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AN UPPER BOUND OF THE RECIPROCAL SUMS OF GENERALIZED SUBSET–SUM–DISTINCT SEQUENCE

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Abstract. In this paper, we present an upper bound of the reciprocal sums of generalized subset-sum-distinct sequences with respect to the first terms of the sequences. And we show the suggested upper bound is best possible. This is a kind of generalization of [1] which contains similar result for classical subset-sum-distinct sequences.

1. Introduction

We call an infinte strictly increasing sequence of positive integers a subset-sum-distinct sequence if every one of its finite subsets is uniquely determined by its sum. This traditional concept has been extenced to a generalized subset-sum-distinct sequence in [3] and [4]. Here we give the precise definition.

Definition 1.1.

(i) For a set A of real numbers, we say that A has the k-fold subset-sum-distinct property (briefly k-SSD-property) if for any two finite subsets X, Y of A,

$$\sum_{x\in X} \epsilon_x \cdot x = \sum_{y\in Y} \epsilon_y \cdot y \text{ for some } \epsilon_x,\, \epsilon_y \in \{1, 2, \cdots, k\} \quad \text{implies} \quad X = Y.$$

Also, we say that A is k-SSD or A is a k-SSD-set if it has the k-SSD-property.

(ii) An increasing sequence of positive integers $\{a_n\}_{n=1}^{\infty}$ is called a k-fold subset-sum-distinct sequence (briefly, k-SSD-sequence) if it has the k-SSD-property.

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For example, $\{109, 147, 161, 166, 168, 169\}$ is 2-SSD. In fact, it is the unique 2-SSD-set which has the least maximal element among all 2-SSD-sets of six elements of positive integers (See [3] or [4]). A classical subset-sum-distinct sequence is just a 1-SSD-sequence. Note that the greedy algorithm produces the k-SSD-sequence $1, k+1, (k+1)^2, (k+1)^3, \cdots$.

After a little preliminaries in the next section, for a k-SSD-sequence $\{a_n\}_{n=1}^{\infty}$, we present an upper bound of $\sum_{n=1}^{\infty} \frac{1}{a_n}$ with respect to a_1. This sort of reciprocal sum has been widely investigated for classical subset-sum-distinct sequence (see [1], [2], [3], [11]).

Regarding classical SSD-sequences, the most famous unsolved problem is Erdös' conjecture on a lower bound of the n-th term. For this subject, one may refer [6], [7], [8], [9]. For another widely known Conway-Guy conjecture, which is now a theorem proved by T. Bowman [5] in 1996, one may consult [4], [5], [10], [12].

2. Preliminaries

The following four lemmas will be used in the proof of the main theorems of the paper.

LEMMA 2.1. *Let* $\{a_n\}_{n=1}^{\infty}$ *be a* k*-SSD-sequence. Then*

$$a_1 + a_2 + \cdots + a_n \ \geq \ \frac{(k+1)^n - 1}{k}$$

for every $n \geq 1$.

Proof. See Lemma 2.2 in [3]. □

LEMMA 2.2. *If* $\{b_1, b_2, b_3, \cdots, b_m\}$ *is* k*-SSD and* $K > k(b_1 + b_2 + \cdots + b_m)$, *then also the set*

$$A := \{K + b_1, K + b_2, K + b_3, \cdots, K + b_m\}$$

is k*-SSD.*

Proof. Suppose that A is not k-SSD. By definition, there are two distinct subsets I, J of $\{1, 2, 3, \cdots, m\}$ such that $\sum_{i \in I} \epsilon_i (K+b_i) = \sum_{j \in J} \epsilon_j (K+$

$b_j)$ where $\epsilon_i, \epsilon_j \in \{1, 2, \cdots, k\}$. Since $\{b_1, b_2, \cdots, b_m\}$ is k-SSD, we have $\sum_{i\in I} \epsilon_i \neq \sum_{j\in J} \epsilon_j$. So, one may assume $\sum_{j\in J} \epsilon_j > \sum_{i\in I} \epsilon_i$. But then we have

$$\begin{aligned} K &\leq \left(\sum_{j\in J} \epsilon_j - \sum_{i\in I} \epsilon_i\right) K \\ &= \sum_{i\in I} \epsilon_i b_i - \sum_{j\in J} \epsilon_j b_j \leq k(b_1 + b_2 + \cdots + b_m) < K, \end{aligned}$$

a clear contradiction. □

Other two lemmas are from trivial observations on calculus.

Lemma 2.3. *Let f and g are decreasing functions on an interval. Then*

(i) $\alpha \cdot f + \beta \cdot g$ is also decreasing for fixed $\alpha > 0$, $\beta > 0$.

(ii) $f \cdot g$ is decreasing if f, g are both nonnegative on the interval.

Proof. (i) is obvious. For (ii), let x, y be in the interval with $x < y$. Then

$$f(x)g(x) - f(y)g(y) = f(x)\left(g(x) - g(y)\right) + g(y)\left(f(x) - f(y)\right) \leq 0.$$

□

Lemma 2.4. *The function*

$$f(x) = \frac{\log(2x)}{\log(x+1)}$$

is positive decreasing on $[4, \infty)$.

Proof. Differentiating f, we have

$$f'(x) = \frac{(x+1)\log(x+1) - x\log(2x)}{x(x+1)}.$$

Hence it's enough to show that $(x+1)\log(x+1) \leq x\log(2x)$ on $[4, \infty)$. Observe that

$$\begin{aligned} &(x+1)\log(x+1) \leq x\log(2x) \\ \Longleftrightarrow\ &(x+1)^{x+1} \leq (2x)^x \iff \left(1 + \frac{1}{x}\right)^x \leq \frac{2^x}{x+1}. \end{aligned}$$

But the last inequality follows immediately from the fact that $2^x/(x+1)$ is increasing on $[4,\infty)$ and, for $x \geq 4$,

$$\left(1+\frac{1}{x}\right)^x \leq e \leq \frac{16}{5} \leq \frac{2^x}{x+1}.$$

$\square$

3. An upper bound

Now we present a kind of optimal upper bound of $\sum_{n=1}^{\infty} \frac{1}{a_n}$ for k-SSD-sequences $\{a_n\}_{n=1}^{\infty}$. The first theorem states the upper bound and the second one shows the optimality.

THEOREM 3.1. *Let* $\mathbf{a} = \{a_n\}_{n=1}^{\infty}$ *be a k-SSD-sequence with* $a_1 > 1$. *Then*

$$\sum_{n=1}^{\infty} \frac{1}{a_n} \;\leq\; C \cdot \frac{\log a_1}{a_1}$$

where C *is either of*

(i) $C = \dfrac{2}{\log 2}\left(1+\dfrac{2k\log(2k)}{(2k-1)\log(k+1)}\right)$ *a constant that depends on* k,

(ii) $C = \dfrac{6}{\log 2}$, *an absolute constant.*

Proof. Let $b_j = a_{2j} - a_{2j-1}$ for $j = 1,2,3,\cdots$. Since the sequence $\mathbf{a}$ is k-SSD, the set $\{b_1, b_2, b_3, \cdots\}$ is k-SSD too. We claim that

$$a_{2j+1} \;\geq\; a_1 + b_1 + b_2 + \cdots + b_j, \qquad j = 1,2,3,\cdots. \tag{3.1}$$

We use induction on j. Since, by definition, $b_1 = a_2 - a_1$, we have $a_3 > a_2 = a_1 + b_1$ which satisfies the claim (3.1) for $j = 1$. Now assume that

$$a_{2j+1} \;\geq\; a_1 + b_1 + b_2 + \cdots + b_j\,.$$

By definition, $b_{j+1} = a_{2j+2} - a_{2j+1}$, and so $a_{2j+2} = a_{2j+1} + b_{j+1}$. Thus

$$a_{2j+3} \;\geq\; a_{2j+2} \;= a_{2j+1} + b_{j+1} \;\geq\; a_1 + b_1 + b_2 + \cdots + b_j + b_{j+1}$$

and this completes the proof of the claim (3.1). Applying Lemma 2.1 to the set $\{b_1, b_2, b_3, \cdots b_j\}$, we obtain

$$a_{2j+1} \;\geq\; a_1 + b_1 + b_2 + \cdots + b_j \;\geq\; a_1 + \frac{(k+1)^j - 1}{k}$$

for $j = 0, 1, 2, 3, \cdots$. Therefore we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{a_n} &= \sum_{j=0}^{\infty} \left(\frac{1}{a_{2j+1}} + \frac{1}{a_{2j+2}} \right) \le 2 \sum_{j=0}^{\infty} \frac{1}{a_{2j+1}} \\
&\le 2 \sum_{j=0}^{\infty} \frac{k}{ka_1 + (k+1)^j - 1} \le \frac{2}{a_1} + 2 \cdot \int_0^{\infty} \frac{k}{ka_1 + (k+1)^x - 1}\, dx \\
&= \frac{2}{a_1} + \frac{2k}{\log(k+1)} \cdot \frac{\log(ka_1)}{ka_1 - 1} = g(a_1) \cdot \frac{\log a_1}{a_1}
\end{aligned}$$

where

$$\begin{aligned}
g(x) &= \frac{x}{\log x} \left(\frac{2}{x} + \frac{2k \log(kx)}{(kx-1)\log(k+1)} \right) \\
&= \frac{2}{\log x} + \frac{2k \log k}{\log(k+1)} \cdot \frac{x}{(kx-1)\log x} + \frac{2k}{\log(k+1)} \cdot \frac{x}{(kx-1)}.
\end{aligned}$$

Since $\frac{1}{\log x}$ and $\frac{x}{(kx-1)} = \frac{1}{k}\left(1 + \frac{1}{kx-1}\right)$ are positive decreasing on $[2, \infty)$, by Lemma 2.3, $g(x)$ is decreasing on $[2, \infty)$. Hence

$$g(a_1) \le g(2) = \frac{2}{\log 2} \left(1 + \frac{2k \log(2k)}{(2k-1)\log(k+1)} \right)$$

and we may take C as in (i). To obtain the absolute constant in (ii), let

$$h(k) = \frac{2k \log(2k)}{(2k-1)\log(k+1)}.$$

Note $2x/(2x-1)$ is positive decreasing on $[1, \infty)$ and, by Lemma 2.4,

$$\frac{\log(2x)}{\log(x+1)}$$

is decreasing on $[4, \infty)$. Applying Lemma 2.3, we have

$$\max\{h(k) \;:\; k = 1, 2, 3, \cdots\} = \max\{h(1), h(2), h(3), h(4)\}$$

which is $h(1) = 2$ by calculation. Thus

$$g(a_1) \le g(2) = \frac{2}{\log 2}(1 + h(k)) \le \frac{6}{\log 2}$$

and we can take $C = 6/\log 2$. □

Finally, we show that the inequality in Theorem 3.1 is essentially best possible in the following sense:

Theorem 3.2. *Let $f(x)$ be a positive real valued function that is defined on $(1, \infty)$ such that*

$$f(x) \cdot \frac{\log x}{x} \longrightarrow \infty \tag{3.2}$$

as $x \to \infty$. Then for any $T > 0$, there exists a k-SSD-sequence $\{a_n\}_{n=1}^{\infty}$ such that

$$a_1 > 1 \qquad \textit{and} \qquad f(a_1) \sum_{n=1}^{\infty} \frac{1}{a_n} > T .$$

Proof. For k-SSD-sequences $\mathbf{a}(1)$, $\mathbf{a}(2)$, $\mathbf{a}(3)$, $\cdots$, let us use the notations

$$\mathbf{a}(m) = \{a_{mn}\}_{n=1}^{\infty} \quad \text{for} \quad m = 1, 2, 3, \cdots .$$

We are to construct k-SSD-sequences $\mathbf{a}(m)$ for $m = 1, 2, 3, \cdots$ so that $a_{m1} > 1$ and

$$f(a_{m1}) \sum_{n=1}^{\infty} \frac{1}{a_{mn}} \longrightarrow \infty$$

as $m \to \infty$. We know $\{1, k+1, (k+1)^2, (k+1)^3, \cdots, (k+1)^{m-1}\}$ is k-SSD. Applying Lemma 2.2 with $K = (k+1)^m$, we obtain k-SSD property of the set

$$\{K+1, K+(k+1), K+(k+1)^2, \cdots, K+(k+1)^{m-1}\} .$$

Now, for a given positive integer m, we define

$$a_{mn} = \begin{cases} K + (k+1)^{n-1} , & \text{if} \quad 1 \le n \le m \\ (k+1) \displaystyle\sum_{i=1}^{n-1} a_{mi} , & \text{if} \quad n > m . \end{cases}$$

From the construction, it's obvious that $\mathbf{a}(m)$ is k-SSD and $a_{m1} > 1$.

Moreover,

$$
\begin{aligned}
f(a_{m1})\sum_{n=1}^{\infty}\frac{1}{a_{mn}} &\geq f(a_{m1})\sum_{n=1}^{m}\frac{1}{a_{mn}} \\
&= f(a_{m1})\sum_{n=1}^{m}\frac{1}{a_{m1}+(k+1)^{n-1}-1} \\
&\geq f(a_{m1})\int_0^m \frac{1}{a_{m1}+(k+1)^x-1}\,dx \\
&= f(a_{m1})\cdot\frac{1}{\log(k+1)}\cdot\frac{\log a_{m1}-\log 2}{a_{m1}-1} \\
&\geq \alpha\cdot f(a_{m1})\cdot\frac{\log a_{m1}}{a_{m1}}
\end{aligned}
$$

for some positive α. Thus the theorem follows from (3.2) since $a_{m1} = (k+1)^m+1 \to \infty$ as $m \to \infty$.

$\square$

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