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JOINT WEAK SUBNORMALITY OF OPERATORS

Jun Ik Lee* and Sang Hoon Lee**

Abstract. We introduce jointly weak subnormal operators. It is shown that if $\mathbf{T} = (T_1, T_2)$ is subnormal then $\mathbf{T}$ is weakly subnormal and if f $\mathbf{T} = (T_1, T_2)$ is weakly subnormal then $\mathbf{T}$ is hyponormal. We discuss the flatness of weak subnormal operators.

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded linear operators on $\mathcal{H}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *normal* if $T^*T = TT^*$, *hyponormal* if $T^*T \geq TT^*$ and *subnormal* if $T = N|_{\mathcal{H}}$, where N is normal on some Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. Thus the operator T is subnormal if and only if there exist operators A and B such that $\widehat{T} := \begin{pmatrix} T & A \\ 0 & B \end{pmatrix}$ is normal, i.e.,

$$(1.1) \qquad \begin{cases} [T^*, T] := T^*T - TT^* = AA^* \\ A^*T = BA^* \\ [B^*, B] + A^*A = 0. \end{cases}$$

An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *weakly subnormal* ([5]) if there exist operators $A \in \mathcal{B}(\mathcal{H}, \mathcal{H}')$ and $B \in \mathcal{B}(\mathcal{H}')$ such that the first two conditions in (1.1) hold: $[T^*, T] := T^*T - TT^* = AA^*$ and $A^*T = BA^*$, or equivalently, there is an extension $\widehat{T}$ of T such that

$$(1.2) \qquad \widehat{T}^*\widehat{T}f = \widehat{T}\widehat{T}^*f \quad \text{for all} \quad f \in \mathcal{H}.$$

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The operator $\widehat{T}$ is said to be a *partially normal extension* (briefly, p.n.e.) of T. Note that the condition (1.2) implies $||\widehat{T}f|| = ||\widehat{T}^*f||$ for all $f \in \mathcal{H}$, and that if (1.2) holds for all $f \in \mathcal{H} \oplus \mathcal{H}'$, then $\widehat{T}$ becomes normal, so T is in that case subnormal. We also say that $\widehat{T} \in \mathcal{B}(\mathcal{K})$ is a *minimal partially normal extension* (briefly, m.p.n.e.) of a weakly subnormal operator T if $\mathcal{K}$ has no proper subspace containing $\mathcal{H}$ to which the restriction of $\widehat{T}$ is also a partially normal extension of T. It is known ([5, Lemma 2.5]) that if $\widehat{T}$ is a partially normal extension of $T \in \mathcal{B}(\mathcal{H})$ on $\mathcal{K}$ then $\widehat{T}$ is minimal if and only if $\mathcal{K} = \bigvee\{\widehat{T}^{*k}h : h \in \mathcal{H}, \quad k = 0, 1\}$. Clearly, subnormal $\Longrightarrow$ weakly subnormal $\Longrightarrow$ hyponormal; however, the converses are not true in general (cf. [5]).

2. Weak subnormality

For $S, T \in \mathcal{B}(\mathcal{H})$, let $[S, T] := ST - TS$. We say that an n-tuple $\mathbf{T} = (T_1, \cdots, T_n)$ of operators on $\mathcal{H}$ is (jointly) *hyponormal* if the operator matrix

$$[\mathbf{T}^*, \mathbf{T}] := \begin{pmatrix} [T_1^*, T_1] & [T_2^*, T_1] & \cdots & [T_n^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] & \cdots & [T_n^*, T_2] \\ \vdots & \vdots & \ddots & \vdots \\ [T_1^*, T_n] & [T_2^*, T_n] & \cdots & [T_n^*, T_n] \end{pmatrix}$$

is positive on the direct sum of n copies of $\mathcal{H}$ (cf. [1], [6]). The n-tuple $\mathbf{T}$ is said to be *normal* if $\mathbf{T}$ is commuting and each T_i is normal, and $\mathbf{T}$ is *subnormal* if $\mathbf{T}$ is the restriction of a normal n-tuple to a common invariant subspace. Clearly, normal $\Rightarrow$ subnormal $\Rightarrow$ hyponormal. But the converses are not true in general. We now introduce:

Definition 2.1. An n-tuple $\mathbf{T} = (T_1, \cdots, T_n)$ of operators on $\mathcal{H}$ is said to be *jointly weak subnormal* if each T_i is weakly subnormal and has a doubly commuting partially normal extension.

If $\mathbf{T} = (T_1, \cdots, T_n)$ is weakly subnormal, then there exist $\widehat{\mathbf{T}} = (\widehat{T_1}, \cdots, \widehat{T_n})$ such that (i) $\widehat{T}_i\widehat{T}_j = \widehat{T}_j\widehat{T}_i$, (ii) $\widehat{T}_i\widehat{T}_j^{\,*} = \widehat{T}_j^{\,*}\widehat{T}_i$, (iii) $\widehat{T}_i = p.n.e(T_i)$ for each i, j. We then have:

Theorem 2.2. (i) *If $\mathbf{T} = (T_1, T_2)$ is subnormal, then $\mathbf{T}$ is weakly subnormal.*

(ii) *If $\mathbf{T} = (T_1, T_2)$ is weakly subnormal, then $\mathbf{T}$ is hyponormal.*

Proof. (i) If $\mathbf{T}$ is subnormal, then there is a commuting normal extension of $\mathbf{T}$. By Fuglede's Theorem, it clearly is a double commuting extension. For (ii), observe that

$$[\mathbf{T}^*,\mathbf{T}] = \begin{pmatrix} [T_1^*,T_1] & [T_2^*,T_1] \\ [T_1^*,T_2] & [T_2^*,T_2] \end{pmatrix} = \begin{pmatrix} A_1A_1^* & A_1A_2^* \\ A_2A_1^* & A_2A_2^* \end{pmatrix},$$

where $\widehat{T_i} := \begin{pmatrix} T_i & A_i \\ 0 & B_i \end{pmatrix}$ is a p.n.e of T_i for $i = 1, 2$. By [5], we can take $A_1 := [T_1^*, T_1]^{\frac{1}{2}}$. Using Smul'jan's theorem ([9]) which stats that if $A \geq 0$ and $B = A^{\frac{1}{2}}V$, then $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0 \Leftrightarrow C \geq V^*V$, we can see that $[\mathbf{T}^*,\mathbf{T}] \geq 0$. □

A tuple $\mathbf{T} = (T_1, \cdots, T_n)$ is said to be *jointly quasinormal* if T_i commutes with $T_j^*T_j$ for all i, j ([7]), which is equivalent to requiring that the different parts of the polar decompositions of the individual operators to all commute. Observe that $[\mathbf{T}^*,\mathbf{T}]\mathbf{T} = 0$ for quasinormal operator $\mathbf{T}$, and hence Ker$[\mathbf{T}^*,\mathbf{T}]$ is invariant for $\mathbf{T}$. For a single weakly subnormal operator, the same property hold ([5]).

3. Flatness

If $A = A^* \in \mathcal{B}(\mathcal{H}_1)$, then an operator matrix (whose entries have possibly infinite-matrix representations)

$$\widetilde{A} = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} : \mathcal{H}_1 \oplus \mathcal{H}_2 \longrightarrow \mathcal{H}_1 \oplus \mathcal{H}_2$$

is called an *extension* of A. If A is of finite rank, we refer to a rank-preserving extension $\widetilde{A}$ of A as a flat extension of A. It is known ([CF2]) that if A is of finite rank and $A \geq 0$, then $\widetilde{A}$ is a flat extension of A if and only if $\widetilde{A}$ is of the form

$$\widetilde{A} = \begin{pmatrix} A & AV \\ V^*A & V^*AV \end{pmatrix}$$

for an operator $V : \mathcal{H}_2 \longrightarrow \mathcal{H}_1$. Moreover $\widetilde{A}$ is positive whenever A is positive. We shall introduce the notion of flatness for a pair of operators.

Definition 3.1. Let $\mathbf{T} = (T_1, T_2)$ be a pair of operators on $\mathcal{H}$. Then we shall say that $\mathbf{T}$ is a *flat pair* if $[\mathbf{T}^*,\mathbf{T}]$ is flat relative to $[T_1^*, T_1]$ or $[T_2^*, T_2]$.

Remark 3.2. ([4]) The following facts are evident from the definition.

(i) Flatness of (T_1, T_2) is not affected by permuting the operators T_i.
(ii) If (T_1, T_2) is flat, then so is $(\lambda_1 T_1, \lambda_2 T_2)$ for every $\lambda_1, \lambda_2 \in \mathbb{C}$.
(iii) If (T_1, T_2) is flat, then so is $(T_1 - \lambda_1 I, T_2 - \lambda_2 I)$ for every $\lambda_1, \lambda_2 \in \mathbb{C}$.
(iv) If $S \in \mathcal{B}(\mathcal{H})$ is hyponormal with finite-rank self-commutator then $(\mu_1 S - \mu_2 I, \lambda_1 S - \lambda_2 I)$ is flat for every $\mu_1, \mu_2, \lambda_1, \lambda_2 \in \mathbb{C}$.
(v) If T_1 or T_2 is hyponormal and if (T_1, T_2) is flat, then (T_1, T_2) is hyponormal.

Proposition 3.3. *(cf. [4]) Let $A \geq 0$ be of finite rank. Then $\widetilde{A} = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ is flat if and only if $R(B) \subseteq R(A)$ and $C = B^* A^{\#} B$, where $A^{\#}$ is the Moore-Penrose inverse of A, in the sense that $AA^{\#}A = A, A^{\#}AA^{\#} = A^{\#}, (A^{\#}A)^{\#} = A^{\#}A$, and $(AA^{\#})^{\#} = AA^{\#}$.*

Proof. Write $A = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix} : R(A) \oplus N(A) \longrightarrow R(A) \oplus N(A)$, where A_0 is invertible. Then $A^{\#} = \begin{pmatrix} A_0^{-1} & 0 \\ 0 & 0 \end{pmatrix}$. If $\widetilde{A}$ is flat, it follows from [Smu] that there exists $V : \mathcal{H}_2 \longrightarrow R(A)$ such that $B = AV$. Since $R(V) \subseteq R(A)$, V is uniquely determined by $V = A^{\#}B$, so $C = V^*AV = B^*A^{\#^*}AA^{\#}B = B^*A^{\#}B$. The converse is trivial.(cf. [4, Lemma1.2]). □

Corollary 3.4. *If $\mathbf{T} = (T_1, T_2)$ is a hyponormal pair and if $[T_1^*, T_1]$ is of finite rank, then $\mathbf{T}$ is flat if and only if $[T_2^*, T_2] = [T_1^*, T_2][T_1^*, T_1]^{\#}[T_2^*, T_1]$.*

Proof. This follows from Proposition 3.3. □

Theorem 3.5. *Every weakly subnormal pair $\mathbf{T} = (T_1, T_2)$ satisfying the inclusion $R([T_2^*, T_2]) \subseteq R([T_1^*, T_1])$ and rank $[T_1^*, T_1] < \infty$ is flat.*

Proof. Let $\widehat{T_i} := \begin{pmatrix} T_i & A_i \\ 0 & B_i \end{pmatrix}$ be a p.n.e of T_i for $i = 1, 2$. Then

$$[\mathbf{T}^*, \mathbf{T}] = \begin{pmatrix} [T_1^*, T_1] & [T_2^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] \end{pmatrix} = \begin{pmatrix} A_1 A_1^* & A_1 A_2^* \\ A_2 A_1^* & A_2 A_2^* \end{pmatrix}.$$

Since $R([T_2^*, T_2]) \subseteq R([T_1^*, T_1])$, it follows that $R(A_2) \subseteq R(A_1)$ and $rank(A_2 A_2^*) \leq rank(A_1 A_1^*) < \infty$. Since $A_1 A_1^*$ is of finite rank, $A_1 A_1^*$ has Moore-Penrose inverse $(A_1 A_1^*)^{\#}$, and hence so have both A_1 and A_1^*. Moreover, $(A_1 A_1^*)^{\#} = (A_1^{\#})^* A_1^{\#}$. Since $(A_1^{\#} A_1)^* = A_1^{\#} A_1$, it follows that

$$\begin{aligned}(A_2A_1^*)(A_1A_1^*)^{\#}(A_1A_2^*) &= A_2(A_1^*A_1^{\#*})(A_1^{\#}A_1)A_2^* \\ &= A_2(A_1^{\#}A_1A_1^{\#}A_1)A_2^* \\ &= A_2(A_1^{\#}A_1)A_2^*.\end{aligned}$$

Since we can take $A_1 := [T_1^*, T_1]^{\frac{1}{2}}$ and $A_2 := [T_2^*, T_2]^{\frac{1}{2}}$ by [5], we have $R(A_2^*) \subseteq R(A_1^*)$. Since $A_1^{\#}A_1$ is the projection onto $R(A_1^*)$, it follows that $A_2(A_1^{\#}A_1)A_2^* = A_2A_2^*$, which implies that

$$[T_2^*, T_2] = [T_1^*, T_2][T_1^*, T_1]^{\#}[T_2^*, T_1].$$

Therefore by Corollary 3.4, $\mathbf{T} = (T_1, T_2)$ is flat. □

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Department of Mathematics Education
Sangmyung University
Seoul 110-743, Republic of Korea
E-mail: jilee@smu.ac.kr

**
Department of Mathematics
Chungnam National University
Daejeon 305-764, Republic of Korea
E-mail: shlee@math.cnu.ac.kr