f'(\theta))^{m-1}}.$$

THEOREM 1.4. *(Kalantari et al. [5]). For each $m \geq 1$, we have*

$$D'_m = \frac{m+1}{f}(f' D_m - D_{m+1}).$$

2. Recursive formula for L_m

In the sequel, we denote the k-th derivative of $f(z)$ by $f^{(k)}(z)$ and suppress the variable z in $f^{(k)}(z)$, $L_k(z)$ for simplicity.

THEOREM 2.1. *For each $m \geq 1$, we have*

$$\begin{aligned} (8) \quad L'_m &= \frac{m}{f}(f' L_m - L_{m+1}) \\ &= m\Big(\sum_{i=2}^{m} \frac{(-1)^i}{i!} f^{i-2} f^{(i)} L_{m+1-i} + (-1)^{m+1} \frac{f^{m-1} f^{(m+1)}}{m!}\Big). \end{aligned}$$

Proof. Since (2) is equivalent to

$$(9) \qquad f' L_{m-1} - L_m = \sum_{i=2}^{m-1} \frac{(-1)^i}{i!} f^{i-1} f^{(i)} L_{m-i} + \frac{(-1)^m}{(m-1)!} f^{m-1} f^{(m)},$$

the second equality in the theorem follows from (9). We use a mathematical induction on m. For $m = 1$, $L_1 = f'$ and $L_2 = f'^2 - f f''$, and thus $\frac{1}{f}(f' L_1 - L_2) = f'' = L_1'$. Hence, the theorem is true for $m = 1$. Assume (8) is true for $m-1$. Equation (2) is

$$L_m = \sum_{i=1}^{m-1} \frac{(-1)^{i-1}}{i!} f^{i-1} f^{(i)} L_{m-i} + \frac{(-1)^{m-1}}{(m-1)!} f^{m-1} f^{(m)}.$$

Differentiate L_m and then we have

$$L_m' = A + B + C + D + E$$

where

$$\begin{aligned}
A &= \sum_{i=2}^{m-1} \frac{(-1)^{i-1}}{i!}(i-1) f^{i-2} f' f^{(i)} L_{m-i} \\
B &= \sum_{i=1}^{m-1} \frac{(-1)^{i-1}}{i!} f^{i-1} f^{(i+1)} L_{m-i} = \sum_{i=2}^{m} \frac{(-1)^{i-2}}{(i-1)!} f^{i-2} f^{(i)} L_{m+1-i} \\
C &= \sum_{i=1}^{m-1} \frac{(-1)^{i-1}}{i!} f^{i-1} f^{(i)} L_{m-i}' \\
D &= \tfrac{(-1)^{m-1}}{(m-2)!} f^{m-2} f' f^{(m)} \\
E &= \tfrac{(-1)^{m-1}}{(m-1)!} f^{m-1} f^{(m+1)} .
\end{aligned}$$

Using the induction hypothesis, $C = C_1 + C_2$ where

$$\begin{aligned}
C_1 &= \sum_{i=1}^{m-1} \frac{(-1)^{i-1}}{i!}(m-i) f^{i-2} f' f^{(i)} L_{m-i} \\
&= \tfrac{m-1}{f} f'^2 L_{m-1} + \textstyle\sum_{i=2}^{m-1} \tfrac{(-1)^{i-1}}{i!}(m-i) f^{i-2} f' f^{(i)} L_{m-i} \\
C_2 &= -\tfrac{m-1}{f} f' L_m + \sum_{i=2}^{m-1} \frac{(-1)^{i}}{i!}(m-i) f^{i-2} f^{(i)} L_{m+1-i}.
\end{aligned}$$

Now, note that

$$A + C_1 = m \sum_{i=2}^{m-1} (-1)^{i-1} \frac{f^{i-2} f' f^{(i)}}{i!} L_{m-i} + \frac{m-1}{f} f'^2 L_{m-1}$$

and

$$B + C_2 = -\frac{m-1}{f} f' L_m + m \sum_{i=2}^{m-1} \frac{(-1)^{i-2}}{i!} f^{i-2} f^{(i)} L_{m+1-i} + \frac{(-1)^{m-2}}{(m-1)!} f^{m-2} f^{(m)} L_1.$$

Thus,

$$\begin{aligned} L'_m &= A + C_1 + D + B + C_2 + E \\ &= \tfrac{(m-1)}{f}\, f' \Big(f'\, L_{m-1} - L_m + \textstyle\sum_{i=2}^{m-1} \tfrac{(-1)^{i-1}}{i!}\, f^{i-1} f^{(i)} L_{m-i} + (-1)^{m-1} \tfrac{f^{m-1}\, f^{(m)}}{(m-1)!} \Big) \\ &+ m \Big(\textstyle\sum_{i=2}^{m} \tfrac{(-1)^i}{i!}\, f^{i-2}\, f^{(i)} L_{m+1-i} + (-1)^{m+1} \tfrac{f^{m-1} f^{(m+1)}}{m!} \Big) \end{aligned}$$

since (9) implies the sum of first four terms is zero. Hence, we have

$$\begin{aligned} L'_m &= m \Big(\textstyle\sum_{i=2}^{m} \tfrac{(-1)^i}{i!}\, f^{i-2}\, f^{(i)} L_{m+1-i} + (-1)^{m+1} \tfrac{f^{m-1} f^{(m+1)}}{m!} \Big) \\ &= \tfrac{m}{f} \left(f'\, L_m - L_{m+1} \right) \end{aligned}$$

and thus

$$L_{m+1} \;=\; L_m\, f' - \frac{1}{m}\, f\, L'_m \tag{10}$$

holds for all $m \geq 1$. □

Now let $F_1 = f$ and $F_m = f\, L_{m-1}^{-\frac{1}{m-1}}$ for $m \geq 2$. Then by (10)

$$F'_m = L_{m-1}^{-\frac{m}{m-1}} \left(f' L_{m-1} - \tfrac{1}{m-1}\, f\, L'_{m-1} \right) = L_{m-1}^{-\frac{m}{m-1}}\, L_m$$

and thus $\frac{F_m}{F'_m} = f\, \frac{L_{m-1}}{L_m}$. Also, $\frac{F_m}{F'^{1/m}_m} = \frac{f}{L_m^{1/m}} = F_{m+1}$. Thus $K_m = z - f\, \frac{L_{m-1}}{L_m}$ has mth-order of convergence by Theorem 1.1. Hence, we have the rational formulation for K_m:

Theorem 2.2. *Let $F_1 = f$ and for each $m \geq 2$, recursively define $F_m = f\, L_{m-1}^{-\frac{1}{m-1}}$. Then $K_m(z) = z - f(z)\, \frac{L_{m-1}(z)}{L_m(z)}$ defines an iteration function whose order of convergence for a simple zero is m.*

Theorem 2.3. *Let $A_1(z) = \frac{f'(z)}{f(z)}$ and, for each $m \geq 2$, recursively define*

$$A_m(z) = -\frac{1}{m-1} A'_{m-1}(z). \tag{11}$$

Then the function

$$z - \frac{A_{m-1}(z)}{A_m(z)} = z + (m-1)\, \frac{A_{m-1}(z)}{A'_{m-1}(z)} \tag{12}$$

defines iterates that converge to a simple zero with order m.

Proof. Construct a set of functions $\hat{A}_m$,

$$\hat{A}_1 = \frac{1}{A_1} = \frac{f}{f'}\,, \qquad \hat{A}_{m-1} = \frac{1}{A_{m-1}^{\frac{1}{m-1}}}\,, \quad m \geq 2, \tag{13}$$

using the A_m defined in (11). Direct differentiation yields

$$\hat{A}'_{m-1} = -\tfrac{1}{m-1}\, A_{m-1}^{-\frac{1}{m-1}-1}\, A'_{m-1} = A_{m-1}^{-1}\, A_m\, \hat{A}_m \tag{14}$$

$$= A_m\, A_{m-1}^{-\frac{m}{m-1}}. \tag{15}$$

From (15), we have

$$\frac{\hat{A}_{m-1}}{\hat{A}'^{\frac{1}{m}}_{m-1}} = \frac{A_{m-1}^{-\frac{1}{m-1}}}{A_m^{\frac{1}{m}}\, A_{m-1}^{-\frac{1}{m-1}}} = \frac{1}{A_m^{\frac{1}{m}}} = \hat{A}_m. \tag{16}$$

Using (13) and (14) we obtain

$$\frac{\hat{A}_{m-1}}{\hat{A}'_{m-1}} = \frac{\hat{A}_m}{A_{m-1}^{-1}\, A_m\, \hat{A}_m} = \frac{A_{m-1}}{A_m}\,. \tag{17}$$

Therefore, from (16) and Theorem 1.1, the method defined as in (12) has mth-order convergence. □

It is easily seen that $\frac{A_1(z)}{A_2(z)}$ and $\frac{A_2(z)}{A_3(z)}$ can be obtained by applying Newton's method and the Halley's iteration function ([3]) to the function f/f'. We show the relation between L_m and A_m for $m \geq 1$..

THEOREM 2.4. *Suppose that f is an analytic function. For each $m \geq 1$, $A_m(z)$ and $L_m(z)$ are related by*

$$L_m(z) = f^m(z)\, A_m(z)\,. \tag{18}$$

Proof. We use a mathematical induction on m. For $m = 1$, $f\, A_1 = f' = L_1$. For $m = 2$, $f^2\, A_2 = f'^2 - f\, f''$ which is equal to L_2. Assume that (18) is true for m. Then $A_{m+1} = -\frac{1}{m} A'_m$ and, by the induction hypothesis,

$$L'_m = (A_m\, f^m)' = A'_m\, f^m + m\, A_m\, f^{m-1} f' = -m\, A_{m+1}\, f^m + m\, A_m\, f^{m-1} f'.$$

By the recursion formula (10), we have

$$A_{m+1}\, f^{m+1} = A_m\, f^m f' - \frac{1}{m}\, f L'_m = f' L_m - \frac{1}{m} f L'_m = L_{m+1}.$$

Therefore, (18) holds for all $m \geq 1$. □

We see that by using Theorem 2.4 that $\frac{A_{m-1}(z)}{A_m(z)} = f(z)\frac{L_{m-1}(z)}{L_m(z)}$ for $m \geq 2$. Hence $z - f(z)\,\frac{Lm-1(z)}{L_m(z)}$ have mth order convergence. We note that Pomentale [11] constructed the mth order of convergence iteration as following:

THEOREM 2.5. *(Pomentale* [11]*) Suppose f is analytic. Define*

$$\Phi_m(z) = -\frac{f}{f' - \frac{\phi'_{m-2}}{(m-1)\phi_{m-2}}f},\ m = 2, 3, \ldots$$

where ϕ_m are defined by the following recurrence relation:

$$\phi_{m-1}(z) = \phi'_{m-2}(z)f(z) - (m-1)\phi_{m-2}(z)f'(z),\ \ \phi_0(z) = f'(z). \tag{19}$$

Then the function

$$z + (m-1)\frac{(f'/f)^{(m-2)}}{(f'/f)^{(m-1)}} = z + (m-1)f\frac{\phi_{m-2}(z)}{\phi_{m-1}(z)} \tag{20}$$

defines iteration of the mth order of convergence.

THEOREM 2.6. *Suppose that f is an analytic function. For each $m \geq 0$, $\phi_m(z)$ and $L_m(z)$ are related by*

$$\phi_m = (-1)^m m!\, L_{m+1},\ \ m \geq 0 \tag{21}$$

Proof. We use a mathematical induction on m. For $m = 0$, $\phi_0 = f' = L_1$. For $m = 1$, $\phi_1 = \phi_0{}'f - \phi_0 f' = f\,f'' - f'^2$ which is equal to $-L_2$. Assume that (21) is true for m. Then by the induction hypothesis,

$$\begin{aligned}\phi_{m+1} &= \phi'_m f - (m+1)\phi_m f' \\ &= (-1)^m m!\, L'_{m+1}f - (m+1)(-1)^m m!\, L_{m+1}f' \\ &= (-1)^{m+1}(m+1)!\left(L_{m+1}f' - \tfrac{1}{m+1}L'_{m+1}f\right).\end{aligned}$$

By the recursion formula (10), we have

$$\phi_{m+1} = (-1)^{m+1}(m+1)!\, L_{m+2}$$

and thus (21) holds for all $m \geq 0$. □

Hence, (20) is equivalent to

$$z + (m-1)f\frac{\phi_{m-2}(z)}{\phi_{m-1}(z)} = z - f\,\frac{L_{m-1}(z)}{L_m(z)}$$

which is the mth order of convergence. In this case, the closed form of the iteration $\Phi_m(z)$ is not yet given and thus we shall give a closed form of K_m as in (4).

3. Construction of a Toeplitz matrix

Suppose that f is an analytic function with a simple zero at α. Let $P(z) = \frac{f(z)}{f'(z)}$ and then $P(z)$ is also analytic with a simple zero at α. Let $H_0(z) = 1$ and define for $m \geq 1$, $H_m(z)$ corresponds to the determinant of a Toeplitz matrix in (6) defined with respect to the normalized derivatives of $\frac{f(z)}{f'(z)}$, i.e., we may say that $H_m(z) = D_m(P(z); z)$. Also, we consider $\hat{H}_{m,k}(z) = \hat{D}_{m,k}(P(z); z)$.

By Theorem 1.1 and Theorem 1.3, we have

$$B_m(P(z); z) = z - \frac{f(z)}{f'(z)} \frac{H_{m-2}(z)}{H_{m-1}(z)}. \tag{22}$$

A closed form expression for a basic family B_m can be found in Theorem 1.3. From Theorem 1.2,

$$G_m(P(z); z) = z - \frac{f(z)}{f'(z)} \frac{Q_m(z)}{Q_{m+1}(z)}$$

where $Q_2(z) = 1$ and for $m \geq 2$, $Q_{m+1}(z) = \left(\frac{f(z)}{f'(z)}\right)' Q_m(z) - \frac{1}{m-1} \frac{f(z)}{f'(z)} Q'_m(z)$. Both of B_m and G_m have order of convergence m and it was shown that $B_m = G_m$ for each $m \geq 2$ in [5]. By Theorem 1.4 and the recursion formula $\{Q_m\}_{m=2}^{\infty}$, we have

$$H_{m-1}(z) = \left(\frac{f(z)}{f'(z)}\right)' H_{m-2}(z) - \frac{1}{m-1} \frac{f(z)}{f'(z)} H'_{m-2}(z) \tag{23}$$

for $m \geq 2$. We now have the following key result.

THEOREM 3.1. *Suppose that f is an analytic function with a simple zero at α. For each $m \geq 1$, $H_{m-1}(z)$ and $L_m(z)$ are related by*

$$f'(z)^m H_{m-1}(z) = L_m(z). \tag{24}$$

Proof. Use an induction on m. Since $H_0 = 1$, (24) is true for $m = 1$. For $m = 2$, $L_2 = f'^2 - ff'' = f'^2 \frac{f'^2 - ff''}{f'^2} = f'^2 \left(\frac{f}{f'}\right)' = f'^2 H_1$. Hence, (24) is true for $m = 2$. Assume that (24) is true for $m - 1$, i.e., $L_{m-1} =$

$(f')^{m-1}\,H_{m-2}$. By Theorem 2.1, $L_m = f'\,L_{m-1} - \frac{f}{m-1}\,L'_{m-1}$ for all $m \geq 2$. Using the induction hypothesis and (23), then we obtain

$$\begin{aligned} L'_m &= f'(f')^{m-1}\,H_{m-2} - \tfrac{f}{m-1}\Big((f')^{m-1}H_{m-2}\Big)' \\ &= (f')^m\,H_{m-2} - \tfrac{f}{m-1}\Big((m-1)(f')^{m-2}f''\,H_{m-2} + (f')^{m-1}H'_{m-2}\Big) \\ &= (f')^m\,H_{m-2} - f\,f'^{m-2}f''\,H_{m-2} - f(f')^{m-1}\tfrac{f}{P}(P'H_{m-2} - H_{m-1}) \\ &= (f')^m\,H_{m-1} + H_{m-2}\Big((f')^m - f\,(f')^{m-2}f'' - \tfrac{P'}{P}(f')^{m-1}\,f\Big) \\ &= (f')^m\,H_{m-1} + H_{m-2}(f')^{m-2}\,\Big((f')^2 - f\,f'' - \tfrac{P'}{P}\,f'\Big) \\ &= (f')^m\,H_{m-1} \end{aligned}$$

since $\frac{P'}{P}\,f' = (f')^2 - f\,f''$. Hence, (24) holds for all $m \geq 1$. □

We note that the relationship between L_m and H_m in (24) can be obtained by recursive row operations. We also show (10) follows from (24).

THEOREM 3.2. *If $\{L_m\}_{m=1}^{\infty}$ satisfies (24), then the recursion formula (10) holds for each $m \geq 1$.*

Proof. We use a mathematical induction on m. For $m = 1$, the right-hand side of (10) is $f'\,L_1 - f\,L'_1 = f'^2 - f\,f''$ which is equal to L_2. Assume that (10) is true for $m-1$. Applying Theorem 3.1, then

$$\begin{aligned} &f'\,L_m - \tfrac{1}{m}\,f\,L'_m \\ &\quad = f'\,(f')^m H_{m-1} - \tfrac{1}{m}((f')^m\,H_{m-1})' \\ &\quad = (f')^{m+1}H_{m-1} - \tfrac{1}{m}\big(m\,(f')^{m-1}f''\,H_{m-1} + (f')^m\,H'_{m-1}\big) \\ &\quad = (f')^{m+1}\Big(\tfrac{f'^2 - f\,f''}{(f')^2}H_{m-1} - \tfrac{1}{m}\tfrac{f}{f'}\,H'_{m-1}\Big) \\ &\quad = (f')^{m+1}\Big(P'H_{m-1} - \tfrac{1}{m}P\,H'_{m-1}\Big) \\ &\quad = (f')^{m+1}H_m \\ &\quad = L_{m+1} \end{aligned}$$

Therefore, (10) holds for all $m \geq 1$. □

4. Convergence analysis

For each $m \geq 2$, define $K_m(z)$ as in (4). We use the recursion formula for L_m to show the fixed-iteration $z_{n+1} = K_m(z_n)$, $n = 1, 2 \ldots$ converges with order m to a simple zero of $f(z)$. We give closed form expansions of $K_m(z)$ and $U_m(z)$ and show that iterations defined by $U_m(z)$ also have mth order of convergence.

THEOREM 4.1. *Let $f(z)$ be an analytic function over the entire complex plane with a simple zero at α. For each $m \geq 2$, define $K_m(z)$ as in (4). Then, $K_m(z)$ satisfies the following*

$$K_m(z) = \alpha + \sum_{k=m}^{\infty} (-1)^m \frac{\hat{H}_{m-1,k}(z)}{H_{m-1}(z)} (\alpha - z)^k.$$

Moreover, the fixed point iteration defined by $z_{n+1} = K_m(z_n), n = 1, 2, \ldots$ converges to α in some neighborhood of α with mth-order of convergence and the asymptotic error constant is

$$\lim_{n\to\infty} \frac{\alpha - z_{n+1}}{(\alpha - z_n)^m} = (-1)^m \frac{\hat{H}_{m-1,m}(\alpha)}{H_{m-1}(\alpha)} = (-1)^m \hat{H}_{m-1,m}(\alpha) \tag{25}$$

where, for any $m \geq 1$ and for each $k \geq m+1$, $\hat{H}_{m,k}(z)$ is defined by

$$\hat{H}_{m,k}(z) = \hat{D}_{m,k}(P(z); z)$$

where $\hat{D}_{m,k}$ is defined in (7).

Proof. For $m \geq 2$, (22) implies that

$$\begin{aligned} B_m(P(z); z) &= z - P(z) \tfrac{H_{m-2}(z)}{H_{m-1}(z)} = z - \tfrac{f(z)}{f'(z)} \frac{\frac{L_{m-1}(z)}{f'(z)^{m-1}}}{\frac{L_m(z)}{f'(z)^m}} \\ &= z - f(z) \tfrac{L_{m-1}(z)}{L_m(z)} = K_m(z). \end{aligned} \tag{26}$$

By [6], the closed form of $\{B_m(P(z); z)\}_{m=2}^{\infty}$ is given, (25) is obtained and the fixed-point iteration $z_{n+1} = K_m(z_n)\,, n = 1, 2, \ldots$ has mth order of convergence with

$$\lim_{n\to\infty} \frac{\alpha - z_{n+1}}{(\alpha - z_n)^m} = (-1)^m \frac{\hat{H}_{m-1,m}(\alpha)}{H_{m-1}(\alpha)} = (-1)^m \hat{H}_{m-1,m}(\alpha)$$

since $H_{m-1}(\alpha) = (P')^{m-1}(\alpha) = 1$. □

We shall show that iteration using (5) also give mth order of convergence.

THEOREM 4.2. *Let $f(z)$ be an analytic function with a simple zero at α. Suppose v is a complex constant with $v \neq \alpha$. For each $m \geq 2$, define $U_m(v,z)$ as in (5). Then $U_m(v,z)$ satisfies the expansion*

$$\begin{aligned} U_m(v,z) &= \alpha + \big((-1)^m T_{m-1,m}(z) + \tfrac{(-1)^{m-1}}{\alpha - v} T_{m-2,m-1}(z)\big)(\alpha - z)^m \\ &\quad + \textstyle\sum_{k=m+1}^{\infty} S_k(z)(\alpha - z)^{k+1} \end{aligned}$$

for some function $S_k(z)$, $k \geq m+1$. The iterations

$$z_{n+1} = U_m(v, z_n),\ n = 1, 2, \ldots$$

converge to α and the order of convergence is m.

Proof. From (4), we have $f(z)L_{m-1}(z) = L_m(z)\,(z - K_m(z))$ for $m \geq 2$ and, plugging into (5), then we have

$$\begin{aligned} U_m(v,z) &= z - f(z)\tfrac{(v-z)L_{m-1}(z) + f(z)L_{m-2}(z)}{(v-z)L_m(z) + f(z)L_{m-1}(z)} \\ &= z - f(z)\tfrac{L_{m-1}(z)}{L_m(z)}\ \tfrac{K_{m-1}(z) - v}{K_m(z) - v}. \end{aligned}$$

$K_m(z)$ is rewritten as $K_m(z) = \alpha + \sum_{k=m}^{\infty}(-1)^m T_{m-1,k}(z)(\alpha - z)^k$ where $T_{m-s,k}(z) = \frac{\hat{H}_{m-s,k}(z)}{H_{m-s}(z)}$ for some positive integers m, s and k. Hence

$$\tfrac{K_{m-1}(z) - v}{K_m(z) - v} = 1 + \textstyle\sum_{k=m-1}^{\infty} C_k(z)\,(\alpha - z)^k \tag{27}$$

for some function $C_k(z)$. For $k = m-1$, $C_{m-1}(z) = \frac{(-1)^{m-1}}{\alpha - v} T_{m-2,m-1}(z)$. Substituting (27) to (5), then we have

$$\begin{aligned} U_m(v,z) &= z - f(z)\tfrac{L_{m-1}}{L_m}\ \tfrac{K_{m-1} - v}{K_m - v} \\ &= z - f(z)\tfrac{L_{m-1}}{L_m} + \big(z - f(z)\tfrac{L_{m-1}}{L_m}\big)\textstyle\sum_{k=m-1}^{\infty} C_k(z)\,(\alpha - z)^k \\ &\quad - z\textstyle\sum_{k=m-1}^{\infty} C_k(z)\,(\alpha - z)^k \\ &= \alpha + \textstyle\sum_{k=m}^{\infty}(-1)^m T_{m-1,k}(\alpha - z)^k + \sum_{k=m-1}^{\infty} C_k(z)\,(\alpha - z)^{k+1} \\ &\quad + \big(\textstyle\sum_{k=m}^{\infty}(-1)^m T_{m-1,k}(\alpha - z)^k)\big)\big(\textstyle\sum_{k=m-1}^{\infty} C_k(z)\,(\alpha - z)^k)\big) \\ &= \alpha + \big((-1)^m T_{m-1,m}(z) + \tfrac{(-1)^{m-1}}{\alpha - v} T_{m-2,m-1}(z)\big)(\alpha - z)^m \\ &\quad + \textstyle\sum_{k=m+1}^{\infty} S_k(z)(\alpha - z)^{k+1} \end{aligned}$$

where

$$\sum_{k=m+1}^{\infty} S_k(z)(\alpha - z)^{k+1} = \sum_{k=m+1}^{\infty} \big((-1)^m T_{m-1,k} + C_{k-1}(z)\big)(\alpha - z)^{k+1} + \big(\sum_{k=m}^{\infty}(-1)^m T_{m-1,k}(\alpha - z)^k\big)\big(\sum_{k=m-1}^{\infty} C_k(z)\,(\alpha - z)^k\big).$$

Hence, the iterations $z_{n+1} = U_m(v, z_n), n \geq 1$ converge to α with order m. □

References

[1] W.F. Ford and J.A. Pennline, Accelerated convergence in Newton's method, *SIAM Rev.,* 1996, **38**(4):658-659

[2] J. Gerlach, Accelerated convergence in Newton's method, *SIAM Rev.,* 1994, **36**(2): 272-276

[3] E. Halley, A new, exact, and easy method of finding roots of any equations generally, and that without any previous eduction (Latin), *Philos. Trans. Roy. Soc. London,*, 1694 **18**:136-148.

[4] B. Kalantari, Generalization of Taylor's theorem and Newton's method via a new family of determinantal interpolation formulas and its applications , *J. Comput. Appl. Math.,* 2000, **126**(1-2): 287-318.

[5] B. Kalantari and J. Gerlach, Newton's method and generation of a determinantal family of iteration functions, *J. Comput. Appl. Math.,* 2000, **116**(1): 195-200.

[6] B. Kalantari, I. Kalantari and R. Zaare-Nahandi, A basic family of iteration functions for polynomial root finding and its characterizations, *J. Comput. Appl. Math.,* 1997, **80**(2): 209-226.

[7] B. Kalantari and Y. Jin, On extraneous fixed-points of the basic family of iteration functions, *BIT*, 2003, **43**(2): 453-458.

[8] B. Kalantari, An infinite family of bounds on zeros of analytic functions and relationship to Smale's bound, *Math. of Computations*, 2005, **74**(250): 841-852

[9] S. Kulik, On the Laguerre method for separating the roots of algebraic equations, *Proceedings of AMS*, 1957 **8**(5): 841-843.

[10] E. N. Laguerre, Oeuvres de Laguerre, *Gauthier-Villars*, Paris, 1880, **1**: 87-103.

[11] T. Pomentale, A class of iterative methods for holomorphic functions, *Numer. Math.*, 1971 **18**(3): 193-203.

Department of Mathematics
Kyonggi University
Suwon 443-760, Republic of Korea
E-mail: ymham@kyonggi.ac.kr

Department of Mathematics
Sungkyunkwan University
Suwon 440-746, Republic of Korea
E-mail: sglee@skku.edu

Department of Mathematics
University of Northern Iowa
Cedar Falls, IA 50614-0506, USA
E-mail: jerry.ridenhour@uni.edu