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ON SPECTRAL SUBSPACES OF SEMI–SHIFTS

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Abstract. In this paper, we show that for a semi-shift the analytic spectral subspace coincides with the algebraic spectral subspace. Using this result, we have the following result. Let T be a decomposable operator on a Banach space $\mathcal{X}$ and let S be a semi-shift on a Banach space $\mathcal{Y}$. Then every linear operator $\theta : \mathcal{X} \to \mathcal{Y}$ with $S\theta = \theta T$ is necessarily continuous.

1. Preliminaries

Throughout this paper we shall use the standard notions and some basic results on the theory of local spectral theory and automatic continuity theory. Let $\mathcal{X}$ be a Banach space over the complex plane $\mathbb{C}$. And let $\mathcal{L}(\mathcal{X})$ denote the Banach algebra of all bounded linear operators on a Banach space $\mathcal{X}$. Given an operator $T \in \mathcal{L}(\mathcal{X})$, $\mathrm{Lat}(T)$ denotes the collection of all closed T-invariant linear subspaces of $\mathcal{X}$, and for an $\mathcal{Y} \in \mathrm{Lat}(T)$, $T|\mathcal{Y}$ denotes the restriction of T on $\mathcal{Y}$.

Definition 1. Let $T : \mathcal{X} \to \mathcal{X}$ be a linear operator on a Banach space $\mathcal{X}$. Let F be a subset of the complex plane $\mathbb{C}$. Consider the class of all linear subspaces $\mathcal{Y}$ of $\mathcal{X}$ which satisfy $(T-\lambda)\mathcal{Y} = \mathcal{Y}$ for all $\lambda \notin F$, let $E_T(F)$ denote the algebraic linear span of all such subspaces $\mathcal{Y}$ of $\mathcal{X}$. Equivalently, we may define $E_T(F)$ as maximal among all linear subspaces $\mathcal{Y}$ of $\mathcal{X}$ for which $(T-\lambda)\mathcal{Y} = \mathcal{Y}$ for which $\lambda \notin F$. $E_T(F)$ is called an *algebraic spectral subspace* of T.

We always have that

$$E_T(A) \subseteq \bigcap_{\lambda \notin A, n \in \mathbb{N}} (T-\lambda)^n \mathcal{X}$$

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and sometimes the equality holds.

In the next proposition, we collect a number of results on algebraic spectral subspaces. These results are found in [6].

Proposition 2. *Let T be a linear operator on a Banach space $\mathcal{X}$ and let $F \subseteq \mathbb{C}$. Then the following assertions hold:*
(1) $E_T(F)$ is a hyperinvariant subspace, that is, for any bounded linear operator $S : \mathcal{X} \to \mathcal{X}$ for which $ST = TS$ we have $SE_T(F) \subseteq E_T(F)$.
(2) $E_T(F) = E_T(F \cap \sigma(T))$.
(3) $E_T(F)$ is absorbent, that is, if $\lambda \in A$ and $(T - \lambda)x \in E_T(F)$ for some $x \in \mathcal{X}$, then $x \in E_T(F)$. In particular, $\ker(T - \lambda) \subseteq E_T(F)$ for every $\lambda \in A$.
(4) $E_T(\bigcap F_\alpha) = \bigcap E_T(F_\alpha)$ for any family of subsets $\{F_\alpha : \alpha \in A\}$ of $\mathbb{C}$. In particular, if $F_1 \subseteq F_2$, then $E_T(F_1) \subseteq E_T(F_2)$.

A linear subspace $\mathcal{Z}$ of $\mathcal{X}$ is called a T*-divisible subspace* if

$$(T - \lambda)\mathcal{Z} = \mathcal{Z} \quad \text{for all} \quad \lambda \in \mathbb{C}.$$

Hence $E_T(\emptyset)$ is precisely the largest T-divisible subspace. There exists a compact and quasi-nilpotent operator T on a Banach space $\mathcal{X}$ such that T has a non-trivial divisible subspace.

Example. Let $\mathcal{X} = \mathrm{C}[0,1]$ be the complex valued continuous functions on unit interval $[0,1]$ with pointwise addition, pointwise multiplications and supremum norm. And let $T \in \mathcal{L}(\mathcal{X})$ denote the Volterra operator defined by

$$(Tf)(s) = \int_0^s f(t)dt \quad \text{for all } f \in \mathrm{C}[0,1] \text{ and } s \in [0,1].$$

Then T is both compact and quasi-nilpotent. But T has the following non-trivial divisible subspace

$$\mathcal{Y} = \{f \in \mathrm{C}^\infty[0,1] : f^{(k)}(0) = 0 \;\; \text{for all} \;\; k = 0, 1, \dots\}.$$

On the other hand, many important operators do not have non trivial divisible subspaces. For example, hyponormal operators on Hilbert spaces do not have non-trivial divisible subspaces.

For a given $T \in \mathcal{L}(\mathcal{X})$, let $\sigma(T)$, $\sigma_p(T)$ and $\rho(T)$ denote the spectrum, the point spectrum and the resolvent set of T, respectively. The *local resolvent set* $\rho_T(x)$ of T at the point $x \in \mathcal{X}$ is defined as the union of all open subsets U of $\mathbb{C}$ for which there is an analytic function $f : U \to \mathcal{X}$ which satisfies $(T - \lambda)f(\lambda) = x$ for all $\lambda \in U$. The *local spectrum* $\sigma_T(x)$ of T at x is then defined as

$$\sigma_T(x) = \mathbb{C} \setminus \rho_T(x).$$

Clearly, the local resolvent set $\rho_T(x)$ is open, and the local spectrum $\sigma_T(x)$ is closed. For each $x \in \mathcal{X}$, the function $f(\lambda) : \rho(T) \to \mathcal{X}$ defined by $f(\lambda) = (T - \lambda)^{-1}x$ is analytic on $\rho(T)$ and satisfies $(T - \lambda)f(\lambda) = x$ for all $\lambda \in \rho(T)$. Hence the resolvent set $\rho(T)$ is always subset of $\rho_T(x)$ and hence $\sigma_T(x)$ is always subset of $\sigma(T)$. The analytic solutions occurring in the definition of the local resolvent set may be thought of as local extensions of the function $(T - \lambda)^{-1}x : \rho(T) \to \mathcal{X}$. There is no uniqueness implied. Thus we need the following definition. An operator $T \in \mathcal{L}(\mathcal{X})$ is said to have the *single-valued extension property*, abbreviated SVEP, if for every open set $U \subseteq \mathbb{C}$, the only analytic solution $f : U \to \mathcal{X}$ of the equation $(T - \lambda)f(\lambda) = 0$ for all $\lambda \in U$ is the zero function on U. Hence if T has the SVEP, then for each $x \in \mathcal{X}$ there is the maximal analytic extension of $(T - \lambda)^{-1}x$ on $\rho_T(x)$.

For a closed subset F of $\mathbb{C}$, $\mathcal{X}_T(F) = \{x \in X : \sigma_T(x) \subseteq F\}$ is said to be an *analytic spectral subspace* of T. It is easy to see that $\mathcal{X}_T(F)$ is a hyperinvariant subspace of $\mathcal{X}$, while generally not closed. Analytic spectral subspaces have been fundamental in the recent progress of local spectral theory, for instance in connection with functional models and invariant subspaces and also in the theory of spectral inclusions for operators on Banach spaces.

In the next proposition, we collect a number of results on analytic spectral subspaces. These results are found in [6].

Proposition 3. *Let T be a bounded linear operator on a Banach space $\mathcal{X}$ and let $F \subseteq \mathbb{C}$. Then the following assertions hold:*

(1) $\mathcal{X}_T(F) = \mathcal{X}_T(F \cap \sigma(T))$.
(2) For all $\lambda \notin F$, $(T - \lambda)\mathcal{X}_T(F) = \mathcal{X}_T(F)$. *This implies that* $\mathcal{X}_T(F) \subseteq E_T(F)$ *for all* $F \subseteq \mathbb{C}$.
(3) If $\{F_\alpha\}$ *is a family of subsets of* $\mathbb{C}$, *then* $\mathcal{X}_T(\cap F_\alpha) = \cap \mathcal{X}_T(F_\alpha)$.
(4) T *has the SVEP if and only if* $\mathcal{X}_T(\emptyset) = \{0\}$.

Let θ be a linear operator from a Banach space $\mathcal{X}$ into a Banach space $\mathcal{Y}$. The space

$$\mathfrak{S}(\theta) = \{y \in \mathcal{Y} : \text{ there is a sequence } x_n \to 0 \text{ in } \mathcal{X} \text{ and } \theta x_n \to y\}$$

is called the *separating space* of θ. It is easy to see that $\mathfrak{S}(\theta)$ is a closed linear subspace of $\mathcal{Y}$. By the closed graph theorem, θ is continuous if and only if $\mathfrak{S}(\theta) = \{0\}$. The following lemma is found in [9].

LEMMA 4. *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be Banach spaces. If* R *is a continuous linear operator from* $\mathcal{Y}$ *to a Banach space* $\mathcal{Z}$, *and if* $\theta : \mathcal{X} \to \mathcal{Y}$ *is a linear operator, then* $(R\mathfrak{S}(\theta))^- = \mathfrak{S}(R\theta)$. *In particular,* $R\theta$ *is continuous if and only if* $R\mathfrak{S}(\theta) = \{0\}$.

The next lemma states that a certain descending sequence of separating space which obtained from θ via a countable family of continuous linear operators is eventually constant. It is proved in [9].

STABILITY LEMMA. *Let* $\theta : \mathcal{X}_0 \to \mathcal{Y}$ *be a linear operator between the Banach spaces* $\mathcal{X}_0$ *and* $\mathcal{Y}$ *with separating space* $\mathfrak{S}(\theta)$, *and let* $\langle \mathcal{X}_i : i = 1, 2, \dots \rangle$ *be a sequence of Banach spaces. If each* $T_i : \mathcal{X}_i \to \mathcal{X}_{i-1}$ *is continuous linear operator for* $i = 1, 2, \dots,$ *then there is an* $n_0 \in \mathbb{N}$ *for which* $\mathfrak{S}(\theta T_1 T_2 \dots T_n) = \mathfrak{S}(\theta T_1 T_2 \dots T_{n_0})$ *for all* $n \geq n_0$.

An operator $T \in \mathcal{L}(\mathcal{X})$ is called *decomposable* if, for every open covering $\{U, V\}$ of the complex plane $\mathbb{C}$, there exist $\mathcal{Y}, \mathcal{Z} \in \mathrm{Lat}(T)$ such that $\sigma(T|\mathcal{Y}) \subseteq U$, $\sigma(T|\mathcal{Z}) \subseteq V$ and $\mathcal{Y} + \mathcal{Z} = \mathcal{X}$. Decomposable operators are rich. For example, normal operators, spectral operators in the sense of Dunford, operators with totally disconnected spectrums and hence compact operators are decomposable.

Given a topological space Ω and a topological vector space $\mathcal{X}$, we denote by $\mathfrak{F}(\Omega)$ the collection of all closed subsets of Ω, and by $\mathcal{S}(\mathcal{X})$ the collection of all closed linear subspaces of $\mathcal{X}$. A mapping $\mathcal{E}(\cdot) : \mathfrak{F}(\Omega) \to \mathcal{S}(\mathcal{X})$ is said to be a *precapacity* if $\mathcal{E}(\emptyset) = \{0\}$ and $\mathcal{E}(F) \subseteq \mathcal{E}(G)$ for all closed sets F, $G \subseteq \Omega$ with $F \subseteq G$. Given a precapacity $\mathcal{E}(\cdot) : \mathfrak{F}(\Omega) \to \mathcal{S}(\mathcal{X})$, we say that $\mathcal{E}(\cdot)$ is *decomposable* if

$$\mathcal{X} = \mathcal{E}(\overline{U}) + \mathcal{E}(\overline{V}) \quad \text{for every open cover} \quad \{U, V\} \text{ of } \Omega,$$

and that $\mathcal{E}(\cdot)$ is *stable* if arbitrary intersections are preserved, that is,

$$\mathcal{E}(\bigcap F_\alpha) = \bigcap \mathcal{E}(F_\alpha)$$

for every family of closed subsets $\{F_\alpha : \alpha \in A\}$ of Ω. A stable map is called a *spectral capacity* if $\mathcal{E}(\cdot)$ satisfies the following condition:

$$X = \sum_\alpha \mathcal{E}(\overline{G_\alpha}) \quad \text{for every finite open cover} \quad \{G_\alpha : \alpha \in A\} \quad \text{of} \quad \mathbb{C}.$$

If Ω is second countable, then it follows easily from Lindelöf's covering theorem that a precapacity on $\mathfrak{F}(\Omega)$ is stable whenever intersections of countable families of closed sets are preserved. We say that $\mathcal{E}(\cdot)$ is *order preserving* if it preserves the inclusion order. Clearly a stable map is order preserving. It is well known that T is decomposable if and only if there exists a spectral capacity $\mathcal{E}(\cdot)$ such that $\mathcal{E}(F) \in \operatorname{Lat}(T)$ and $\sigma(T|\mathcal{E}(F)) \subseteq F$ for each closed set $F \subseteq \mathbb{C}$. In this case the spectral capacity of a closed subset F of $\mathbb{C}$ is uniquely determined and it is the analytic spectral subspace $\mathcal{X}_T(F)$.

A typical example of stable precapacity is provided by

$$\mathcal{E}(F) = \{f \in \mathrm{C}(\Omega) : \operatorname{supp}(f) \subseteq F\} \quad \text{for all closed sets} \quad F \subseteq \Omega,$$

where $\operatorname{supp}(f)$ is the support of the function f and $\mathrm{C}(\Omega)$ is the space of all continuous complex valued functions on Ω endowed with any vector space topology that is finer than the topology of pointwise convergence. By Urysohn's lemma, precapacity is decomposable if the topological space Ω is normal.

The following lemma, known as *localization of the singularities*, has appeared in various form. We adopted in [4].

LEMMA 5. *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces. Suppose that $\mathcal{E}_{\mathcal{X}} : \mathcal{F}(\mathbb{C}) \to \mathcal{S}(\mathcal{X})$ is an order preserving map such that $\mathcal{X} = \mathcal{E}_{\mathcal{X}}(\overline{U}) + \mathcal{E}_{\mathcal{X}}(\overline{V})$ whenever $\{U, V\}$ is an open cover of $\mathbb{C}$. And suppose that $\mathcal{E}_{\mathcal{Y}} : \mathcal{F}(\mathbb{C}) \to \mathcal{S}(Y)$ is a stable map. If $\theta : \mathcal{X} \to \mathcal{Y}$ is a linear operator for which $\mathfrak{S}(\theta|\mathcal{E}_{\mathcal{X}}(F)) \subseteq \mathcal{E}_{\mathcal{Y}}(F)$ for every $F \in \mathcal{F}(\mathbb{C})$, then there is a finite set $\Lambda \subseteq \mathbb{C}$ for which $\mathfrak{S}(\theta) \subseteq \mathcal{E}_{\mathcal{Y}}(\Lambda)$.*

The following theorem is a variation of the Mittag-Leffler Theorem of Bourbaki. The theorem is found in [9].

MITTAG-LEFFLER THEOREM. *Let $\langle \mathcal{X}_n : n = 0, 1, 2, \dots \rangle$ be a sequence of complete metric spaces, and for $n = 1, 2, \dots$, let $f_n : \mathcal{X}_n \to \mathcal{X}_{n-1}$ be a continuous map with $f_n(\mathcal{X}_n)$ dense in $\mathcal{X}_{n-1}$. Let $g_n = f_1 \circ \cdots \circ f_n$. Then $\bigcap_{n=1}^{\infty} g_n(\mathcal{X}_n)$ is dense in $\mathcal{X}_0$.*

2. Intertwiners with decomposable operators and semi-shifts

An isometry T on a Banach space $\mathcal{X}$ is said to be a *semi-shift* if

$$\bigcap_{n=1}^{\infty} T^n \mathcal{X} = \{0\}.$$

Evidently, a semi-shift on a non trivial Banach space is a non invertible isometry. And it is well known that non invertible isometry is not decomposable but has the single-valued extension property.[6] Hence every semi-shift can not be decomposable and has the single valued extension property. Natural examples of semi-shifts include, for any $1 \le p \le \infty$, the unilateral right shift operators of arbitrary multiplicity on the sequence space $\ell^p(\mathbb{N})$ as well as the right translation operators on the Lebesgue spaces $L^p(\mathbb{R}^+)$ on the positive half line $\mathbb{R}^+ = [0, \infty)$.

PROPOSITION 6. *Let T be a semi-shift on a Banach space $\mathcal{X}$. Then T has no eigenvalues.*

Proof. Let T be a semi-shift on a Banach space $\mathcal{X}$. And let $Tx = \lambda x$ for some $x \in \mathcal{X}$ and $\lambda \in \mathbb{C}$. Then $T^n x = \lambda^n x$ for each $n \in \mathbb{N}$. Hence $\lambda^n x \in T^n \mathcal{X}$. Since $T^n \mathcal{X}$ is a linear subspace of $\mathcal{X}$, $x \in T^n \mathcal{X}$ for all $n \in \mathbb{N}$. Therefore we have,

$$x \in \bigcap_{n=1}^{\infty} T^n \mathcal{X} = \{0\}.$$

Hence T has no eigenvalue. □

The following proposition is in [6].

PROPOSITION 7. *Let T be a bounded linear operator on a Banach space $\mathcal{X}$. Suppose that $E_T(F)$ is closed for all closed sets $F \subseteq \mathbb{C}$. Then the identity $\mathcal{X}_T(F) = E_T(F)$ holds for all closed sets $F \subseteq \mathbb{C}$*

We denote by $\mathrm{C}^\infty(\mathbb{C})$ the Fréchet algebra of all infinitely differentiable complex valued functions $\varphi(z)$, $z = x_1 + ix_2$, x_1, $x_2 \in \mathbb{R}$, defined on the complex plane $\mathbb{C}$ with the topology of uniform convergence of every derivative on each compact subset of $\mathbb{C}$. That is, with the topology generated by a family of pseudo-norm

$$|\varphi|_{K,m} = \max_{|p| \le m} \sup_{z \in K} |D^p \varphi(z)|,$$

where K is an arbitrary compact subset of $\mathbb{C}$, m a non-negative integer, $p = (p_1, p_2)$, p_1, $p_2 \in \mathbb{N}$, $|p| = p_1 + p_2$ and

$$D^p \varphi = \frac{\partial^{|p|} \varphi}{\partial {x_1}^{p_1} \partial {x_2}^{p_2}}\,, \quad z = x_1 + ix_2.$$

An operator $T \in \mathcal{L}(\mathcal{X})$ is called a *generalized scalar operator* if there exists a continuous algebra homomorphism $\Phi : \mathrm{C}^\infty(\mathbb{C}) \to \mathcal{L}(\mathcal{X})$ satisfying $\Phi(1) = I$, the identity operator on $\mathcal{X}$, and $\Phi(z) = T$ where z denotes the identity function on $\mathbb{C}$. Such a continuous function Φ is in fact an operator valued distribution and it is called a *spectral distribution* for T. The class of generalized scalar operators was introduced by Colojoară and Foiaş [1]. Every linear operator on a finite dimensional space as well as every spectral operator of finite type are generalized scalar operators. For more examples and properties of generalized scalar operators one may refer to [1]. It is well known that if T is invertible isometry then T is a generalized scalar operator. For a generalized scalar operator it is well known that $\mathcal{X}_T(F) = E_T(F)$ for all closed sets $F \subseteq \mathbb{C}$. Hence if T is invertible isometry then $\mathcal{X}_T(F) = E_T(F)$ for all closed sets $F \subseteq \mathbb{C}$. Moreover, the identity $\mathcal{X}_T(F) = E_T(F)$ holds on an isometry.

PROPOSITION 8. *Let T be an isometry on a Banach space $\mathcal{X}$. Then for any closed set F of $\mathbb{C}$,*

$$\mathcal{X}_T(F) = E_T(F).$$

In particular, for a semi-shift T the identity $\mathcal{X}_T(F) = E_T(F)$ holds for any closed set F of $\mathbb{C}$.

Proof. If T is an invertible isometry, then T is a generalized scalar operator. Hence The identity $\mathcal{X}_T(F) = E_T(F)$ holds for any closed set F of $\mathbb{C}$. Thus we may assume that T is a non invertible isometry. By Proposition 7, it is enough to show that $E_T(F)$ is closed for any closed set F of $\mathbb{C}$. Let $F \subseteq \mathbb{C}$ be a given closed set. Suppose that there is a $\lambda \notin F$ with $|\lambda| < 1$. If $E_T(F) = \{0\}$ then the space $E_T(F)$ is closed. Hence we may assume that $E_T(F)$ is non trivial. Let $\mathcal{W} = \overline{E_T(F)}$. Since $T - \lambda$ is bounded below, $(T - \lambda)(\mathcal{W})$ is closed. Therefore, we have

$$(T - \lambda)(\mathcal{W}) = \mathcal{W}.$$

Hence $(T - \lambda)|\mathcal{W}$ is invertible. And hence $\lambda \notin \sigma(T|\mathcal{W})$. It is well known that for the spectrum of a non invertible isometry is the entire unit disk. Since $|\lambda| < 1$, $T|\mathcal{W}$ can not to be a non invertible isometry. Hence $T|\mathcal{W}$ is an invertible isometry. Thus $E_{T|\mathcal{W}}(F)$ is closed in $\mathcal{W}$. Since $\mathcal{W}$ is closed, $E_{T|\mathcal{W}}(F)$ is closed in $\mathcal{X}$. It is clear that

$$E_{T|\mathcal{W}}(F) = E_T(F) \cap \mathcal{W} = E_T(F).$$

Therefore, $E_T(F)$ is closed in $\mathcal{X}$. If there is no $\lambda \notin F$ with $|\lambda| < 1$, then $\{\lambda \in \mathbb{C} : |\lambda| < 1\} \subseteq F$. Since T is non invertible isometry,

$$\sigma(T) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\} \subseteq F.$$

Therefore we have $E_T(F) = \mathcal{X}$. Hence $E_T(F)$ is closed in $\mathcal{X}$. In any case $E_T(F)$ is closed for all closed $F \subseteq \mathbb{C}$. □

Hence for an isometry, the above proposition allows us to combine the analytic tools associated with the space $\mathcal{X}_T(F)$ with the algebraic tools associated with the space $E_T(F)$.

Let T and S be bounded linear operators on Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, respectively. A linear operator $\theta : \mathcal{X} \to \mathcal{Y}$ is said to be an *intertwiner or intertwining linear operator* with T and S if $S\theta = \theta T$.

Proposition 9. *Suppose that T has the single-valued extension property on a Banach space $\mathcal{X}$ and that S is a semi-shift on a Banach space $\mathcal{Y}$. Then every linear transformation $\theta : X \to \mathcal{Y}$ with the property $S\theta = \theta T$ necessarily satisfies the following:*

$$\theta\mathcal{X}_T(F) \subseteq \mathcal{Y}_S(F)$$

for all closed subsets F of $\mathbb{C}$.

Proof. Let F be a given closed subset of $\mathbb{C}$. Since $\mathcal{X}_T(F) \subseteq E_T(F)$, $\theta\mathcal{X}_T(F) \subseteq \theta E_T(F)$. For every $\lambda \notin F$, we have

$$\theta E_T(F) = \theta(T - \lambda)E_T(F) = (S - \lambda)\theta E_T(F)$$

This shows that

$$\theta E_T(F) \subseteq E_S(F).$$

By Proposition 8, $E_S(F) = \mathcal{Y}_S(F)$, hence we have

$$\theta\mathcal{X}_T(F) \subseteq \mathcal{Y}_S(F).$$

This completes the proof. □

THEOREM 10. *Suppose that T is a decomposable operator on a Banach space $\mathcal{X}$ and that S is a semi-shift on a Banach space $\mathcal{Y}$. Then every linear operator $\theta : \mathcal{X} \to \mathcal{Y}$ for which $\theta T = S\theta$ is necessarily continuous.*

Proof. Consider an arbitrary linear operator $\theta : \mathcal{X} \to \mathcal{Y}$ satisfying $S\theta = \theta T$. To prove the continuity of θ, it suffices to construct a non-trivial polynomial p such that $p(S)\mathfrak{S}(\theta) = \{0\}$. Indeed if we do so, since S has no eigenvalues, by Proposition 7, all factors $S - \lambda$ of $p(S)$ is injective, hence we have

$$\mathfrak{S}(\theta) = \{0\}.$$

From Proposition 9, we infer that $\theta\mathcal{X}_T(F) \subseteq \mathcal{Y}_S(F)$ for all closed subsets F of $\mathbb{C}$. Since $\mathcal{X}_T(F)$ is the spectral capacity and $\mathcal{Y}_S(F)$ is stable, by Lemma 5, there is a finite set Λ of $\mathbb{C}$ such that $\mathfrak{S}(\theta) \subseteq \mathcal{Y}_S(\Lambda)$. An application of the Stability Lemma to the sequence $T - \lambda$, where $\lambda \in \Lambda$, yields a polynomial p for which

$$\mathfrak{S}(\theta p(T)) = \mathfrak{S}(\theta p(T)(T - \lambda)) \quad \text{for every} \quad \lambda \in \Lambda.$$

Since θ intertwines T and S, this means that by Lemma 4

$$((S - \lambda)p(S)\mathfrak{S}(\theta))^- = (p(S)\mathfrak{S}(\theta))^- \quad \text{for every} \ \lambda \in \Lambda.$$

If we apply Mittag-Leffler Theorem to the above identity then, there exists a dense subspace $\mathcal{W} \subseteq (p(S)\mathfrak{S}(\theta))^-$ for which $(S - \lambda)\mathcal{W} = \mathcal{W}$ for every $\lambda \in \Lambda$. This means that $\mathcal{W} \subseteq E_S(\mathbb{C} \setminus \Lambda)$ by the definition of algebraic spectral subspaces. Since $\mathcal{W} \subseteq \mathfrak{S}(\theta) \subseteq E_S(\Lambda)$, we obtain that

$$\begin{aligned} \mathcal{W} &\subseteq E_S(\Lambda) \cap E_S(\mathbb{C} \setminus \Lambda) \\ &= E_S(\emptyset) \\ &= \mathcal{Y}_S(\emptyset) \\ &= \{0\}. \end{aligned}$$

Therefore, we have $\mathcal{W} = \{0\}$. Consequently, $p(S)\mathfrak{S}(\theta) = \{0\}$. Hence θ is continuous. $\square$

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