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ON THE HYERS-ULAM-RASSIAS STABILITY OF A CAUCHY-JENSEN FUNCTIONAL EQUATION II

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Abstract. In this paper, we obtain the Hyers-Ulam-Rassias stability of a Cauchy-Jensen functional equation

$$f(x+y,z)-f(x,z)-f(y,z)=0,$$
$$2f(x,\frac{y+z}{2})-f(x,y)-f(x,z)=0$$

in the spirit of Th. M. Rassias.

1. Introduction

In 1940, S.M. Ulam [11] raised a question concerning the stability of homomorphisms: Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exists a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality

$$d(h(xy),h(x)h(y)) < \delta$$

for all $x, y \in G_1$ then there is a homomorphism $H : G_1 \to G_2$ with

$$d(h(x),H(x)) < \varepsilon$$

for all $x \in G_1$? The case of approximately additive mappings was solved by D.H.Hyers [2] under the assumption that G_1 and G_2 are Banach spaces. In 1978, Th.M.Rassias [10] gave a generalization. Since then, the further generalization has been extensively investigated by a number of mathematicians[1, 4-6, 8].

Throughout this paper, let X be a normed space and Y a Banach space. A mapping $g : X \to Y$ is called a Cauchy mapping (respectively, a Jensen mapping) if g satisfies the functional equation $g(x+y) = g(x)+g(y)$ (respectively, $2g(\frac{x+y}{2}) = g(x)+g(y)$). For given mappings

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$f, f_1, f_2, f_3, f_4 : X \times X \to Y$, we define

$$C_1 f(x,y,z) := f(x+y,z) - f(x,z) - f(y,z),$$
$$C_2 f(x,y,z) := f(x,y+z) - f(x,y) - f(x,z),$$
$$J_2 f(x,y,z) := 2f(x, \frac{y+z}{2}) - f(x,z) - f(y,z)$$

for all $x, y, z, w \in X$. A mapping $f : X \times X \to Y$ is called a biadditive (respectively Cauchy-Jensen) mapping if f satisfies the functional equations $C_1 f = 0$ and $C_2 f = 0 (C_1 f = 0$ and $J_2 f = 0$, respectively).

In 2006, Park and Bae [9] obtained the generalized Hyers-Ulam stability of the Cauchy-Jensen functional equation. From Theorem 6 in [9], we get the following theorem:

Theorem 1.1. *Let $0 \le p, q < 1$ and $\varepsilon > 0$. Let $f : X \times X \to Y$ be a mapping such that*

$$\|C_1 f(x,y,z)\| \le \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p),$$
$$\|J_2 f(x,y,z)\| \le \varepsilon(\|x\|^q + \|y\|^q + \|z\|^q)$$

for all $x, y, z \in X$. Then there exist two Cauchy-Jensen mappings $F_C, F_J : X \times X \to Y$ such that

$$\|f(x,y) - F_C(x,y)\| \le (\frac{2}{2-2^p} + 3)\varepsilon\|x\|^p + (\frac{3}{3-3^p} + 1)\varepsilon\|y\|^p,$$
$$\|f(x,y) - f(x,0) - F_J(x,y)\| \le (\frac{4}{2-2^p} + 1)\varepsilon\|x\|^p$$
$$+(\frac{3+3^p}{3-3^p} + 6 + 2\cdot 3^p)\varepsilon\|y\|^p$$

for all $x, y \in X$. The mappings $F_C, F_J : X \times X \to Y$ are given by

$$F_C(x,y) := \lim_{j\to\infty} \frac{1}{2^j} f(2^j x, y), \quad F_J(x,y) := \lim_{j\to\infty} \frac{1}{3^j} f(x, 3^j y)$$

for all $x, y \in X$.

In 2007, Lee [7] obtained the Hyers-Ulam-Rassias stability of the Cauchy-Jensen functional equation. From Theorem 2.1, 2.2, 2.3 and 2.4 in [7], we get the following theorem:

Theorem 1.2. *Let $0 \le p, q (\neq 1)$ and $\varepsilon > 0$. Let $f : X \times X \to Y$ be a mapping such that*

$$\|C_1 f(x,y,z)\| \le \varepsilon(\|x\|^p + \|y\|^p)\|z\|^q,$$
$$\|J_2 f(x,y,z)\| \le \varepsilon\|x\|^p(\|y\|^q + \|z\|^q)$$

for all $x, y, z \in X$*. Then there exists a unique Cauchy-Jensen mapping* $F : X \times X \to Y$ *such that*

$$\|f(x,y) - F_1(x,y)\| \le \frac{2\varepsilon}{|2-2^p|}\|x\|^p\|y\|^q$$

for all $x, y \in X$*.*

In this paper, we investigate the stability of a Cauchy-Jensen functional equation in the sense of Th.M.Rassias. We have better stability results than that of Theorem 1.1. We improved Theorem 1.2 under different inequality condition.

2. Stability of a Cauchy-Jensen functional equation

The authors and Cho[3] established the basic properties of a Cauchy-Jensen mapping in the following lemma.

Lemma 2.1. *Let* $f : X \times X \to Y$ *be a Cauchy-Jensen mapping. Then*

$$\begin{aligned} f(x,y) &= 2^n f(\frac{x}{2^n}, y), \\ f(x,y) &= 4^n f(\frac{x}{2^n}, \frac{y}{2^n}) - (4^n - 2^n) f(\frac{x}{2^n}, 0), \\ f(x,y) &= \frac{f(2^n x, 2^n y)}{4^n} + (2^n - 1) f(\frac{x}{2^n}, 0) \end{aligned}$$

for all $x, y \in X$ *and* $n \in \mathbb{N}$*.*

Lemma 2.2. *Let* $f : X \times X \to Y$ *be a mapping such that*

$$C_1 f(x,y,z) = 0, \quad J_2 f(x,y,z) = 0$$

for all $x, y, z \in X\backslash\{0\}$ *and* $f(0,0) = 0$*. Then*

$$C_1 f(x,y,z) = 0, \quad J_2 f(x,y,z) = 0$$

for all $x, y, z \in X$*.*

Proof. Since

$$\begin{aligned} f(0,y) &= -C_1 f(2x,-2x,y) + 2C_1 f(x,-x,y) - C_1 f(x,x,y) - C_1 f(-x,-x,y) \\ &= 0 \end{aligned}$$

for all $y \in X\backslash\{0\}$, we get

$$\begin{aligned} C_1f(x,y,0) &= \frac{1}{2}[J_2f(x+y,z,-z) - J_2f(x,z,-z) \\ &\quad - J_2f(y,z,-z) + C_1f(x,y,z) + C_1F(x,y,-z)] = 0, \\ J_2f(x,y,0) &= J_2f(x,0,y) = -J_2f(x,\frac{y}{2},\frac{3y}{2}) - \frac{1}{2}J_2f(x,y,2y) \\ &\quad + \frac{1}{2}J_2f(x,-y,2y) - \frac{1}{2}J_2f(x,y,-y) = 0, \\ C_1f(x,0,0) &= C_1f(0,y,0) = C_1f(0,0,z) = C_1f(0,0,0) \\ &= C_1f(x,0,z) = C_1f(0,y,z) = 0, \\ J_2f(0,y,z) &= J_2f(0,y,0) = J_2f(0,0,z) = J_2f(x,0,0) \\ &= J_2f(0,0,0) = 0 \end{aligned}$$

for all $x, y, z \in X\backslash\{0\}$ as desired. □

THEOREM 2.3. *Let $p < 1$ and $\varepsilon > 0$. Let $f : X \times X \to Y$ be a mapping such that*

$$\|C_1f(x,y,z)\| \le \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p) \tag{2.1}$$

$$\|J_2f(x,y,z)\| \le \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p) \tag{2.2}$$

for all $x, y, z \in X\backslash\{0\}$. Then there exists a unique Cauchy-Jensen mapping $F : X \times X \to Y$ such that

$$\|f(x,y) - F(x,y)\| \le \frac{2\varepsilon}{2-2^p}\|x\|^p + \varepsilon\|y\|^p \tag{2.3}$$

for all $x, y \in X\backslash\{0\}$. The mapping $F : X \times X \to Y$ is given by

$$F(x,y) := \lim_{j\to\infty} \frac{1}{2^j} f(2^j x, y)$$

for all $x, y \in X$.

Proof. By (2.1), we get

$$\begin{aligned} \|\frac{1}{2^j}f(2^jx,y) - \frac{1}{2^{j+1}}f(2^{j+1}x,y)\| &= \frac{1}{2^{j+1}}\|C_1f(2^jx,2^jx,y)\| \\ &\le \frac{\varepsilon}{2^j}2^{jp}\|x\|^p + \frac{\varepsilon}{2^{j+1}}\|y\|^p \end{aligned}$$

for all $x, y \in X\backslash\{0\}$ and $j \in \mathbb{N}$. For given integers l, m $(0 \le l < m)$,

$$\|\frac{f(2^lx,y)}{2^l} - \frac{f(2^mx,y)}{2^m}\| \le \sum_{j=l}^{m-1}[\frac{\varepsilon}{2^j}2^{jp}\|x\|^p + \frac{\varepsilon}{2^{j+1}}\|y\|^p] \tag{2.4}$$

for all $x, y \in X\backslash\{0\}$. By $p < 1$, the sequence $\{\frac{1}{2^j} f(2^j x, y)\}$ is a Cauchy sequence for all $x, y \in X\backslash\{0\}$. Since Y is complete, the sequence $\{\frac{1}{2^j} f(2^j x, y)\}$ converges for all $x, y \in X\backslash\{0\}$. Define $F_1 : X \times X \to Y$ by

$$F_1(x, y) := \lim_{j\to\infty} \frac{1}{2^j} f(2^j x, y)$$

for all $x, y \in X\backslash\{0\}$. Putting $l = 0$ and taking $m \to \infty$ in (2.4), one can obtain the inequalities

$$\|f(x, y) - F_1(x, y)\| \leq \frac{2^p \varepsilon}{2 - 2^p}\|x\|^p + \varepsilon\|y\|^p$$

for all $x, y \in X\backslash\{0\}$. By (2), we obtain

$$\begin{aligned} 2 \lim_{j\to\infty} \frac{1}{2^j} f(2^j x, 0) \quad &= F_1(x, y) + F_1(x, -y) + \lim_{j\to\infty} \frac{1}{2^j} J_2 f(2^j x, y, -y) \\ &= F_1(x, y) + F_1(x, -y) \end{aligned}$$

for all $x \in X\backslash\{0\}$. Hence we can define $F : X \times X \to Y$ by

$$F(x, y) := \lim_{j\to\infty} \frac{1}{2^j} f(2^j x, y)$$

for all $x, y \in X$. Using (2.1) and (2.2), we get

$$\begin{aligned} C_1 F(x, y, z) &= \lim_{j\to\infty} \frac{1}{2^j} C_1 f(2^j x, 2^j y, z) = 0, \\ J_2 F(x, y, z) &= \lim_{j\to\infty} \frac{1}{2^j} J_2 f(2^j x, y, z) = 0 \end{aligned}$$

for all $x, y, z \in X\backslash\{0\}$. Since $F(0, 0) = 0$, we can apply Lemma 2.2. Hence, F is a Cauchy-Jensen mapping satisfying (2.3). Now, let $F' : X \times X \to Y$ be another Cauchy-Jensen mapping satisfying (2.3). Then we have

$$\begin{aligned} \|F(x, y) - F'(x, y)\| &\leq \frac{1}{2^n}\|F(2^n x, y) - f(2^n x, y)\| \\ &\quad + \frac{1}{2^n}\|f(2^n x, y) - F'(2^n x, y)\| \\ &\leq (\frac{2^p}{2})^n \frac{4\varepsilon}{2 - 2^p}\|x\|^p + \frac{2\varepsilon}{2^n}\|y\|^p \end{aligned}$$

for all $x, y \in X\backslash\{0\}$ and $n \in \mathbb{N}$. As $n \to \infty$, we may conclude that $F(x, y) = F'(x, y)$ for all $x, y \in X\backslash\{0\}$. Since F, F' are Cauchy-Jensen mappings,

$$F(0,y) = 0 = F'(0,y),$$
$$F(x,0) = \frac{1}{2}[F(x,y) + F(x,-y)] = \frac{1}{2}[F'(x,y) + F'(x,-y)] = F'(x,0)$$

for all $x, y \in X\backslash\{0\}$. Thus such a Cauchy-Jensen mapping $F : X \times X \to Y$ is unique. □

Corollary 2.4. *Let ε, f be as in Theorem 2.3 and let $p < 0$. Then there exists a unique Cauchy-Jensen mapping $F : X \times X \to Y$ such that*

$$\|f(x,y) - F(x,y)\| \le \min\{(\frac{2}{2-2^p} + \frac{1}{2})\varepsilon\|x\|^p, 3\varepsilon\|y\|^p\}$$

for all $x, y \neq 0$.

Proof. From (2.1)-(2.3), we get

$$\begin{aligned}\|f(x,y) - F(x,y)\| &= \|C_1 f((k+1)x, -kx, y) + (f-F)((k+1)x, y) \\ &\quad + (f-F)(-kx, y) - C_1 F((k+1)x, -kx, y)\| \\ &\le (\frac{2}{2-2^p} + 1)((k+1)^p + k^p)\varepsilon\|x\|^p + 3\varepsilon\|y\|^p, \\ \|f(x,y) - F(x,y)\| &= \frac{1}{2}\|J_2 f(x, (k+2)y, -ky) + (f-F)(x, (k+2)y) \\ &\quad + (f-F)(x, -ky) - J_2 F(x, (k+2)y, -ky)\| \\ &\le (\frac{2}{2-2^p} + \frac{1}{2})\varepsilon\|x\|^p + ((k+2)^p + k^p)\varepsilon\|y\|^p\end{aligned}$$

for all $x, y \neq 0$ and $k \in \mathbb{N}$. Since k is an arbitrary positive integer and $p < 0$, we get the desired result. □

We can prove the following theorem by the similar method used to prove Theorem 2.3.

Theorem 2.5. *Let $p < 1$, $\varepsilon > 0$ and $q \in \mathbb{R}$. Let $f : X \times X \to Y$ be a mapping such that*

$$\|C_1 f(x,y,z)\| \le \varepsilon(\|x\|^p + \|y\|^p)\|z\|^q$$
$$\|J_2 f(x,y,z)\| \le \varepsilon\|x\|^p(\|y\|^q + \|z\|^q)$$

for all $x, y, z \in X\backslash\{0\}$. Then there exists a unique Cauchy-Jensen mapping $F : X \times X \to Y$ such that

$$\|f(x,y) - F(x,y)\| \le \frac{2\varepsilon}{2-2^p}\|x\|^p\|y\|^q \tag{2.5}$$

for all $x, y \in X\backslash\{0\}$. *The mapping* $F : X \times X \to Y$ *is given by*

$$F(x,y) := \lim_{j\to\infty} \frac{1}{2^j} f(2^j x, y)$$

for all $x, y \in X$.

COROLLARY 2.6. *Let* ε, f *be as in Theorem 2.5 and let* $p, q < 0$. *Then there exists a unique Cauchy-Jensen mapping* $F : X \times X \to Y$ *such that*

$$f(x,y) = F(x,y)$$

for all $(x,y) \neq (0,0)$.

Proof. From (2.5), we get

$$\begin{aligned}\|(f-F)(x,y)\| &= \|C_1 f((k+1)x, -kx, y) + (f-F)((k+1)x, y)\\ &\quad +(f-F)(-kx,y) - C_1F((k+1)x,-kx,y)\|\\ &\leq \frac{4-2^p}{2-2^p}((k+1)^p + k^p)\varepsilon\|x\|^p\|y\|^q,\\ \|(f-F)(0,y)\| &= \|C_1 f(kx,-kx,y) + (f-F)(kx,y) + (f-F)(-kx,y)\|\\ &\leq (\frac{4}{2-2^p}+2)k^p\varepsilon\|x\|^p\|y\|^q,\\ \|(f-F)(x,0)\| &= \frac{1}{2}\|J_2 f(x,ky,-ky) + (f-F)(x,ky) + (f-F)(x,-ky)\|\\ &\leq \frac{4-2^p}{2-2^p}k^q\varepsilon\|x\|^p\|y\|^q\end{aligned}$$

for all $x, y \neq 0$ and $k \in \mathbb{N}$. Since k is an arbitrary positive integer and $p, q < 0$, we get the desired result. □

THEOREM 2.7. *Let* $2 < p$ *and* $\varepsilon > 0$. *Let* $f : X \times X \to Y$ *be a mapping such that*

$$\|C_1 f(x,y,z)\| \leq \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p) \tag{2.6}$$

$$\|J_2 f(x,y,z)\| \leq \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p) \tag{2.7}$$

for all $x, y, z \in X$. *Then there exists a unique Cauchy-Jensen mapping* $F : X \times X \to Y$ *such that*

$$\|f(x,y) - F(x,y)\| \leq (\frac{6\varepsilon}{2^p-4} + \frac{2\varepsilon}{2^p-2})\|x\|^p + \frac{3\cdot 2^p\varepsilon}{2^p-4}\|y\|^p \tag{2.8}$$

for all $x, y \in X$. *The mapping* $F : X \times X \to Y$ *is given by*

$$F(x,y) := \lim_{j\to\infty}[4^j f(\frac{x}{2^j}, \frac{y}{2^j}) - (4^j - 2^j) f(\frac{x}{2^j}, 0)]$$

for all $x, y \in X$.

Proof. By (2.6) and (2.7), we get

$$\|2^j f(\frac{x}{2^j},0) - 2^{j+1} f(\frac{x}{2^{j+1}},0)\| = 2^j \|C_1 f(\frac{x}{2^{j+1}},\frac{x}{2^{j+1}},0)\| \le \frac{2^{j+1}\varepsilon}{2^{(j+1)p}}\|x\|^p$$

and

$$\begin{aligned}&\|4^j(f(\frac{x}{2^j},\frac{y}{2^j}) - f(\frac{x}{2^j},0)) - 4^{j+1}(f(\frac{x}{2^{j+1}},\frac{y}{2^{j+1}}) - f(\frac{x}{2^{j+1}},0))\|\\&= 4^j\|C_1 f(\frac{x}{2^{j+1}},\frac{x}{2^{j+1}},\frac{y}{2^j}) - 2J_2 f(\frac{x}{2^{j+1}},\frac{y}{2^j},0) - C_1 f(\frac{x}{2^{j+1}},\frac{x}{2^{j+1}},0)\|\\&\le \frac{3\cdot 4^j\varepsilon}{2^{jp}}(\frac{2}{2^p}\|x\|^p + \|y\|^p)\end{aligned}$$

for all $x, y \in X$ and $j \in \mathbb{N}$. For given integers l, m $(0 \le l < m)$,

$$\|2^l f(\frac{x}{2^l},0) - 2^m f(\frac{x}{2^m},0) \le \sum_{j=l}^{m-1} \frac{2^{j+1}}{2^{(j+1)p}}\varepsilon\|x\|^p, \tag{2.9}$$

and

$$\begin{aligned}&\|4^l(f(\frac{x}{2^l},\frac{y}{2^l}) - f(\frac{x}{2^l},0)) - 4^m(f(\frac{x}{2^m},\frac{y}{2^m}) - f(\frac{x}{2^m},0))\|\\&\qquad \le \sum_{j=l}^{m-1}\frac{3\cdot 4^j\varepsilon}{2^{jp}}(\frac{2}{2^p}\|x\|^p + \|y\|^p)\end{aligned}$$

for all $x, y \in X$. By $p > 2$, the sequences $\{2^n f(\frac{x}{2^n},0)\}$ and $\{4^n(f(\frac{x}{2^n},\frac{y}{2^n}) - f(\frac{x}{2^n},0))\}$ are Cauchy sequences for all $x, y \in X$. Since Y is complete, the sequences $\{2^n f(\frac{x}{2^n},0)\}$ and $\{4^n(f(\frac{x}{2^n},\frac{y}{2^n}) - f(\frac{x}{2^n},0))\}$ converge for all $x, y \in X$. Define $F_1, F_2 : X \times X \to Y$ by

$$F_1(x,y) := \lim_{j\to\infty} 2^j f(\frac{x}{2^j},0)$$

and

$$F_2(x,y) := \lim_{j\to\infty} 4^j(f(\frac{x}{2^j},\frac{y}{2^j}) - f(\frac{x}{2^j},0))$$

for all $x, y \in X$. Putting $l = 0$ and taking $m \to \infty$ in (2.9) and (2.10), one can obtain the inequalities

$$\|f(x,0) - F_1(x,y)\| \le \frac{2\varepsilon}{2^p - 2}\|x\|^p$$

and

$$\|f(x,y) - f(x,0) - F_2(x,y)\| \le \frac{6\varepsilon}{2^p-4}\|x\|^p + \frac{3\cdot 2^p\varepsilon}{2^p-4}\|y\|^p$$

for all $x, y \in X$. By (2.6), (2.7) and the definitions of F_1 and F_2, we get

$$C_1F_1(x,y,z) = \lim_{j\to\infty} 2^j C_1 f(\frac{x}{2^j}, \frac{y}{2^j}, 0) = 0,$$
$$J_2F_1(x,y,z) = 0,$$
$$C_1F_2(x,y,z) = \lim_{j\to\infty} 4^j [C_1 f(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}) - C_1 f(\frac{x}{2^j}, \frac{y}{2^j}, 0)] = 0,$$
$$J_2F_2(x,y,z) = \lim_{j\to\infty} 4^j J_2 f(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}) = 0$$

for all $x, y, z \in X$ and so F is a Cauchy-Jensen mapping satisfying (2.8) where F is given by

$$F(x,y) = F_1(x,y) + F_2(x,y).$$

Cauchy-Jensen mappings. Now, let $F' : X \times X \to Y$ be another Cauchy-Jensen mapping satisfying (2.8). Using Lemma 2.1, we have

$$\begin{aligned}
\|(F-F')(x,y)\| &= \|4^n(F-F')(\frac{x}{2^n}, \frac{y}{2^n}) + (2^n - 4^n)(F-F')(\frac{x}{2^n}, 0))\| \\
&\le 4^n\|(F-f)(\frac{x}{2^n}, \frac{y}{2^n})\| + 4^n\|(f-F')(\frac{x}{2^n}, \frac{y}{2^n})\| \\
&\quad +4^n\|(F-f)(\frac{x}{2^n}, 0)\| + 4^n\|(f-F')(\frac{x}{2^n}, 0)\| \\
&\le \frac{4^{n+1}}{2^{np}}(\frac{6\varepsilon}{2^p-4} + \frac{2\varepsilon}{2^p-2})\|x\|^p + \frac{4^n}{2^{np}} \cdot \frac{6 \cdot 2^p\varepsilon}{2^p-4}\|y\|^p
\end{aligned}$$

for all $x, y \in X$ and $n \in \mathbb{N}$. As $n \to \infty$, we may conclude that $F(x,y) = F'(x,y)$ for all $x, y \in X$. Thus such a Cauchy-Jensen mapping $F : X \times X \to Y$ is unique. □

THEOREM 2.8. *Let $1 < p < 2$ and $\varepsilon > 0$. Let $f : X \times X \to Y$ be a mapping such that*

$$\|C_1 f(x,y,z)\| \le \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p) \tag{2.10}$$
$$\|J_2 f(x,y,z)\| \le \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p) \tag{2.11}$$

for all $x, y, z \in X$. Then there exists a unique Cauchy-Jensen mapping $F : X \times X \to Y$ such that

$$\|f(x,y) - F(x,y)\| \le (\frac{2\varepsilon}{2^p-2} + \frac{6\varepsilon}{4-2^p})\|x\|^p + \frac{3 \cdot 2^p\varepsilon}{4-2^p}\|y\|^p \tag{2.12}$$

for all $x, y \in X$. The mapping $F : X \times X \to Y$ is given by

$$F(x,y) := \lim_{j\to\infty} [\frac{f(2^jx, 2^jy) - f(2^jx, 0)}{4^j} + 2^j f(\frac{x}{2^j}, 0)]$$

for all $x, y \in X$.

Proof. Let F_1 be as in Theorem 2.7. By (2.11) and (2.12), we get

$$\begin{aligned}&\|\frac{1}{4^j}(f(2^jx,2^jy)-f(2^jx,0))-\frac{1}{4^{j+1}}(f(2^{j+1}x,2^{j+1}y)-f(2^{j+1}x,0))\|\\&=\frac{1}{4^{j+1}}\|C_1f(2^jx,2^jx,2^{j+1}y)-2J_2f(2^jx,2^{j+1}y,0)-C_1f(2^jx,2^jx,0)\|\\&\le\frac{3\cdot 2^{jp}\varepsilon}{4^{j+1}}(2\|x\|^p+2^p\|y\|^p)\end{aligned}$$

for all $x, y \in X$ and $j \in \mathbb{N}$. For given integers l, m $(0 \le l < m)$,

$$\begin{aligned}&\|\frac{1}{4^l}(f(2^lx,2^ly)-f(2^lx,0))-\frac{1}{4^m}(f(2^mx,2^my)-f(2^mx,0))\|\\&\le\sum_{j=l}^{m-1}\frac{3\cdot 2^{jp}\varepsilon}{4^{j+1}}(2\|x\|^p+2^p\|y\|^p)\end{aligned}\tag{2.13}$$

for all $x, y \in X$. By $1 < p < 2$, the sequence $\{\frac{1}{4^n}(f(2^nx,2^ny)-f(2^nx,0))\}$ is a Cauchy sequence for all $x, y \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n}(f(2^nx,2^ny)-f(2^nx,0))\}$ converges for all $x, y \in X$. Define $F_2 : X \times X \to Y$ by

$$F_2(x,y) := \lim_{j\to\infty}\frac{1}{4^j}[f(2^jx,2^jy)-f(2^jx,0)]$$

for all $x, y \in X$. Putting $l = 0$ and taking $m \to \infty$ in (14), one can obtain the inequality

$$\|f(x,y)-f(x,0)-F_2(x,y)\| \le \frac{6\varepsilon}{4-2^p}\|x\|^p+\frac{3\cdot 2^p\varepsilon}{4-2^p}\|y\|^p$$

for all $x, y \in X$. By (2.12), (2.13) and the definition of F_2, we get

$$\begin{aligned}C_1F_2(x,y,z)&=\lim_{j\to\infty}\frac{1}{4^j}[C_1f(2^jx,2^jy,2^jz)-C_1f(2^jx,2^jy,0)]=0,\\J_2F_2(x,y,z)&=\lim_{j\to\infty}\frac{1}{4^j}J_2f(2^jx,2^jy,2^jz)=0\end{aligned}$$

for all $x, y, z \in X$ and so F is a Cauchy-Jensen mapping satisfying (2.13) where F is given by

$$F(x,y) = F_1(x,y) + F_2(x,y).$$

Cauchy-Jensen mappings. Now, let $F' : X\times X \to Y$ be another Cauchy-Jensen mapping satisfying (2.13). Using Lemma 2.1, we have

$$\begin{aligned}\|F(x,y) &- F'(x,y)\| \\ &= \|\frac{(F-F')(2^n x, 2^n y)}{4^n} + (2^n-1)(F-F')(\frac{x}{2^n},0)\| \\ &\le \frac{\|(F-f)(2^n x,2^n y)\| + \|(f-F')(2^n x,2^n y)\|}{4^n} \\ &+2^n\|(F-f)(\frac{x}{2^n},0)\| + 2^n\|(f-F')(\frac{x}{2^n},0)\| \\ &\le (\frac{2^{np}}{4^n}+\frac{2^n}{2^{np}})(\frac{4\varepsilon}{2^p-2}+\frac{12\varepsilon}{4-2^p})\|x\|^p + \frac{2^{np}}{4^n}\frac{6\cdot 2^p\varepsilon}{4-2^p}\|y\|^p\end{aligned}$$

for all $x,y\in X$ and $n\in\mathbb{N}$. As $n\to\infty$, we may conclude that $F(x,y) = F'(x,y)$ for all $x,y\in X$. Thus such a Cauchy-Jensen mapping $F : X\times X\to Y$ is unique. □

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