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ON h-STABILITY OF LINEAR DIFFERENCE SYSTEMS VIA n_∞- QUASISIMILARITY

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Abstract. In this paper, we study h-stability for linear difference systems by using the notion of n_∞-quasisimilarity and discrete Gronwall's inequality.

1. Introduction

Let $\mathbb{Z}_+$ be the set of nonnegative integers and $M_n(\mathbb{R})$ be the set of $n \times n$ matrices over $\mathbb{R}$. We define the following sets:

$$\begin{aligned}
\mathcal{M}_n &= \{A \mid A : \mathbb{Z}_+ \to M_n(\mathbb{R}) \text{ is a matrix} - \text{valued function}\},\\
\mathcal{S} &= \{S \in \mathcal{M}_n \mid S \text{ and } S^{-1} \text{ are bounded}\},\\
\mathcal{I} &= \{F \in \mathcal{M}_n \mid \sum_{m=0}^{\infty} F(m) \text{ exists}\},\\
\mathcal{A} &= \{F \in \mathcal{M}_n \mid \sum_{m=0}^{\infty} |F(m)| \text{ exists}\},
\end{aligned}$$

where $|A|$ is some norm of matrix A.

We consider two linear difference systems

$$\Delta x(m) = A(m)x(m),\ m \in \mathbb{Z}_+, \tag{1.1}$$

and

$$\Delta y(m) = B(m)y(m),\ m \in \mathbb{Z}_+, \tag{1.2}$$

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where Δ is the forward difference operator, and $I+A(m)$ and $I+B(m)$ are invertible on $\mathbb{Z}_+$. Then we recall that $X, Y \in \mathcal{M}_n$ defined by

$$X(m) = \prod_{i=0}^{m-1} (I + A(i)), \ \ Y(m) = \prod_{i=0}^{m-1} (I + B(i)),$$

are called *fundamental matrices* for (1.1) and (1.2), respectively. Also we see that if m_0 is a fixed nonnegative integer, then the solutions of (1.1) and (1.2) satisfy

$$\begin{aligned} x(m) &= X(m)X^{-1}(m_0)x(m_0), \\ y(m) &= Y(m)Y^{-1}(m_0)y(m_0), \ m \geq m_0, \end{aligned}$$

respectively.

Trench [11] introduced t_∞-quasisimilarity that is not symmetric or transitive, but preserves strict and uniform stability of linear differential systems, and has linear asymptotic equilibrium. He also introduced the notion of n_∞-summable similarity which is the corresponding t_∞-quasisimilarity for the discrete case and gave the analogs of some of results in [6, 11] for difference systems.

In this paper, we study h-stability for linear difference systems by using the notion of n_∞-quasisimilarity and discrete Gronwall's inequality.

2. Main results

The following lemma is the discrete Gronwall-tpye inequality to need to prove our main results.

LEMMA 2.1. [8] *Let $u(j), b(j)$ be nonnegative sequences defined on $\mathbb{Z}_+$ and c a positive constant, and suppose that*

$$u(j) \leq c + \sum_{m=m_0}^{j-1} b(m)u(m), \ j \geq m_0.$$

Then we have

$$u(j) \leq c \exp\Big(\sum_{m=m_0}^{j-1} b(m)\Big), \ j \geq m_0.$$

LEMMA 2.2. [10] *Let $X(m)$ be a fundamental matrix for (1.1) with $X(0) = I$. Then (1.1) is* *(i) uniformly stable if and only if there is a positive constant C such that*

$$|X(j)X^{-1}(i)| \leq C, \ 0 \leq i \leq j.$$

(ii) exponential stable if and only if there are positive constants C and ρ with $0 < \rho < 1$ such that

$$|X(j)X^{-1}(i)| \leq C\rho^{j-i}, \ 0 \leq i \leq j.$$

Now, we recall the definition of h-stability introduced by Medina and Pinto [9].

Definition 2.3. (1.1) is *h-stable* if there exist a constant $c > 0$ and a positive bounded function $h : \mathbb{Z}_+ \to \mathbb{R}$ such that for any $m_0 \in \mathbb{Z}_+$ and $x_0 \in \mathbb{R}^n$, the corresponding solution $x(m, m_0, x_0)$ satisfies

$$|x(m, m_0, x_0)| \leq c|x_0|h(m)h(m_0)^{-1}, \ m \geq m_0, \tag{2.1}$$

where $h(m)^{-1} = \frac{1}{h(m)}$.

Lemma 2.4. *If (1.1) is h-stable if and only if there exist a positive bounded function h defined on $\mathbb{Z}_+$ and a constant $c \geq 1$ such that*

$$|X(j)X^{-1}(i)| \leq ch(j)h(i)^{-1}, \ j \geq i,$$

where $X(j)$ is a fundamental matrix for (1.1) with $X(0) = I$.

We recall the notion of n_∞-quasisimilarity in [10] as a discrete analog of Trench's definition of t_∞-quasisimilarity in [11].

Definition 2.5. [10] Let $A, B \in \mathcal{M}_n$. Then B is *n_∞-quasisimilar* to A if there is an $S \in \mathcal{S}$ that the $n \times n$ matrix function $F^{(0)}$ defined by

$$F^{(0)}(m) = \Delta S(m) + S(m+1)B(m) - A(m)S(m) \tag{2.2}$$

is in $\mathcal{I}$. Either $F^{(0)} \in \mathcal{A}$, or there is a positive integer p such that the $n \times n$ matrix functions $F^{(1)}, \cdots, F^{(p)}$ defined by

$$Q^{(r)}(m) = \sum_{k=m}^{\infty} F^{(r-1)}(k)$$

and

$$F^{(r)}(m) = Q^{(r)}(m+1)B(m) - A(m)Q^{(r)}(m), \ 1 \leq r \leq p$$

are in $\mathcal{I}$, and $F^{(p)} \in \mathcal{A}$.

REMARK 2.6. n_∞-quasisimilarity with $p = 0$ in the definition 2.5 becomes n_∞-similarity (or summable similarity [10]) which is an equivalence relation preserving linear asymptotic equilibrium and uniform, exponential, and strict stability.

We need the the following lemma [10] in order to prove our main result.

LEMMA 2.7. [10, Lemma 1] *Suppose that B is n_∞-quasisimilar to A. Define*

$$\Gamma^{(0)} + I \text{ and } \Gamma^{(r)} = I + S^{-1}\sum_{l=1}^{r} Q^{(r)},\ 1 \le r \le p.$$

Then

$$\begin{aligned}\Gamma^{(p)}(j)Y(j) &= S^{-1}(j)X(j)[X^{-1}(i)S(i)\Gamma^{(p)}(i)Y(i) \\ &+ \sum_{m=i}^{j-1} X^{-1}(m+1)F^{(p)}(m)Y(m)],\ 0 \le i \le j.\end{aligned}$$

THEOREM 2.8. *Suppose that (1.1) is h-stable and B is n_∞-quasisimilar to A with $\sum_{m=0}^{\infty} \frac{h(m)}{h(m+1)}|F^{(p)}(m)| < \infty$. Then (1.2) is h-stable.*

Proof. From Lemma2.4, there exist a positive bounded function $h : \mathbb{Z}_+ \to \mathbb{R}$ and a constant $c \ge 1$ such that

$$|X(j)X^{-1}(i)| \le ch(j)h(i)^{-1},\ j \ge i, \tag{2.3}$$

where $X(j)$ is a fundamental matrix for (1.1). From Lemma 2.7

$$\begin{aligned}Y(j)Y^{-1}(i) &= (\Gamma^{(p)}(j))^{-1}S^{-1}(j)X(j)[X^{-1}(i)S(i)\Gamma^{(p)}(i) \\ &+ \sum_{m=i}^{j-1} X^{-1}(m+1)F^{(p)}(m)Y(m)Y^{-1}(i)],\ 0 \le i \le j.\end{aligned}$$

Note that $\Gamma^{(p)}, S, (\Gamma^{(p)})^{-1}$, and S^{-1} are bounded. Then this and (2.3) implies that there are positive constants c_1, c_2 such that

$$\begin{aligned}&|Y(j)Y^{-1}(i)| \le c_1 h(j)h(i)^{-1} \\ &+c_2 \sum_{m=i}^{j-1} h(j)h(m+1)^{-1}|F^{(p)}(m)||Y(m)Y^{-1}(i)|], 0 \le i \le j.\end{aligned} \tag{2.4}$$

Dividing (2.4) by $h(j)$ yields the inequality

$$\frac{|Y(j)Y^{-1}(i)|}{h(j)} \le c_1 h(i)^{-1} + c_2 \sum_{m=i}^{j-1} \frac{h(m)}{h(m+1)}|F^{(p)}(m)|\frac{|Y(m)Y^{-1}(i)|}{h(m)}],$$

for $j \geq i \geq 0$. From Lemma 2.1, we obtain

$$\begin{aligned} |Y(j)Y^{-1}(i)| &\leq c_1 h(j)h(i)^{-1} \exp\left(c_2 \sum_{m=i}^{j-1} \frac{h(m)}{h(m+1)} |F^{(p)}(m)| \right) \\ &\leq ch(j)h(i)^{-1}, \ \ j \geq i \geq 0, \end{aligned}$$

where $c = c_1 \exp(c_2 \sum_{m=0}^{\infty} \frac{h(m)}{h(m+1)} |F^{(p)}(m)|)$. Hence (1.2) is h-stable. This completes the proof. □

REMARK 2.9. If $h(j)$ is a positive bounded function on $\mathbb{Z}_+$, then $\frac{h(j)}{h(j+1)}$ is not bounded in general. For example, see [4, Remark 3.1].

COROLLARY 2.10. *Suppose that B is n_∞-quasisimilar to A and (1.1) is h-stable with bounded function $\frac{h(j)}{h(j+1)}$. Then (1.2) is h-stable.*

COROLLARY 2.11. *If the function h is constant or is given by $h(j) = \rho^j$ in Theorem 2.8, then (1.2) is uniformly stable or exponentially stable.*

THEOREM 2.12. *Suppose that*

$$\sum_{m=0}^{\infty} |A(m)| < \infty \tag{2.5}$$

and there is an $S \in \mathcal{S}$ such that the $n \times n$ matrix function K_0 defined by

$$K_0(m) = \Delta S(m) + S(m+1)(B(m) - A(m))$$

is in $\mathcal{A}$, or it is in $\mathcal{I}$ and there is a positive integer p such that the $n \times n$ matrix functions $K_1, \cdots, K_p$ defined by

$$K_r(m) = \left(\sum_{k=m+1}^{\infty} K_{r-1}(k) \right) (B(m) - A(m)), \ \ 1 \leq r \leq p, \tag{2.6}$$

are in $\mathcal{I}$, and $K_p \in \mathcal{A}$. Then (1.2) is h-stable.

Proof. We note that the solution $x(m)$ of (1.1) with the initial value $x(m_0) = x_0$ satisfies the relation

$$x(m, m_0, x_0) = x_0 + \sum_{k=m_0}^{m-1} A(k)x(k), \ m \geq m_0.$$

In view of the condition (2.5) of A and Lemma 2.1, we have

$$|x(m, m_0, x_0)| \leq |x_0| \exp\left(\sum_{k=m_0}^{m-1} |A(k)| \right) = |x_0| h(m)h(m_0)^{-1}, \ m \geq m_0,$$

where $h(m) = \exp(\sum_{k=0}^{m-1} |A(k)|)$ is a positive bounded function on $\mathbb{Z}_+$. Thus (1.1) is h-stable. We easily see that $\frac{h(m)}{h(m+1)}$ is bounded on $\mathbb{Z}_+$.

Next, we show that B is n_∞-quasisimilar to A. (2.2) becomes

$$\begin{aligned} F^{(0)}(m) &= \Delta S(m) + S(m+1)B(m) - A(m)S(m) \\ &= \Delta S(m) + S(m+1)(B(m) - A(m)) + S(m+1)A(m) \\ &\qquad -A(m)S(m). \end{aligned}$$

It follows from (2.5) that $F^{(0)} \in \mathcal{A}$. There is a positive integer p such that the $n \times n$ matrix functions $F^{(1)}, \cdots, F^{(p)}$ defined by

$$\begin{aligned} Q^{(r)}(m) &= \sum_{k=m}^{\infty} F^{(r-1)}(k), \\ F^{(r)}(m) &= Q^{(r)}(m+1)B(m) - A(m)Q^{(r)}(m) \\ &= Q^{(r)}(m+1)(B(m) - A(m)) + Q^{(r)}(m+1)A(m) \\ &\qquad -A(m)Q^{(r)}(m) \\ &= K_r(m) + Q^{(r)}(m+1)A(m) - A(m)Q^{(r)}(m), \; 1 \le r \le p \end{aligned}$$

are in $\mathcal{I}$, and $F^{(p)} \in \mathcal{A}$. This implies that B is n_∞-quasisimilar to A. Hence (1.2) is h-stable in view of Theorem 2.8. This completes the proof. □

If $A = 0$ in (1.1), then we obtain easily the following corollary by Theorem 2.12. We also can give another proof of the corollary.

COROLLARY 2.13. *Suppose that there is an $S \in \mathcal{S}$ such that the $n \times n$ matrix function $F^{(0)}$ defined by*

$$F^{(0)}(m) = \Delta S(m) + S(m+1)B(m) \tag{2.7}$$

is in $\mathcal{A}$, or it is in $\mathcal{I}$ and there is a positive integer p such that the $n \times n$ matrix functions $F^{(1)}, \cdots, F^{(p)}$ defined by

$$F^{(r)}(m) = \left(\sum_{k=m+1}^{\infty} F^{(r-1)}(k) \right) B(m), \;\; 1 \le r \le p. \tag{2.8}$$

are in $\mathcal{I}$, and $F^{(p)} \in \mathcal{A}$. Then (1.2) is h-stable.

Proof. We easily see that the fundamental matrix X of (1.1) with $A = 0$ is given by $X(j) = I$. This and the argument used in the proof of Theorem 2.8 implies that

$$\begin{aligned} |Y(j)Y^{-1}(i)| &\leq c_1 \exp\left(c_2 \sum_{m=i}^{j-1} |F^{(p)}(m)| \right) \\ &\leq c_1 h(j)h(i)^{-1}, \;\; j \geq i \geq 0, \end{aligned}$$

where $h(j) = \exp(c_2 \sum_{m=0}^{j-1} |F^{(p)}(m)|)$ is a positive bounded function. Hence (1.2) is h-stable by Lemma 2.4. This completes the proof. □

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