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NOTE ON THE OPERATOR $\widehat{P}$ ON $L^p(\partial D)$

Ki Seong Choi*

Abstract. Let ∂D be the boundary of the open unit disk D in the complex plane and $L^p(\partial D)$ the class of all complex, Lebesgue measurable function f for which $\{\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(\theta)|^p d\theta\}^{1/p} < \infty$. Let P be the orthogonal projection from $L^p(\partial D)$ onto $\cap_{n<0}\ker a_n$. For $f \in L^1(\partial D)$, $\hat{f}(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t-\theta)f(\theta)d\theta$ is the harmonic extension of f. Let $\widehat{P}$ be the composition of P with the harmonic extension. In this paper, we will show that if $1 < p < \infty$, then $\widehat{P} : L^p(\partial D) \to H^p(D)$ is bounded. In particular, we will show that $\widehat{P}$ is unbounded on $L^\infty(\partial D)$.

1. Introduction

Let D be the open unit disk in the complex plane and ∂D the boundary of the open unit disk D. Let $d\theta$ be the arc-length measure on ∂D. We define $L^p(\partial D)$, for $1 \le p < \infty$, to be the class of all complex, Lebesgue measurable, 2π periodic functions on R for which the norm

$$\| f \|_p = \left[\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(\theta)|^p d\theta\right]^{1/p}$$

is finite.

The function

$$P_r(\theta) = \sum_{-\infty}^{\infty} r^{|n|}e^{in\theta} \ (0 \le r < 1,\ \theta \ real)$$

is called the Poisson kernel. If $z = re^{it}$, we can easily show that

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$$P_r(t-\theta) = Re\frac{e^{i\theta}+z}{e^{i\theta}-z} = \frac{1-r^2}{1-2r\cos(t-\theta)+r^2}.$$

If $f \in L^1(\partial D)$ and

$$\hat{f}(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t-\theta)f(\theta)d\theta,$$

then the function $\hat{f}$ so defined in D is called the Poisson integral of f. It follows that $\hat{f}(z)$ is harmonic in D for every $f \in L^1(\partial D)$ (See Theorem 11.8).

For any $f \in L^1(\partial D)$, we define the Fourier coefficients of f by the formula

$$a_n(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}d\theta \ (n \in Z)$$

where Z is the set of all integers. The Fourier series of f is

$$\sum_{-\infty}^{\infty} a_n(f)e^{in\theta}.$$

It is easy to prove that $a_n(f) \to 0$ as $n \to \infty$ for every $f \in L^1(\partial D)$. Since $L^2(\partial D) \subset L^1(\partial D)$, $a_n(f)$ can be applied to every $f \in L^2(\partial D)$. The Riesz-Fischer theorem asserts that if $\{c_n\}$ is a sequence of complex numbers such that

$$\sum_{-\infty}^{\infty} |c_n|^2 < \infty,$$

then there exists an $f \in L^2(\partial D)$ such that

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}d\theta \ (n \in Z).$$

Since a_n is bounded linear functional on $L^p(\partial D)$ for any fixed n,

$$\cap_{n<0} \ker a_n = \cap_{n<0}\{f \in L^p(\partial D) : a_n(f) = 0\}$$

is a closed subspace of $L^p(\partial D)$ and hence a Banach space.

Let $p \geq 1$. Since $\cap_{n<0} \ker a_n$ is a closed subspace of $L^p(\partial D)$, there exists an orthogonal projection from $L^p(\partial D)$ onto $\cap_{n<0} \ker a_n$. We shall denote this projection by P.

If $a_n(f)$ are the Fourier coefficients of f, then

$$\hat{f}(z) = \sum_{n=0}^{\infty} a_n(f) z^n + \sum_{n=1}^{\infty} a_{-n}(f) \overline{z}^n.$$

This implies that if $f \in \cap_{n<0} \ker a_n$, then $\hat{f}(z)$ is analytic in D.

Let the function f_r on ∂D, associated to f in D be given by

$$f_r(e^{i\theta}) = f(re^{i\theta}) \ (0 \leq r < 1).$$

We set

$$\| f_r \|_p = \left[\int_{-\pi}^{\pi} |f_r(\theta)|^p d\theta \right]^{1/p} \quad (0 < p < \infty),$$
$$\| f_r \|_\infty = \sup_\theta |f(re^{i\theta})|.$$

Let $H^p(D)$ be the space of analytic functions on D which are harmonic extensions of functions in $\cap_{n<0} \ker a_n$ such that

$$\| f \|_p = \sup\{\| f_r \|_p \colon 0 \leq r < 1\} < \infty.$$

Let $\widehat{P}$ be the composition of P with the harmonic extension, that is, $\widehat{P}f = \widehat{Pf}$ for all $f \in L^p(\partial D)$. In section II, we will show that if $1 < p < \infty$, then $\widehat{P} : L^p(\partial D) \to H^p(D)$ is bounded. In particular, we will show that $\widehat{P}$ is unbounded on $L^\infty(\partial D)$ in Section III.

2. Bounded operator $\widehat{P}$ on $L^p(\partial D)$, $1 \leq p < \infty$

THEOREM 2.1. *If $f \in L^1(\partial D)$, then*

$$\lim_{r \to 1} \hat{f}(re^{i\theta}) = f(e^{i\theta}) \ a.e \ \theta.$$

Proof. See Corollay of Theorem 11.2 in [10]. □

THEOREM 2.2. *If $1 \leq p \leq \infty$ and $f \in L^p(\partial D)$, then*

$$\| \hat{f}_r \|_p \leq \| f \|_p$$

where $0 \leq r < 1$. If $1 \leq p < \infty$, then

$$\lim_{r \to 1} \| \hat{f}_r - f \|_p = 0.$$

Proof. See Theorem 3.3.4 in [10]. □

Lemma 2.3. *If $f, g \in L^p(\partial D)$ and $\hat{f} = \hat{g}$, then $f = g$ a.e.*

Proof. Since $f \in L^p(\partial D)$ and $g \in L^p(\partial D)$,

$$\lim_{r \to 1} \| \hat{f}_r - f \|_p = 0,$$

and

$$\lim_{r \to 1} \| \hat{g}_r - g \|_p = 0.$$

Since

$$\begin{aligned} \| f - g \|_p &= \| f - \hat{f}_r + \hat{f}_r - \hat{g}_r + \hat{g}_r - g \|_p \\ &\leq \| f - \hat{f}_r \|_p + \| \hat{f}_r - \hat{g}_r \|_p + \| \hat{g}_r - g \|_p \\ &= \| f - \hat{f}_r \|_p + \| \hat{g}_r - g \|_p, \end{aligned}$$

$$\| f - g \|_p = \frac{1}{2\pi} \int_0^{2\pi} |f(\theta) - g(\theta)|^p d\theta = 0.$$

This implies that $f = g$ *a.e.* □

Theorem 2.4. *To each p such that $1 < p < \infty$, there corresponds a constant A_p such that the inequality*

$$\| v \|_p \leq A_p \| u \|_p$$

holds for every real harmonic function u in D if v is harmonic conjugate of u.

Proof. See Theorem 17.26 in [12]. □

Theorem 2.5. *If $1 < p < \infty$, then $\widehat{P} : L^p(\partial D) \to H^p(D)$ is bounded.*

Proof. Since $\cap_{n<0} \ker a_n$ is a closed subspace of $L^p(\partial D)$ and $f \in L^p(\partial D)$, there exist $g \in \cap_{n<0} \ker a_n$ and $h \in (\cap_{n<0} \ker a_n)^{\perp}$ such that $f = g + h$. $\hat{g}(z)$ is analytic and $\hat{g}(z) \in H^p(D)$. Put

$$g(e^{i\theta}) = U(e^{i\theta}) + iV(e^{i\theta})$$

and

$$\hat{g}(z) = u(z) + iv(z).$$

By Theorem 2.1,

$$\lim_{r \to 1} \hat{g}(re^{i\theta}) = \lim_{r \to 1} \widehat{Pf}(re^{i\theta}) = Pf(e^{i\theta}) = g(e^{i\theta}),$$

$$\lim_{r\to 1} u(re^{i\theta}) = U(e^{i\theta}).$$

By Theorem 2.2,

$$\lim_{r\to 1} \frac{1}{2\pi}\int_{-\pi}^{\pi} |\hat{g}(re^{i\theta})|^p d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} |g(e^{i\theta})|^p d\theta.$$

This implies that

$$\lim_{r\to 1} \frac{1}{2\pi}\int_{-\pi}^{\pi} |u(re^{i\theta})|^p d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} |U(e^{i\theta})|^p d\theta.$$

By Theorem 2.4, there is a constant c_p such that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |g(re^{i\theta})|^p d\theta \le c_p \frac{1}{2\pi}\int_{-\pi}^{\pi} |u(re^{i\theta})|^p d\theta.$$

This implies that

$$\begin{aligned}
&\sup_{0<r<1} \frac{1}{2\pi}\int_{-\pi}^{\pi} |g(re^{i\theta})|^p d\theta \\
&\le c_p \sup_{0<r<1} \frac{1}{2\pi}\int_{-\pi}^{\pi} |u(re^{i\theta})|^p d\theta \\
&\le c_p \frac{1}{2\pi}\int_{-\pi}^{\pi} |U(e^{i\theta})|^p d\theta
\end{aligned}$$

since $\frac{1}{2\pi}\int_{-\pi}^{\pi} |u(re^{i\theta})|^p d\theta$ is monotonically increasing function of r in $[0,1]$(See Theorem 17.6 in [11]). This implies that $\widehat{Pf} \in H^p(D)$ for $f \in L^p(\partial D)$. Also,

$$\begin{aligned}
&\sup_{0<r<1} \frac{1}{2\pi}\int_{-\pi}^{\pi} |\widehat{Pf}(re^{i\theta})|^p d\theta \\
&= \lim_{r\to 1} \frac{1}{2\pi}\int_{-\pi}^{\pi} |\hat{g}(re^{i\theta})|^p d\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} |g(e^{i\theta})|^p d\theta \\
&\le \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(e^{i\theta})|^p d\theta.
\end{aligned}$$

This implies that $\widehat{P} : L^p(\partial D) \to H^p(D)$ is bounded if $1 < p < \infty$. $\square$

3. Unbounded operator $\widehat{P}$ on $L^\infty(D)$

THEOREM 3.1. $\widehat{Pf}(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(\theta)}{1-ze^{-i\theta}}d\theta$.

Proof. Note that $\{e^{in\theta} : n \in N\}$ is an orthonormal basis for $L^2(\partial D)$ and $\{z^n : n \in N_0\}$ forms an orthonormal basis for $H^2(D)$. If $f \in L^2(\partial D)$ and $z \in D$, then

$$\begin{aligned}\widehat{Pf}(z) &= \sum_{n=0}^{+\infty}\langle \widehat{Pf}, z^n\rangle z^n \\ &= \sum_{n=0}^{+\infty}\langle Pf, e^{in\theta}\rangle z^n \\ &= \sum_{n=0}^{+\infty}\langle f, e^{in\theta}\rangle z^n \\ &= \frac{1}{2\pi}\sum_{n=0}^{+\infty} z^n \int_{-\pi}^{\pi} f(\theta)e^{-in\theta}d\theta.\end{aligned}$$

The above integral formula for $\widehat{P}$ also extends the domain of $\widehat{P}$ to $L^1(\partial D)$. □

THEOREM 3.2. *If the sequence a_n is monotone decreasing with*

$$\lim_{n\to\infty} a_n = 0$$

and the partial sums of $\sum b_n$ are bounded, then $\sum a_n b_n$ converges

Proof. See Theorem 3.4.2 in [9]. □

LEMMA 3.3.

$$\sum_{k=1}^{n} \sin kx = \frac{\cos(x/2) - \cos(n+1/2)x}{2\sin(x/2)}$$

for all x with $\sin(x/2) \neq 0$.

Proof. Since

$$2\sin A \sin B = \cos(B-A) - \cos(B+A),$$

$$\begin{aligned}
&\sin(x/2)\sum_{k=1}^{n}\sin kx \\
&= \sin(x/2)\sin x + \sin(x/2)\sin 2x + \cdots + \sin(x/2)\sin nx \\
&= (\cos(x/2) - \cos(3x/2)) + (\cos(3x/2) - \cos(5x/2)) + \\
&\cdots + (\cos(n-1/2)x - \cos(n+1/2)x) \\
&= \cos(x/2) - \cos(n+1/2)x.
\end{aligned}$$

□

Corollary 3.4. *$\sum_{n=1}^{\infty}\frac{\sin n\theta}{n}$ converges for every θ. In particular, the series converges uniformly for $a \le x \le \pi$, provided $a > 0$.*

Proof. It is now clear that partial sums of $\sum_{n=1}^{\infty}\sin n\theta$ are bounded by $1/|\sin(\theta/2)|$ by Lemma 3.3. By Theorem 3.2, $\sum_{n=1}^{\infty}\frac{\sin n\theta}{n}$ converges for all θ except possibly those for which $\sin(\theta/2) = 0$. However, these are the values $x = 0, \pm 2\pi, \cdots$ and the series is clearly convergent for these also.

If $a > 0$ and $a \le |x| \le \pi$, then

$$\left|\frac{\cos(x/2) - \cos n(n+1/2)x}{2\sin(x/2)}\right| \le \frac{1}{\sin(a/2)}$$

for all n. □

Theorem 3.5. *If a periodic function $f(x)$ with period 2π is piecewise continuous in the interval $-\pi \le x \le \pi$ and has a left-hand derivative and right-hand derivative at each point of that interval, then the Fourier series of $f(x)$ is convergent. Its sum is $f(x)$, except at a point x_0 at which $f(x)$ is discontinuous.*

Proof. See Theorem 1(p575) in [7]. □

Theorem 3.6. *$f(\theta) = \sum_{n=1}^{\infty}\frac{\sin n\theta}{n}$ is in $L^{\infty}(\partial D)$.*

Proof. Let

$$F(x) = \begin{cases} \dfrac{\pi}{2} - \dfrac{x}{2} - \dfrac{x^2}{4\pi} & if \ -\pi \le x \le 0, \\ \dfrac{\pi}{2} + \dfrac{x}{2} - \dfrac{x^2}{4\pi} & if \ 0 \le x \le \pi, \end{cases}$$

and let F be repeated periodically outside this interval. The resulting function $F(x)$ is continuous for all x and is piecewise smooth. It's Fourier series is

$$\frac{2\pi}{3} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}.$$

By Theorem 3.5, a series in the above converges uniformly to $F(x)$. By Corollary 3.4,

$$F'(x) = \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin nx}{n}$$

for $-\pi \le x \le \pi$, except for $x = 0$.

Also, $F(x)$ is the periodic function of period 2π such that $F(0) = 0$ and

$$F(x) = \begin{cases} -\frac{1}{2} - \frac{x}{2\pi} & if -\pi \le x < 0, \\ \frac{1}{2} - \frac{x}{2\pi} & if 0 \le x \le \pi. \end{cases}$$

Above two result implies that

$$F(x) = \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin nx}{n}.$$

We have now proved that $f(\theta) = \sum_{n=1}^{\infty}\frac{\sin n\theta}{n}$ is in $L^{\infty}(\partial D)$. □

THEOREM 3.7. *If* $f(\theta) = \sum_{n=1}^{\infty}\frac{\sin n\theta}{n}$, *then* $\widehat{Pf}(z) = \frac{1}{2i}\sum_{n=1}^{\infty}\frac{z^n}{n}$.

Proof.

$$\begin{aligned}\widehat{Pf}(z) &= \frac{1}{2\pi}\int_0^{2\pi}\frac{f(\theta)}{1-ze^{-i\theta}}d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi}\sum_{n=1}^{\infty}\frac{\sin n\theta}{n}\frac{1}{1-ze^{-i\theta}}d\theta \\ &= \sum_{n=1}^{\infty}\frac{1}{n}\frac{1}{2\pi}\int_0^{2\pi}\frac{\sin n\theta}{1-ze^{-i\theta}}d\theta \\ &= \sum_{n=1}^{\infty}\frac{1}{n}\frac{1}{2\pi}\int_0^{2\pi}\sin n\theta\sum_{m=0}^{\infty}(ze^{-i\theta})^m d\theta \\ &= \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{1}{n}\frac{1}{2\pi}z^m\int_0^{2\pi}\sin n\theta\{\cos m\theta - i\sin m\theta\}d\theta.\end{aligned}$$

This implies that

$$\begin{aligned}\widehat{Pf}(z) &= \sum_{n=1}^{\infty} \frac{1}{n}\frac{1}{2\pi} z^n \int_0^{2\pi} \sin m\theta \cos n\theta d\theta \\ &= \sum_{n=1}^{\infty} \frac{1}{2i}\frac{z^n}{n} \\ &= \frac{1}{2i}\sum_{n=1}^{\infty}\frac{z^n}{n}.\end{aligned}$$

□

THEOREM 3.8. *$\widehat{P}$ is unbounded in $L^\infty(\partial D)$.*

Proof. By Theorem 3.6,

$$f(\theta) = \sum_{n=1}^{\infty} \frac{\sin n\theta}{n}$$

in in $L^\infty(\partial D)$. Also

$$\widehat{Pf}(z) = \frac{1}{2i}\sum_{n=1}^{\infty}\frac{z^n}{n}$$

by Theorem 3.7. If z is a real number, $\widehat{Pf}$ is not in $H^\infty(D)$. □

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Department of Information Security
Konyang University
Nonsan 320-711, Republic Korea
E-mail: ksc@konyang.ac.kr