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GENERALIZED ANALYTIC FEYNMAN INTEGRALS INVOLVING GENERALIZED ANALYTIC FOURIER–FEYNMAN TRANSFORMS AND GENERALIZED INTEGRAL TRANSFORMS

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Abstract. In this paper, we use a generalized Brownian motion process to define a generalized analytic Feynman integral. We then establish several integration formulas for generalized analytic Feynman integrals, generalized analytic Fourier-Feynman transforms and generalized integral transforms of functionals in the class of functionals $\mathbb{E}_0$. Finally, we use these integration formulas to obtain several generalized Feynman integrals involving the generalized analytic Fourier-Feynman transform and the generalized integral transform of functionals in $\mathbb{E}_0$.

1. Introduction

Let $C_0[0,T]$ denote one-parameter Wiener space; that is the space of real-valued continuous functions x on $[0,T]$ with $x(0)=0$. The concept of L_1 analytic Fourier-Feynman transforms(FFT) was introduced by Brue in [1]. In [4], Cameron and Storvick introduced an L_2 analytic FFT. In [10], Johnson and Skoug developed an L_p analytic FFT theory for $1 \leq p \leq 2$ which extended the results in [4] and gave various relationships between the L_1 and the L_2 theories. In a unifying paper [13], Lee defined an integral transform $\mathcal{F}_{\gamma,\beta}$ of analytic functionals on abstract Wiener spaces. For certain values of the parameters γ and β and for certain classes of functionals, the Fourier-Wiener transform [2], the modified Fourier-Wiener transform

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[3], the Fourier-Feynman transform [1,4] and the Gauss transform are special cases of Lee's integral transform $\mathcal{F}_{\gamma,\beta}$. Also see paper [14] for further work involving integral transforms.

The function space $C_{a,b}[0,T]$ induced by generalized Brownian motion was introduced by J. Yeh in [16,17] and was used extensively by Chang and Chung [6]. The Wiener process used in [1-4,9-14] is stationary in time and is free of drift while the stochastic process used in this paper as well as in [5-8], is nonstationary in time, is subject to a drift $a(t)$, and can be used to explain the position of the Ornstein-Uhlenbeck process in an external force field [15]. However, when $a(t) \equiv 0$ and $b(t) = t$ on $[0,T]$, the general function space $C_{a,b}[0,T]$ reduces to the Wiener space $C_0[0,T]$. In this paper, we obtain several interesting generalized Feynman integrals involving generalized analytic Fourier-Feynman transforms and generalized integral transforms of functionals in $\mathbb{E}_0$.

2. Definitions and preliminaries

Let $D = [0,T]$ and let $(\Omega, \mathcal{B}, P)$ be a probability measure space. A real-valued stochastic process Y on $(\Omega, \mathcal{B}, P)$ and D is called *a generalized Brownian motion process* if $Y(0,\omega)$=0 almost everywhere and for $0 = t_0 < t_1 < \cdots < t_n \le T$, the n-dimensional random vector $(Y(t_1,\omega), \cdots, Y(t_n,\omega))$ is normally distributed with density function

$$(2.1)\quad \begin{aligned} W_n(\vec{t},\vec{\eta}) &= \Big((2\pi)^n \prod_{j=1}^{n}(b(t_j) - b(t_{j-1}))\Big)^{-1/2} \\ &\quad \cdot\ \exp\bigg\{-\frac{1}{2}\sum_{j=1}^{n}\frac{\big((\eta_j - a(t_j)) - (\eta_{j-1} - a(t_{j-1}))\big)^2}{b(t_j) - b(t_{j-1})}\bigg\} \end{aligned}$$

where $\vec{\eta} = (\eta_1, \cdots, \eta_n)$, $\eta_0 = 0$, $\vec{t} = (t_1, \cdots, t_n)$, $a(t)$ is a continuous real-valued function on $[0,T]$ with $a(0) = 0$, and $b(t)$ is a monotone increasing, continuously differentiable real-valued function with $b(0) = 0$ and $b'(t) > 0$ for each $t \in [0,T]$.

As explained in [16, p.18-20], Y induces a probability measure μ on the measurable space $(\mathbb{R}^D, \mathcal{B}^D)$ where $\mathbb{R}^D$ is the space of all real valued functions $x(t)$, $t \in D$, and $\mathcal{B}^D$ is the smallest σ-algebra of subsets of $\mathbb{R}^D$ with respect to which all the coordinate evaluation maps $e_t(x) = x(t)$ defined on $\mathbb{R}^D$ are measurable. The triple $(\mathbb{R}^D, \mathcal{B}^D, \mu)$ is a probability measure space. This measure space is called the function space induced by the generalized Brownian motion process Y determined by $a(\cdot)$ and $b(\cdot)$.

We note that the generalized Brownian motion process Y determined by $a(\cdot)$ and $b(\cdot)$ is a Gaussian process with mean function $a(t)$ and covariance function $r(s,t) = \min\{b(s), b(t)\}$. By Theorem 14.2 [16, p.187], the probability measure μ induced by Y, taking a separable version, is supported by $C_{a,b}[0,T]$ (which is equivalent to the Banach space of continuous functions x on $[0,T]$ with $x(0) = 0$ under the sup norm). Hence $(C_{a,b}[0,T], \mathcal{B}(C_{a,b}[0,T]), \mu)$ is the function space induced by Y where $\mathcal{B}(C_{a,b}[0,T])$ is the Borel σ-algebra of $C_{a,b}[0,T]$.

A subset B of $C_{a,b}[0,T]$ is said to be scale-invariant measurable provided ρB is $\mathcal{B}(C_{a,b}[0,T])$-measurable for all $\rho > 0$, and a scale-invariant measurable set N is said to be a scale-invariant null set provided $\mu(\rho N) = 0$ for all $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere(s-a.e.). If two functionals F and G are equal scale-invariant almost everywhere, we write $F \approx G$.

We denote the function space integral of a $\mathcal{B}(C_{a,b}[0,T])$-measurable functional F by

$$E[F] = \int_{C_{a,b}[0,T]} F(x)d\mu(x)$$

whenever the integral exists.

In this paper, let $K_{a,b}[0,T]$ be the set of all complex-valued continuous functions $x(t)$ defined on $[0,T]$ which vanish at $t = 0$ and whose real and imaginary parts are elements of $C_{a,b}[0,T]$; namely,

$$K_{a,b}[0,T] = \{x : [0,T] \to \mathbb{C} \mid x(0) = 0,$$
$$\mathrm{Re}(x) \in C_{a,b}[0,T] \text{ and } \mathrm{Im}(x) \in C_{a,b}[0,T]\}.$$

Thus $C_{a,b}[0,T]$ is a subspace of $K_{a,b}[0,T]$.

We are now ready to state the definition of the generalized analytic Feynman integral.

Definition 2.1. *Let $\mathbb{C}$ denote the complex numbers. Let $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : Re\lambda > 0\}$ and $\tilde{\mathbb{C}}_+ = \{\lambda \in \mathbb{C} : \lambda \neq 0$ and $Re\lambda \geq 0\}$. Let $F : K_{a,b}[0,T] \longrightarrow \mathbb{C}$ be such that for each $\lambda > 0$, the function space integral*

$$J(\lambda) = \int_{C_{a,b}[0,T]} F(\lambda^{-\frac{1}{2}}x)d\mu(x)$$

exists for all $\lambda > 0$. If there exists a function $J^*(\lambda)$ analytic in $\mathbb{C}_+$ such that $J^*(\lambda) = J(\lambda)$ for all $\lambda > 0$, then $J^*(\lambda)$ is defined to be the analytic function space integral of F over $C_{a,b}[0,T]$ with parameter λ, and for $\lambda \in \mathbb{C}_+$ we write

$$E^{\mathrm{an}_\lambda}[F] \equiv E_x^{\mathrm{an}_\lambda}[F(x)] = J^*(\lambda).$$

Let $q \neq 0$ be a real number and let F be a functional such that $E^{\mathrm{an}_\lambda}[F]$ exists for all $\lambda \in \mathbb{C}_+$. If the following limit exists, we call it the generalized analytic Feynman integral of F with parameter q and we write

$$E^{\mathrm{anf}_q}[F] \equiv E_x^{\mathrm{anf}_q}[F(x)] = \lim_{\lambda \to -iq} E^{\mathrm{an}_\lambda}[F] \tag{2.2}$$

where λ approaches $-iq$ through values in $\mathbb{C}_+$.

Next, we state the definition of the generalized analytic Fourier-Feynman transform(GFFT).

Definition 2.2. *For $\lambda \in \mathbb{C}_+$ and $y \in C_{a,b}[0,T]$, let*

$$T_\lambda(F)(y) = E_x^{\mathrm{an}_\lambda}[F(y+x)].$$

For $p \in (1,2]$, we define the L_p analytic GFFT, $T_q^{(p)}(F)$ of F, by the formula ($\lambda \in \mathbb{C}_+$)

$$T_q^{(p)}(F)(y) = l.i.m._{\lambda \to -iq} T_\lambda(F)(y)$$

if it exists; i.e., for each $\rho > 0$,

$$\lim_{\lambda \to -iq} \int_{C_{a,b}[0,T]} \left| T_\lambda(F)(\rho y) - T_q^{(p)}(F)(\rho y) \right|^{p'} d\mu(y) = 0$$

where $1/p + 1/p' = 1$. We define the L_1 analytic GFFT, $T_q^{(1)}(F)$ of F, by the formula ($\lambda \in \mathbb{C}_+$)

$$T_q^{(1)}(F) = \lim_{\lambda \to -iq} T_\lambda(F)(y) \tag{2.3}$$

if it exists.

We note that for $1 \le p \le 2$, $T_q^{(p)}(F)$ is only defined s-a.e.. We also note that if $T_q^{(p)}(F)$ exists and if $F \approx G$, then $T_q^{(p)}(G)$ exists and $T_q^{(p)}(G) \approx T_q^{(p)}(F)$.

Let $L^2_{a,b}[0,T]$ be the Hilbert space of functions on $[0,T]$ which are Lebesgue measurable and square integrable with respect to the Lebesgue Stieltjes measures on $[0,T]$ induced by $a(\cdot)$ and $b(\cdot)$; i.e.,

$$L^2_{a,b}[0,T] = \Big\{v : \int_0^T v^2(s)db(s) < \infty \text{ and } \int_0^T v^2(s)d|a|(s) < \infty\Big\}$$

where $|a|(t)$ denotes the total variation of the function a on the interval $[0,t]$.

For $u, v \in L^2_{a,b}[0,T]$, let

$$(u,v)_{a,b} = \int_0^T u(t)v(t)d[b(t) + |a|(t)].$$

Then $(\cdot,\cdot)_{a,b}$ is an inner product on $L^2_{a,b}[0,T]$ and $\|u\|_{a,b} = \sqrt{(u,u)_{a,b}}$ is a norm on $L^2_{a,b}[0,T]$. In particular note that $\|u\|_{a,b} = 0$ if and only if $u(t) = 0$ a.e. on $[0,T]$. Furthermore $(L^2_{a,b}[0,T], \|\cdot\|_{a,b})$ is a separable Hilbert space. Note that all functions of bounded variation on $[0,T]$ are elements of $L^2_{a,b}[0,T]$. Also note that if $a(t) \equiv 0$ and $b(t) = t$ on $[0,T]$, then $L^2_{a,b}[0,T] = L^2[0,T]$. In fact,

$$(L^2_{a,b}[0,T], \|\cdot\|_{a,b}) \subset (L^2_{0,b}[0,T], \|\cdot\|_{0,b}) = (L^2[0,T], \|\cdot\|_2)$$

since the two norms $\|\cdot\|_{0,b}$ and $\|\cdot\|_2$ are equivalent.

Let $\{\phi_j\}_{j=1}^\infty$ be a complete orthonormal set of real-valued functions of bounded variation on $[0,T]$ such that

$$(\phi_j, \phi_k)_{a,b} = \begin{cases} 0 & , \ j \ne k \\ 1 & , \ j = k \end{cases},$$

and for each $v \in L^2_{a,b}[0,T]$, let

$$v_n(t) = \sum_{j=1}^n (v, \phi_j)_{a,b}\phi_j(t)$$

for $n = 1, 2, \cdots$. Then for each $v \in L^2_{a,b}[0,T]$, the Paley-Wiener-Zygmund (PWZ) stochastic integral $\langle v, x\rangle$ is defined by the formula

$$\langle v, x\rangle = \lim_{n\to\infty} \int_0^T v_n(t)dx(t)$$

for all $x \in C_{a,b}[0,T]$ for which the limit exists; one can show that for each $v \in L^2_{a,b}[0,T]$, the PWZ integral $\langle v, x\rangle$ exists for μ-a.e. $x \in C_{a,b}[0,T]$.

Let $\{\alpha_1, \alpha_2, \cdots\}$ be any complete orthonormal set of functions in the separable Hilbert space $(L^2_{a,b}[0,T], \|\cdot\|_{a,b})$, and for each $j = 1, 2, \cdots$, let

$$A_j \equiv \int_0^T \alpha_j(t)da(t) \tag{2.4}$$

and

$$B_j \equiv \int_0^T \alpha_j^2(t)db(t). \tag{2.5}$$

We note that for each $j = 1, 2, \cdots$,

$$0 < B_j = \int_0^T \alpha_j^2(t)db(t) \le \int_0^T \alpha_j^2(t)d[b(t) + |a|(t)] = \|\alpha_j\|^2_{a,b} = 1,$$

while A_j may be positive, negative or zero.

The following well-known integration formula (see equation (2.20) on page 2929 of [5]) is used several times in this paper. Let $h : \mathbb{R}^n \to \mathbb{R}$ be Lebesgue measurable and let $H(x) = h(\langle\alpha_1, x\rangle, \cdots, \langle\alpha_n, x\rangle)$. Then

(2.6)

$$\int_{C_{a,b}[0,T]} H(x)d\mu(x) = \left(\prod_{j=1}^n 2\pi B_j\right)^{-\frac{1}{2}} \int_{\mathbb{R}^n} h(u_1, \cdots, u_n) \cdot \exp\left\{-\sum_{j=1}^n \frac{(u_j - A_j)^2}{2B_j}\right\}du_1\cdots du_n$$

in the sense that if either side of (2.6) exists, both sides exist, and equality holds.

Using formula (2.6) we observe that $E[\langle\alpha_j, x\rangle] = A_j, E[\langle\alpha_j, x\rangle^2] = B_j + A_j^2$ and that $Var(\langle\alpha_j, x\rangle) = B_j$ for each $j = 1, 2, \cdots$.

Also note that the complete orthonormal set $\{\alpha_1, \alpha_2, \cdots\}$ in $L^2_{a,b}[0,T]$ is completely at our disposal. For example, we could choose the α_j's to be continuous and of bounded variation on $[0,T]$, or we could choose the α_j's to be the Haar functions on $[0,T]$, etc.

Throughout this paper, we will assume that for all complex number $z \in \tilde{\mathbb{C}}_+$ (or $z^{\frac{1}{2}}$) is always chosen to have positive real parts.

Last, we are ready to state the definition of the generalized integral transform on $K_{a,b}[0,T]$.

DEFINITION 2.4. *Let F be a functional defined on $K_{a,b}[0,T]$. For each pair of nonzero complex numbers γ and β, the generalized integral transforms $\mathcal{F}_{\gamma,\beta}F$ of F is defined by*

$$\mathcal{F}_{\gamma,\beta}F(y) = \int_{C_{a,b}[0,T]} F(\gamma x + \beta y)\, d\mu(x), \quad y \in K_{a,b}[0,T], \tag{2.7}$$

if it exists.

Next we describe the class of functionals that we work with in this paper. Let $\mathbb{E}_0$ be the space of all functionals $F : K_{a,b}[0,T] \to \mathbb{C}$ of the form

$$F(x) = f\big(\langle \alpha_1, x\rangle, \cdots, \langle \alpha_n, x\rangle\big) \tag{2.8}$$

for some positive integer n, where $f(\lambda_1, \cdots, \lambda_n)$ is an entire function of the n complex variables $\lambda_1, \cdots, \lambda_n$ of exponential type; that is to say,

$$|f(\lambda_1, \cdots, \lambda_n)| \le A_F \exp\bigg\{ B_F \sum_{j=1}^{n} |\lambda_j| \bigg\} \tag{2.9}$$

for some positive constants A_F and B_F.

REMARK 2.5. *For each $m = 0, 1, 2, \cdots$ and for each $j = 1, 2, \cdots$, let $H^j_m(u)$ denote the generalized Hermite polynomial*

$$H^j_m(u) \equiv (-1)^m (m!)^{-\frac{1}{2}} (B_j)^{\frac{m}{2}} \exp\bigg\{ \frac{(u - A_j)^2}{2B_j} \bigg\} \frac{d^m}{du^m} \bigg(\exp\bigg\{ -\frac{(u - A_j)^2}{2B_j} \bigg\} \bigg).$$

Then the set, for each $j = 1, 2, \cdots,$

$$\left\{(2\pi B_j)^{-\frac{1}{4}} H_m^j(u) \exp\left\{-\frac{\big(u - A_j\big)^2}{4B_j}\right\} : m = 0, 1, \cdots\right\}$$

is a complete orthonormal set in $L_2(\mathbb{R})$*. Now we define*

$$\phi_{(m,k)}(x) \equiv H_m^k(\langle \alpha_k, x\rangle), \qquad m = 0, 1, 2, \cdots, k = 1, 2, \cdots,$$

and

$$\Phi_{(m_1,\cdots,m_k)}(x) \equiv \phi_{(m_1,1)}(x)\phi_{(m_2,2)}(x)\cdots\phi_{(m_k,k)}(x). \tag{2.10}$$

The functional in (2.10) is called the generalized Fourier-Hermite functional. Chang, Chung and Skoug showed that the generalized Fourier-Hermite functionals forms a complete orthonormal set in $L_2(C_{a,b}[0,T])$*, that is to say, let* F *be any functionals on* $C_{a,b}[0,T]$ *with*

$$\int_{C_{a,b}[0,T]} |F(x)|^2 d\mu(x) < \infty,$$

and for $N = 1, 2, \cdots,$ *let*

$$F_N(x) = \sum_{m_1,\cdots,m_N=0}^{N} A^F_{(m_1,\cdots,m_N)} \Phi_{(m_1,\cdots,m_N)}(x)$$

where $A^F_{(m_1,\cdots,m_N)}$ *is the generalized Fourier-Hermite coefficient,*

$$A^F_{(m_1,\cdots,m_N)} \equiv \int_{C_{a,b}[0,T]} F(x)\Phi_{(m_1,\cdots,m_N)}(x) d\mu(x).$$

Then

$$\int_{C_{a,b}[0,T]} |F_N(x) - F(x)|^2 d\mu(x) \to 0$$

as $N \to \infty$ *and*

$$\begin{aligned} F(x) &= l.i.m._{N\to\infty} F_N(x) \\ &= l.i.m._{N\to\infty} \sum_{m_1,\cdots,m_N=0}^{N} A^F_{(m_1,\cdots,m_N)} \Phi_{(m_1,\cdots,m_N)}(x) \end{aligned}$$

is called the generalized Fourier-Hermite series expansion of F. For more details in [7,8]. Furthermore, we can easily check that F_N is an element of $\mathbb{E}_0$ for all $N = 1, 2, \cdots$ and so the set $\mathbb{E}_0$ is dense in $L_2(C_{a,b}[0,T])$.

To simplify the expressions, we use the following notations. For $\gamma, \beta \in \mathbb{C}$, $\vec{u} = (u_1, \cdots, u_n) \in \mathbb{C}^n$ and $\vec{\lambda} = (\lambda_1, \cdots, \lambda_n) \in \mathbb{C}^n$, we write

$$f\big(\gamma\vec{u} + \beta\vec{\lambda}\big) = f\big(\gamma u_1 + \beta\lambda_1, \cdots, \gamma u_n + \beta\lambda_n\big)$$

and

$$f\big(\langle\vec{\alpha}, x\rangle\big) = f\big(\langle\alpha_1, x\rangle, \cdots, \langle\alpha_n, x\rangle\big).$$

3. Generalized Feynman integrals, L_1 analytic GFFTs and generalized integral transforms of functionals in $\mathbb{E}_0$

In this section, we obtain several integration formulas involving generalized Feynman integrals, L_1 analytic GFFTs and generalized integral transforms of functionals in $\mathbb{E}_0$.

THEOREM 3.1. *Let q_0 be a nonzero real number and let $F \in \mathbb{E}_0$ be given by (2.8). Then for all nonzero real number q with $|q| \geq |q_0|$, the generalized analytic Feynman integral of F exists and is given by the formula*

$$E^{\text{anf}_q}[F] = \Big(\prod_{j=1}^{n} 2\pi B_j\Big)^{-\frac{1}{2}} \int_{\mathbb{R}^n} f\big((-iq)^{-\frac{1}{2}}\vec{u}\big) \exp\Big\{-\sum_{j=1}^{n} \frac{(u_j - A_j)^2}{2B_j}\Big\} d\vec{u}. \tag{3.1}$$

Proof. For $\lambda > 0$, by using formula (2.6) it follows that

$$\int_{C_{a,b}[0,T]} F(\lambda^{-\frac{1}{2}}x)d\mu(x) = \Big(\prod_{j=1}^{n} 2\pi B_j\Big)^{-\frac{1}{2}} \int_{\mathbb{R}^n} f\big(\lambda^{-\frac{1}{2}}\vec{u}\big) \exp\Big\{-\sum_{j=1}^{n} \frac{(u_j - A_j)^2}{2B_j}\Big\} d\vec{u}.$$

Since f is an entire function of exponential type and $B_j > 0$, the last integral exists and so it can be analytically in $\mathbb{C}_+$. Thus we obtain that

$$E^{an_\lambda}[F] = \Big(\prod_{j=1}^{n} 2\pi B_j\Big)^{-\frac{1}{2}} \int_{\mathbb{R}^n} f\big(\lambda^{-\frac{1}{2}}\vec{u}\big) \exp\Big\{-\sum_{j=1}^{n} \frac{(u_j - A_j)^2}{2B_j}\Big\} d\vec{u}.$$

Also the last expression above is a continuous function of λ in $\tilde{\mathbb{C}}_+$ and it is dominated on the region

$$R=\{\lambda\in\tilde{\mathbb{C}}_+ : |\lambda^{-\frac{1}{2}}|\le |2q_0|^{-\frac{1}{2}}\}.$$

Thus by using dominated convergence theorem, we get

$$E^{\mathrm{anf}_q}[F] = \Big(\prod_{j=1}^n 2\pi B_j\Big)^{-\frac{1}{2}}\int_{\mathbb{R}^n} f\big((-iq)^{-\frac{1}{2}}\vec{u}\big)\exp\Big\{-\sum_{j=1}^n \frac{(u_j-A_j)^2}{2B_j}\Big\}d\vec{u}.$$

In fact, by using (2.9) it follows that for all nonzero real number q with $|q|\ge|q_0|$,

$$|E^{\mathrm{anf}_q}[F]| \le \Big(\prod_{j=1}^n 2\pi B_j\Big)^{-\frac{1}{2}}\int_{\mathbb{R}^n} A_F \exp\Big\{-\sum_{j=1}^n \frac{(u_j-A_j)^2}{2B_j} + \frac{B_F}{\sqrt{2|q_0|}}\sum_{j=1}^n |u_j|\Big\}d\vec{u}<\infty.$$

Hence we have the desired result. □

Theorem 3.2. *Let q_0 and F be as in Theorem 3.1. Then for all nonzero real number q with $|q|\ge|q_0|$, the L_1 analytic GFFT $T_q^{(1)}(F)$ of F exists, belongs to $\mathbb{E}_0$ and is given by the formula*

$$T_q^{(1)}(F)(y)=\Gamma_{T_q^{(1)}(F)}(\langle\vec{\alpha},y\rangle) \tag{3.2}$$

for s-a.e. $y\in K_{a,b}[0,T]$, where

$$\Gamma_{T_q^{(1)}(F)}(\vec{\lambda}) = \Big(\prod_{j=1}^n 2\pi B_j\Big)^{-\frac{1}{2}}\int_{\mathbb{R}^n} f\big((-iq)^{-\frac{1}{2}}\vec{u}+\vec{\lambda}\big)\exp\Big\{-\sum_{j=1}^n \frac{(u_j-A_j)^2}{2B_j}\Big\}d\vec{u}. \tag{3.3}$$

Proof. Preceding as in the proof of Theorem 3.1, for all nonzero real number q with $|q|\ge|q_0|$, we get

$$\begin{aligned}T_q^{(1)}(F)(y) &= \Big(\prod_{j=1}^n 2\pi B_j\Big)^{-\frac{1}{2}}\int_{\mathbb{R}^n} f\big((-iq)^{-\frac{1}{2}}\vec{u}+\langle\vec{\alpha},y\rangle\big)\exp\Big\{-\sum_{j=1}^n \frac{(u_j-A_j)^2}{2B_j}\Big\}d\vec{u}\\ &=\Gamma_{T_q^{(1)}(F)}(\langle\vec{\alpha},y\rangle).\end{aligned}$$

Furthermore, by using equation (2.9) it follows that for all nonzero real number q with $|q| \geq |q_0|$,

$$
\begin{aligned}
&|\Gamma_{T_q^{(1)}(F)}(\vec{\lambda})| \\
&\leq \Big(\prod_{j=1}^n 2\pi B_j\Big)^{-\frac{1}{2}} \int_{\mathbb{R}^n} \exp\Big\{B_F \sum_{j=1}^n |\lambda_j|\Big\} \\
&\qquad\qquad \cdot A_F \exp\Big\{-\sum_{j=1}^n \frac{(u_j - A_j)^2}{2B_j} + \frac{B_F}{\sqrt{2|q_0|}} \sum_{j=1}^n |u_j|\Big\} d\vec{u} \\
&= A_{T_q^{(1)}(F)} \exp\Big\{B_{T_q^{(1)}(F)} \sum_{j=1}^n |\lambda_j|\Big\}
\end{aligned}
$$

where

$$
\begin{aligned}
&A_{T_q^{(1)}(F)} \\
&= \Big(\prod_{j=1}^n 2\pi B_j\Big)^{-\frac{1}{2}} \int_{\mathbb{R}^n} A_F \exp\Big\{-\sum_{j=1}^n \frac{(u_j - A_j)^2}{2B_j} + \frac{B_F}{\sqrt{2|q_0|}} \sum_{j=1}^n |u_j|\Big\} d\vec{u} < \infty
\end{aligned}
$$

and

$$
B_{T_q^{(1)}(F)} = B_F
$$

and so $T_q^{(1)}(F) \in \mathbb{E}_0$. Hence we have the desired result. □

THEOREM 3.3. *Let γ and β be nonzero complex numbers and let F be as in Theorem 3.1. Then the generalized integral transform $\mathcal{F}_{\gamma,\beta}F$ of F exists, belongs to $\mathbb{E}_0$ and is given by the formula*

$$
\mathcal{F}_{\gamma,\beta}F(y) = \Gamma_{\mathcal{F}_{\gamma,\beta}F}(\langle \vec{\alpha}, y \rangle) \tag{3.4}
$$

for all $y \in K_{a,b}[0,T]$, where

$$
\begin{aligned}
&\Gamma_{\mathcal{F}_{\gamma,\beta}F}(\vec{\lambda}) \\
&= \Big(\prod_{j=1}^n 2\pi B_j\Big)^{-\frac{1}{2}} \int_{\mathbb{R}^n} f\big(\gamma\vec{u} + \beta\vec{\lambda}\big) \exp\Big\{-\sum_{j=1}^n \frac{(u_j - A_j)^2}{2B_j}\Big\} d\vec{u}.
\end{aligned} \tag{3.5}
$$

Proof. For each $y \in K_{a,b}[0,T]$, by using formula (2.6) it follows that

$$\begin{aligned}
&\mathcal{F}_{\gamma,\beta}F(y)\\
&= \Big(\prod_{j=1}^{n} 2\pi B_j\Big)^{-\frac{1}{2}} \int_{\mathbb{R}^n} f\big(\gamma\vec{u} + \beta\langle\vec{\alpha}, y\rangle\big) \exp\Big\{-\sum_{j=1}^{n} \frac{(u_j - A_j)^2}{2B_j}\Big\} d\vec{u}\\
&= \Gamma_{\mathcal{F}_{\gamma,\beta}F}\big(\langle\vec{\alpha}, y\rangle\big).
\end{aligned}$$

Furthermore, by using (2.6) it follows that

$$\begin{aligned}
&|\Gamma_{\mathcal{F}_{\gamma,\beta}F}(\vec{\lambda})|\\
&\le \Big(\prod_{j=1}^{n} 2\pi B_j\Big)^{-\frac{1}{2}} \int_{\mathbb{R}^n} A_F \exp\Big\{B_F|\beta|\sum_{j=1}^{n}|\lambda_j|\Big\}\\
&\qquad\qquad \cdot \exp\Big\{B_F\sum_{j=1}^{n}|\gamma u_j| - \sum_{j=1}^{n}\frac{(u_j - A_j)^2}{2B_j}\Big\} d\vec{u}\\
&\le A_{\mathcal{F}_{\gamma,\beta}F} \exp\Big\{B_{\mathcal{F}_{\gamma,\beta}F}\sum_{j=1}^{n}|\lambda_j|\Big\}
\end{aligned}$$

where

$$A_{\mathcal{F}_{\gamma,\beta}F} = A_F\Big(\prod_{j=1}^{n} 2\pi B_j\Big)^{-\frac{1}{2}} \int_{\mathbb{R}^n} \exp\Big\{B_F\sum_{j=1}^{n}|\gamma u_j| - \sum_{j=1}^{n}\frac{(u_j - A_j)^2}{2B_j}\Big\} d\vec{u} < \infty$$

and $B_{\mathcal{F}_{\gamma,\beta}F} = B_F|\beta|$. Hence $\mathcal{F}_{\gamma,\beta}F \in \mathbb{E}_0$. □

REMARK 3.4. *(1) In [12], the authors gave some conditions for the existence of integral transforms, convolution products, analytic Fourier-Feynman transforms and first variations of functionals. In this paper we can give some conditions for the existence of our issues. But, for simplicity, we emphasize formulas of Feynman integrals, GFFTs and integral transforms of functionals in* $\mathbb{E}_0$ *rather than the existence.*

(2) Form now on, for simplicity our results, we will assume that generalized Feynman integrals, L_1 *analytic GFFTs and generalized integral transforms in each statements always exist.*

4. Generalized analytic Feynman integrals involving generalized integral transforms and L_1 analytic GFFTs of functionals in $\mathbb{E}_0$

In this section we establish a relationship between the L_1 analytic GFFT and the generalized integral transform. We then obtain several generalized analytic Feynman integrals involving L_1 analytic GFFTs and generalized integral transforms of functionals in $\mathbb{E}_0$.

First we establish a relationship between the L_1 analytic GFFT and the generalized integral transform of functionals in $\mathbb{E}_0$. To obtain this, we need following Lemmas 4.1 and 4.2. These lemmas are follow by equations (3.2)-(3.4) and (3.5).

LEMMA 4.1. *Let γ and β be nonzero complex numbers and let q be a nonzero real number. Let $F \in \mathbb{E}_0$ be given by (2.8). Then the L_1 analytic GFFT $T_q^{(1)}(\mathcal{F}_{\gamma,\beta}F)$ of $\mathcal{F}_{\gamma,\beta}F$ is given by the formula*

$$T_q^{(1)}(\mathcal{F}_{\gamma,\beta}F)(z) = \Gamma_{T_q^{(1)}(\mathcal{F}_{\gamma,\beta}F)}(\langle\vec{\alpha}, z\rangle)$$

for s-a.e. $z \in K_{a,b}[0,T]$, where

$$\begin{aligned}
&\Gamma_{T_q^{(1)}(\mathcal{F}_{\gamma,\beta}F)}(\vec{\lambda}) \\
&= \Big(\prod_{j=1}^{n} 2\pi B_j\Big)^{-1} \int_{\mathbb{R}^{2n}} f\big(\gamma\vec{u} + (-iq)^{-\frac{1}{2}}\beta\vec{v} + \beta\vec{\lambda}\big) \\
&\qquad \cdot \exp\Big\{-\sum_{j=1}^{n} \frac{(u_j - A_j)^2 + (v_j - A_j)^2}{2B_j}\Big\} d\vec{u}d\vec{v}.
\end{aligned} \tag{4.1}$$

LEMMA 4.2. *Let γ, β, q and F be as in Lemma 4.1. Then the generalized integral transform $\mathcal{F}_{\gamma,\beta}T_q^{(1)}(F)$ of $T_q^{(1)}(F)$ is given by the formula*

$$\mathcal{F}_{\gamma,\beta}T_q^{(1)}(F)(z) = \Gamma_{\mathcal{F}_{\gamma,\beta}T_q^{(1)}(F)}(\langle\vec{\alpha}, z\rangle)$$

for s-a.e. $z \in K_{a,b}[0,T]$, where

$$\begin{aligned}
&\Gamma_{\mathcal{F}_{\gamma,\beta}T_q^{(1)}(F)}(\vec{\lambda}) \\
&= \Big(\prod_{j=1}^{n} 2\pi B_j\Big)^{-1} \int_{\mathbb{R}^{2n}} f\big((-iq)^{-\frac{1}{2}}\vec{u} + \gamma\vec{v} + \beta\vec{\lambda}\big) \\
&\qquad \cdot \exp\Big\{-\sum_{j=1}^{n} \frac{(u_j - A_j)^2 + (v_j - A_j)^2}{2B_j}\Big\} d\vec{u}d\vec{v}.
\end{aligned} \tag{4.2}$$

The following theorem is one of main results in this paper.

THEOREM 4.3. *Let γ, β, q and F be as in Lemma 4.1 with β^2 is a nonzero real number. Then*

$$T_q^{(1)}(\mathcal{F}_{\gamma,\beta}F)(z) = \mathcal{F}_{\gamma,\beta}T^{(1)}_{\frac{q}{\beta^2}}(F)(z) \tag{4.3}$$

for s-a.e. $z \in K_{a,b}[0,T]$. Furthermore, both of the expressions in equation (4.3) are given by the expression

$$\Big(\prod_{j=1}^n 2\pi B_j\Big)^{-1}\int_{\mathbb{R}^{2n}} f\big(\beta(-iq)^{-\frac{1}{2}}\vec{u}+\gamma\vec{v}+\beta\langle\vec{\alpha},z\rangle\big) \cdot \exp\bigg\{-\sum_{j=1}^n \frac{(u_j-A_j)^2+(v_j-A_j)^2}{2B_j}\bigg\}d\vec{u}d\vec{v}$$

Proof. By using equations (4.1) and (4.2), we can prove Theorem 4.3. □

The following Theorem 4.4, we obtain several Feynman integrals involving L_1 analytic GFFTs and generalized integral transforms of functionals in $\mathbb{E}_0$. These formulas are given by (4.4)-(4.6) and (4.7).

THEOREM 4.4. *Let γ, β, q and F be as in Lemma 4.1. Then the following generalized analytic Feynman integrals follow quite readily :*

$$\begin{aligned} &E^{\mathrm{anf}_q}[\mathcal{F}_{\gamma,\beta}F] \\ &= \Big(\prod_{j=1}^n 2\pi B_j\Big)^{-1}\int_{\mathbb{R}^{2n}} f\big(\gamma\vec{u}+(-iq)^{-\frac{1}{2}}\beta\vec{v}\big) \cdot \exp\bigg\{-\sum_{j=1}^n \frac{(u_j-A_j)^2+(v_j-A_j)^2}{2B_j}\bigg\}d\vec{u}d\vec{v}, \end{aligned} \tag{4.4}$$

$$\begin{aligned} &E^{\mathrm{anf}_q}[T_q^{(1)}(F)] \\ &= \Big(\prod_{j=1}^n 2\pi B_j\Big)^{-1}\int_{\mathbb{R}^{2n}} f\big((-iq)^{-\frac{1}{2}}(\vec{u}+\vec{v})\big) \cdot \exp\bigg\{-\sum_{j=1}^n \frac{(u_j-A_j)^2+(v_j-A_j)^2}{2B_j}\bigg\}d\vec{u}d\vec{v}, \end{aligned} \tag{4.5}$$

$$E^{\mathrm{anf}_q}[T_q^{(1)}(\mathcal{F}_{\gamma,\beta}F)]$$

$$(4.6)\qquad = \left(\prod_{j=1}^{n} 2\pi B_j\right)^{-\frac{3}{2}} \int_{\mathbb{R}^{3n}} f\big(\gamma\vec{u} + (-iq)^{-\frac{1}{2}}\beta(\vec{v}+\vec{w})\big) \cdot \exp\bigg\{-\sum_{j=1}^{n} \frac{(u_j-A_j)^2+(v_j-A_j)^2+(w_j-A_j)^2}{2B_j}\bigg\} d\vec{u}d\vec{v}d\vec{w},$$

and

$$E^{\mathrm{anf}_q}[\mathcal{F}_{\gamma,\beta}T_q^{(1)}(F)]$$

$$(4.7)\qquad = \left(\prod_{j=1}^{n} 2\pi B_j\right)^{-\frac{3}{2}} \int_{\mathbb{R}^{3n}} f\big((-iq)^{-\frac{1}{2}}\vec{u} + \gamma\vec{v} + (-iq)^{-\frac{1}{2}}\beta\vec{w}\big) \cdot \exp\bigg\{-\sum_{j=1}^{n} \frac{(u_j-A_j)^2+(v_j-A_j)^2+(w_j-A_j)^2}{2B_j}\bigg\} d\vec{u}d\vec{v}d\vec{w}.$$

Proof. By using equations (3.2)-(3.5), (4.1) and (4.2), we can prove Theorem 4.4. □

References

[1] M. D. Brue, *A functional transform for Feynman integrals similar to the fourier transform*, thesis, University of Minnesota, Minneapolis, 1972.

[2] R. H. Cameron and W.T. Martin, *Fourier-Wiener transforms of analytic functionals*, Duke Math. J. **12** (1945), 489-507.

[3] ———, *Fourier-Wiener transforms of analytic functionals belonging to L_2 over the space C*, Duke Math. J. **14** (1947), 99-107.

[4] R. H. Cameron and D.A. Storvick, *An L_2 analytic Fourier-Feynman transform*, Michigan Math. J. **23** (1976), 1-30.

[5] S. J. Chang, J.G. Choi and D. Skoug, *Integration by parts formulas involving generalized Fourier-Feynman transforms on function space*, Trans. Amer. Math. Soc. **355** (2003), 2925-2948.

[6] S. J. Chang and D.M. Chung, *Conditional function space integrals with applications*, Rocky Mountain J. of Math. **26(1)** (1996), 37-62.

[7] S. J. Chang and H.S. Chung, *Generalized Fourier-Wiener function space transforms*, to appear in J. Korean Math. Soc..

[8] S. J. Chang, H.S. Chung and D. Skoug, *Integral transforms of functionals in $L_2(C_{a,b}[0,T])$*, submitted for publication.

[9] T. Huffman, C. Park and D. Skoug, *Analytic Fourier-Feynman transforms and convolution*, Trans. Amer. Math. Soc. **347** (1995), 661-673.

[10] G. W. Johnson and D.L. Skoug, *An L_p Analytic Fourier-Feynman transform*, Michigan Math. J. **26** (1979), 103-127.

[11] ———, *Scale-invariant measurability in Wiener space*, Pacific J. Math **83** (1979), 157-176.

[12] B. J. Kim, B.S. Kim and D. Skoug, *Integral transforms, convolution products and first variations*, Int. J. Math. Math. Soc. **11** (2004), 579-598.
[13] Y. J. Lee, *Integral transforms of analytic functions on abstract Wiener spaces*, J. Funct. Anal. **47** (1982), 153-164.
[14] ______, *Unitary operators on the space of L^2-functions over abstract Wiener spaces*, Soochow. J. Math. **13** (1987), 165-174.
[15] E. Nelson, *Dynamical theories of Brownian motion(2nd edition)*, Math. Notes, Princeton University Press, Princeton (1967).
[16] J. Yeh, *Stochastic Processes and the Wiener Integral*, Marcel Dekker, Inc., New York, 1973.
[17] ______, *Singularity on Gaussian measure on function space induced by Brownian motion process with non-stationary increment*, Illinois. J. Math. **15** (1971), 37-46.

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