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CHARACTERIZATIONS OF THE WEIBULL DISTRIBUTION BY THE INDEPENDENCE OF RECORD VALUES

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Abstract. This paper presents some characterizations of the Weibull distribution by the independence of record values. We prove that $X \sim Weibull(1, \alpha)$, $\alpha > 0$ if and only if $\frac{X_{U(n+1)}}{X_{U(n+1)} - X_{U(n)}}$ and $X_{U(n+1)}$ for $n \geq 1$ are independent. We show that $X \sim Weibull(1, \alpha)$, $\alpha > 0$ if and only if $\frac{X_{U(n)}}{X_{U(n+1)} - X_{U(n)}}$ and $X_{U(n+1)}$ for $n \geq 1$ are independent. And we establish that $X \sim Weibull(1, \alpha)$, $\alpha > 0$ if and only if $\frac{X_{U(n+1)} + X_{U(n)}}{X_{U(n+1)} - X_{U(n)}}$ and $X_{U(n+1)}$ for $n \geq 1$ are independent.

1. Introduction

The record value model was introduced by Chandler [4]. Let X_1, $X_2, \cdots$ be a sequence of independent and identically distributed (i.i.d.) random variables with a cumulative distribution function (cdf) $F(x)$ and a probability density function (pdf) $f(x)$. Suppose $U(1) = 1$ and $U(n) = \min\{j | j > U(n-1)\,,\, X_j > X_{U(n-1)}\,,\, n \geq 2\}$. Then the indices $\{U(n), n \geq 1\}$ are said to be the record times. We say that X_1 is a first upper record value and $\{X_{U(n)},\, n \geq 1\}$ are upper record values.

The cdf of the Weibull distribution is

$$F(x) = \begin{cases} 1 - e^{-x^{\alpha}},\ x > 0,\ \alpha > 0 \\ 0, \text{ otherwise} \end{cases} \tag{1}$$

A notation that designates that X has the cdf (1) is $X \sim Weibull(1, \alpha)$.

Some characterizations by the independence of the upper record values are known. Ahsanullah [2] characterized that $F(x) = 1 - e^{-\frac{x}{\sigma}}$, $x > 0$,

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$\sigma > 0$ if and only if $X_{U(n)} - X_{U(m)}$ and $X_{U(m)}$, $0 < m < n$ are independent. Moreover Ahsanullah [2, 3] characterized if $F(x) = 1 - \left(\frac{\alpha}{x}\right)^{\beta}$, $x \geq \alpha$, α, $\beta > 0$, then $\frac{X_{U(n)}}{X_{U(m)}}$ and $X_{U(m)}$, $0 < m < n$ are independent. Above results are characterized by the simple difference form or the simple quotient form of the upper record values. By the expansion form of denominator, we can get the Weibull distribution.

In this paper, we will give some characterizations of the Weibull distribution by the independence of upper record values.

2. Main theorems

THEOREM 2.1. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with cdf $F(x)$ which is an absolutely continuous with pdf $f(x)$ and $F(0) = 0$ and $F(x) < 1$ for $x > 0$. Then $F(x) = 1 - e^{-x^{\alpha}}$ for $x > 0$ and $\alpha > 0$ if and only if $\frac{X_{U(n+1)}}{X_{U(n+1)} - X_{U(n)}}$ and $X_{U(n+1)}$ are independent for $n \geq 1$.*

Proof. If $F(x) = 1 - e^{-x^{\alpha}}$ for $x > 0$ and $\alpha > 0$, then the joint pdf $f_{n,n+1}(x, y)$ of $X_{U(n)}$ and $X_{U(n+1)}$ is

$$f_{n,n+1}(x, y) = \frac{\alpha^2 x^{\alpha n - 1} y^{\alpha - 1} e^{-y^{\alpha}}}{\Gamma(n)}$$

for $0 < x < y$, $\alpha > 0$ and $n \geq 1$.

Consider the functions $V = \frac{X_{U(n+1)}}{X_{U(n+1)} - X_{U(n)}}$ and $W = X_{U(n+1)}$. It follows that $x_{U(n)} = \frac{(v-1)w}{v}$, $x_{U(n+1)} = w$ and $|J| = \frac{w}{v^2}$. Thus we can write the joint pdf $f_{V,W}(v, w)$ of V and W as

$$f_{V,W}(v, w) = \frac{\alpha^2 (v-1)^{\alpha n - 1} w^{\alpha(n+1) - 1} e^{-w^{\alpha}}}{\Gamma(n) v^{\alpha n + 1}} \tag{2}$$

for $v > 1$, $w > 0$, $\alpha > 0$ and $n \geq 1$.

The marginal pdf $f_V(v)$ of V is given by

$$\begin{aligned} f_V(v) &= \int_0^{\infty} f_{V,W}(v, w)\, dw \\ &= \frac{\alpha^2 (v-1)^{\alpha n - 1}}{\Gamma(n) v^{\alpha n + 1}} \int_0^{\infty} w^{\alpha(n+1) - 1} e^{-w^{\alpha}}\, dw \\ &= \frac{\alpha n (v-1)^{\alpha n - 1}}{v^{\alpha n + 1}} \end{aligned} \tag{3}$$

for $v > 1$, $\alpha > 0$ and $n \geq 1$.

Also, the pdf $f_W(w)$ of W is given by

$$f_W(w) = \frac{(R(w))^n f(w)}{\Gamma(n+1)} = \frac{\alpha w^{\alpha(n+1)-1} e^{-w^\alpha}}{\Gamma(n+1)} \tag{4}$$

for $w > 0$, $\alpha > 0$ and $n \geq 1$.

From (2), (3) and (4), we obtain $f_V(v) f_W(w) = f_{V,W}(v, w)$.

Hence $V = \frac{X_{U(n+1)}}{X_{U(n+1)} - X_{U(n)}}$ and $W = X_{U(n+1)}$ are independent for $n \geq 1$.

Now, the joint pdf $f_{n,n+1}(x, y)$ of $X_{U(n)}$ and $X_{U(n+1)}$ is

$$f_{n,n+1}(x, y) = \frac{(R(x))^{n-1} r(x) f(y)}{\Gamma(n)}$$

for $0 < x < y$, $\alpha > 0$ and $n \geq 1$, where $R(x) = -ln(1 - F(x))$ and $r(x) = \frac{d}{dx}\big(R(x)\big) = \frac{f(x)}{1-F(x)}$.

Let us use the transformation $V = \frac{X_{U(n+1)}}{X_{U(n+1)} - X_{U(n)}}$ and $W = X_{U(n+1)}$. The Jacobian of the transformation is $|J| = \frac{w}{v^2}$. Thus we can write the joint pdf $f_{V,W}(v, w)$ of V and W as

$$f_{V,W}(v, w) = \frac{w}{\Gamma(n) v^2} \left(R\left(\frac{(v-1)w}{v} \right) \right)^{n-1} r\left(\frac{(v-1)w}{v} \right) f(w) \tag{5}$$

for $v > 1$, $w > 0$ and $n \geq 1$.

The pdf $f_W(w)$ of W is given by

$$f_W(w) = \frac{(R(w))^n f(w)}{\Gamma(n+1)} \tag{6}$$

for $w > 0$ and $n \geq 1$.

From (5) and (6), we obtain the pdf $f_V(v)$ of V

$$f_V(v) = \frac{n\, w \left(R\left(\frac{(v-1)w}{v} \right) \right)^{n-1} r\left(\frac{(v-1)w}{v} \right)}{v^2\, (R(w))^n}$$

for $v > 1$, $w > 0$ and $n \geq 1$.

That is,

$$f_V(v) = \frac{\partial}{\partial v} \left(\left(R\left(\frac{(v-1)w}{v} \right) / R(w) \right)^n \right)$$

where $R(x) = -ln(1 - F(x))$ and $r(x) = \frac{d}{dx}\big(R(x)\big) = \frac{f(x)}{1-F(x)}$.

Since V and $W$are independent, we must have

$$R\left(\frac{(v-1)w}{v} \right) = R\left(\frac{v-1}{v} \right) R(w). \tag{7}$$

By the functional equations (see, [1]), the only continous solution of (7) with the boundary condition $R(0) = 0$ is

$$R(x) = x^{\alpha}$$

for $x > 0$ and $\alpha > 0$. Thus we have

$$F(x) = 1 - e^{-x^{\alpha}}$$

for $x > 0$ and $\alpha > 0$.

This completes the proof. □

THEOREM 2.2. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with cdf $F(x)$ which is an absolutely continuous with pdf $f(x)$ and $F(0) = 0$ and $F(x) < 1$ for $x > 0$. Then $F(x) = 1 - e^{-x^{\alpha}}$ for $x > 0$ and $\alpha > 0$ if and only if $\frac{X_{U(n)}}{X_{U(n+1)} - X_{U(n)}}$ and $X_{U(n+1)}$ are independent for $n \geq 1$.*

Proof. In the same manner as Theorem 2.1, we consider the functions $V = \frac{X_{U(n)}}{X_{U(n+1)} - X_{U(n)}}$ and $W = X_{U(n+1)}$. It follows that $x_{U(n)} = \frac{vw}{(v+1)}$, $x_{U(n+1)} = w$ and $|J| = \frac{w}{(v+1)^2}$. Thus we can write the joint pdf $f_{V,W}(v, w)$ of V and W as

$$f_{V,W}(v, w) = \frac{\alpha^2 v^{\alpha n - 1} w^{\alpha(n+1)-1} e^{-w^{\alpha}}}{\Gamma(n)(v+1)^{\alpha n + 1}} \tag{8}$$

for $v > 0$, $w > 0$, $\alpha > 0$ and $n \geq 1$.

The marginal pdf $f_V(v)$ of V is given by

$$f_V(v) = \frac{\alpha n v^{\alpha n - 1}}{(v+1)^{\alpha n + 1}} \tag{9}$$

for $v > 0$, $\alpha > 0$ and $n \geq 1$.

Also, the pdf $f_W(w)$ of W is given by

$$f_W(w) = \frac{\alpha w^{\alpha(n+1)-1} e^{-w^{\alpha}}}{\Gamma(n+1)} \tag{10}$$

for $w > 0$, $\alpha > 0$ and $n \geq 1$.

From (8), (9) and (10), we obtain $f_V(v) f_W(w) = f_{V,W}(v, w)$.

Hence $V = \frac{X_{U(n)}}{X_{U(n+1)} - X_{U(n)}}$ and $W = X_{U(n+1)}$ are independent for $n \geq 1$.

Now, the joint pdf $f_{n,n+1}(x, y)$ of $X_{U(n)}$ and $X_{U(n+1)}$ is

$$f_{n,n+1}(x, y) = \frac{(R(x))^{n-1} r(x) f(y)}{\Gamma(n)}$$

for $0 < x < y$, $\alpha > 0$ and $n \geq 1$.

Let us use the transformation $V = \frac{X_{U(n)}}{X_{U(n+1)} - X_{U(n)}}$ and $W = X_{U(n+1)}$. The Jacobian of the transformation is $|J| = \frac{w}{(v+1)^2}$. Thus we can write the joint pdf $f_{V,W}(v,w)$ of V and W as

$$f_{V,W}(v,w) = \frac{w\left(R\left(\frac{vw}{(v+1)}\right)\right)^{n-1} r\left(\frac{vw}{(v+1)}\right) f(w)}{\Gamma(n)(v+1)^2} \tag{11}$$

for $v > 0$, $w > 0$ and $n \geq 1$.

The pdf $f_W(w)$ of W is given by

$$f_W(w) = \frac{(R(w))^n f(w)}{\Gamma(n+1)} \tag{12}$$

for $w > 0$ and $n \geq 1$.

From (11) and (12), we obtain the pdf $f_V(v)$ of V

$$f_V(v) = \frac{n\, w\left(R\left(\frac{vw}{(v+1)}\right)\right)^{n-1} r\left(\frac{vw}{(v+1)}\right)}{(v+1)^2\,(R(w))^n}$$

for $v > 0$, $w > 0$ and $n \geq 1$.

That is,

$$f_V(v) = \frac{\partial}{\partial v}\left(\left(R\left(\frac{vw}{(v+1)}\right)/R(w)\right)^n\right)$$

where $R(x) = -ln(1 - F(x))$ and $r(x) = \frac{d}{dx}\big(R(x)\big) = \frac{f(x)}{1-F(x)}$.

Since V and $W$are independent, we must have

$$R\left(\frac{vw}{(v+1)}\right) = R\left(\frac{v}{v+1}\right) R(w). \tag{13}$$

By the functional equations (see, [1]), the only continous solution of (13) with the boundary condition $R(0) = 0$ is

$$R(x) = x^\alpha$$

for $x > 0$ and $\alpha > 0$. Thus we have

$$F(x) = 1 - e^{-x^\alpha}$$

for $x > 0$ and $\alpha > 0$.

This completes the proof. □

Theorem 2.3. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with cdf $F(x)$ which is an absolutely continuous with pdf $f(x)$ and $F(0) = 0$ and $F(x) < 1$ for $x > 0$. Then $F(x) = 1 - e^{-x^\alpha}$ for $x > 0$ and $\alpha > 0$ if and only if $\frac{X_{U(n+1)} + X_{U(n)}}{X_{U(n+1)} - X_{U(n)}}$ and $X_{U(n+1)}$ are independent for $n \geq 1$.*

Proof. In the same manner as Theorem 2.1, we consider the functions $V = \frac{X_{U(n+1)}+X_{U(n)}}{X_{U(n+1)}-X_{U(n)}}$ and $W = X_{U(n+1)}$. It follows that $x_{U(n)} = \frac{(v-1)w}{(v+1)}$, $x_{U(n+1)} = w$ and $|J| = \frac{2w}{(v+1)^2}$. Thus we can write the joint pdf $f_{V,W}(v,w)$ of V and W as

$$f_{V,W}(v,w) = \frac{2\alpha^2(v-1)^{\alpha n-1}w^{\alpha(n+1)-1}e^{-w^\alpha}}{\Gamma(n)(v+1)^{\alpha n+1}} \tag{14}$$

for $v > 1$, $w > 0$, $\alpha > 0$ and $n \geq 1$.

The marginal pdf $f_V(v)$ of V is given by

$$f_V(v) = \frac{2\alpha n(v-1)^{\alpha n-1}}{(v+1)^{\alpha n+1}} \tag{15}$$

for $v > 1$, $\alpha > 0$ and $n \geq 1$.

Also, the pdf $f_W(w)$ of W is given by

$$f_W(w) = \frac{\alpha w^{\alpha(n+1)-1}e^{-w^\alpha}}{\Gamma(n+1)} \tag{16}$$

for $w > 0$, $\alpha > 0$ and $n \geq 1$.

From (14), (15) and (16), we obtain $f_V(v)f_W(w) = f_{V,W}(v,w)$.

Hence $V = \frac{X_{U(n+1)}+X_{U(n)}}{X_{U(n+1)}-X_{U(n)}}$ and $W = X_{U(n+1)}$ are independent for $n \geq 1$.

Now, the joint pdf $f_{n,n+1}(x,y)$ of $X_{U(n)}$ and $X_{U(n+1)}$ is

$$f_{n,n+1}(x,y) = \frac{(R(x))^{n-1}r(x)f(y)}{\Gamma(n)}$$

for $0 < x < y$, $\alpha > 0$ and $n \geq 1$.

Let us use the transformation $V = \frac{X_{U(n+1)}+X_{U(n)}}{X_{U(n+1)}-X_{U(n)}}$ and $W = X_{U(n+1)}$. The Jacobian of the transformation is $|J| = \frac{2w}{(v+1)^2}$. Thus we can write the joint pdf $f_{V,W}(v,w)$ of V and W as

$$f_{V,W}(v,w) = \frac{2w\left(R\left(\frac{(v-1)w}{(v+1)}\right)\right)^{n-1} r\left(\frac{(v-1)w}{(v+1)}\right) f(w)}{\Gamma(n)(v+1)^2} \tag{17}$$

for $v > 1$, $w > 0$ and $n \geq 1$.

The pdf $f_W(w)$ of W is given by

$$f_W(w) = \frac{(R(w))^n f(w)}{\Gamma(n+1)} \tag{18}$$

for $w > 0$ and $n \geq 1$.

From (17) and (18), we obtain the pdf $f_V(v)$ of V

$$f_V(v) = \frac{2nw\left(R\left(\frac{(v-1)w}{(v+1)}\right)\right)^{n-1} r\left(\frac{(v-1)w}{(v+1)}\right)}{(v+1)^2\,(R(w))^n}$$

for $v > 1$, $w > 0$ and $n \geq 1$.

That is,

$$f_V(v) = \frac{\partial}{\partial v}\left(-\left(R\left(\frac{(v-1)w}{(v+1)}\right)/R(w)\right)^n\right)$$

where $R(x) = -ln(1 - F(x))$ and $r(x) = \frac{d}{dx}\big(R(x)\big) = \frac{f(x)}{1-F(x)}$.

Since V and $W$are independent, we must have

$$R\left(\frac{(v-1)w}{(v+1)}\right) = R\left(\frac{v-1}{v+1}\right)R(w). \tag{19}$$

By the functional equations (see, [1]), the only continous solution of (19) with the boundary condition $R(0) = 0$ is

$$R(x) = x^{\alpha}$$

for $x > 0$ and $\alpha > 0$. Thus we have

$$F(x) = 1 - e^{-x^{\alpha}}$$

for $x > 0$ and $\alpha > 0$.

This completes the proof. □

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