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DEGREE OF NEARNESS

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Abstract. This paper is a revised version of [5]. In [5], we define 'nearness between two points' in a topological space in many ways and show that a continuous function preserves one-sided nearness. We also show that a T_1-space is characterized by one-sided nearness exactly. In this paper, we introduce extremally disconnected spaces and show that the new topology generated by the set of equivalence classes as a base is extremally disconnected.

1. Pre-ordered set

The meaning of 'a is sufficiently near to b' is up to one's mind case by case. In general, a relation 'a is near to b' is not symmetric. Hence it is neither an equivalence relation nor a partial order. What is the meaning of 'near' in mathematical sense? How can we define the relation with full of meaning? Actually nearness between two persons is neither reflexive nor transitive in general. But we want the relation 'near' in a topological space is at least reflexive and transitive, i.e., a pre-order relation on a space.

Definition 1.1. A *pre-order* or *quasi-order* on a nonempty set P is a binary relation '$\preceq$' satisfying the following:

(1) $x \preceq x$ (reflexivity),
(2) $x \preceq y$ and $y \preceq z$ imply $x \preceq z$ (transitivity).

A set P equipped with a pre-order '$\preceq$' is said to be a *pre-ordered set.*

A pre-order relation '$\preceq$' on P gives rise to a relation '$\prec$' as

$$x \prec y \quad \text{if and only if} \quad x \preceq y \ \text{ and } \ x \neq y.$$

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We write $x \not\preceq y$ if '$x \preceq y$ is false'.

Given a pre-order '$\preceq$' on P, one may define an equivalence relation '$\sim$' on P as

$$x \sim y \quad \text{if and only if} \quad x \preceq y \;\text{ and }\; y \preceq x.$$

Using the relation '$\sim$', it is possible to construct a partial order on the quotient

$$P/\sim = \{[x]_\sim \mid x \in P\},$$

where $[x]_\sim = \{y \mid x \sim y\}$. In case, we can define $[x]_\sim \leq [y]_\sim$ on $P/\sim$ if and only if $x \preceq y$. By the construction of '$\sim$', this definition is independent from the chosen representatives and the corresponding relation is indeed well-defined. It is readily verified that this yields a partially ordered set, say poset.

For terminology not introduced in this paper, we refer to [2].

2. Degree of nearness in a topological space

What is the meaning of 'near'? It may be up to one's mind. Mathematically, $x \in \mathrm{cl}A$, where $\mathrm{cl}A$ is the closure of A, means that A is a set of approximations of x in a topological space. That is, x is near to A or a point $a \in A$ is an approximation of x. Roughly speaking, x nears to A. For example, $A = \{1, 1.4, 1.41, 1.414, \cdots\}$ is a set of approximations of $\sqrt{2}$ in the real line (R, U) with the real line topology, because $\sqrt{2} \in \mathrm{cl}A = A \cup \sqrt{2}$. The sequence $< 1, 1.4, 1.41, 1.414, \cdots >$ converges to $\sqrt{2}$. But for any $a \in A$, $a \notin \mathrm{cl}\{\sqrt{2}\} = \{\sqrt{2}\}$.

Unfortunately, $a \in \mathrm{cl}\{b\}$ does not imply $b \in \mathrm{cl}\{a\}$ if the points are in a non-symmetric topological space. If they are in a Hausdorff space, then $a = b$ if and only if a is near to b.

Roughly speaking, topology is the study of shape without distance. Open sets in the real line with the usual topology are motivations of topological spaces. Specifically, we know the meaning of 'continuity', 'limits' and 'nearness' on the real numbers. These concepts are all defined rigorously in the realm of calculus on the real numbers.

A set is a neighborhood of each of its points if and only if it is open.

The everyday sense of the word 'neighborhood' is such that many of the properties which involve the mathematical idea of neighborhood appear as the mathematical expression of intuitive properties; the choice of this term thus has the advantage of making the language more expressive. For this purpose, it is also permissible to use the expressions 'sufficiently near' and 'as near as we please' in some statements. For example([1]).

A set A is open if and only if for each a in A, all the points sufficiently near a belong to A.

More generally, we shall say that a property holds for all points sufficiently near a point x, if it holds at all points of some neighborhood of x.

How does this notion of 'open sets' help us to characterize the concept of 'nearness' on a space? It helps to think of open sets containing a given point p as 'neighborhoods' of that point. If we have some idea of 'nearness' of points to each other, then every neighborhood of p contains all points within a certain degree of 'nearness' to p. This is why we used open balls in R^2. Open balls give a definitive description of nearness to p, and thus we can say an open set containing p is a 'neighborhood' of p, since it always contains an open ball about p; it always contains all points sufficiently near p. It might contain points 'far' from p as well, but it definitely contains all points that are close to p, given a sufficiently strict definition of 'closeness'. In an indiscrete topology, all points are 'near'.

In summary, when we define a topology of a space, we are implicitly giving the space a notion of continuity and limits. When we deal with a familiar set like the real numbers, this notion of continuity and limits can come directly from a notion of 'distance' between two points. However, the notion of a topology is far more general and can be used to characterize much more abstract spaces. Throughout this paper, a topological space $(X, \mathcal{T})$ is defined on a nonempty set X.

Definition 2.1. Let $(X, \mathcal{T})$ be a topological space and $x, y \in X$. Then we define the following:

(1) *x is said to be one-sided near to y, denoted by $x \to_1 y$*, if every open set containing y contains x.
(2) *x is said to be wholly near to y, denoted by $x \to_w y$*, if there is an open set containing x and y.
(3) *x is said to be G-near to y, denoted by $x \to_G y$*, if $x = y$ or there is an open set $G(\neq X)$ containing x and y.

Remark 2.2. (1) Clearly, the relation $\to_1$ is reflexive.

(2) Let $\beta = \{N_k \mid N_k = \{1, 2, \cdots, k\}, k \in N\}$ be a base for a topology $\mathcal{T}$ on the set of all natural numbers N, then $1 \to_1 n$ for all $n \in N$, but $n \not\to_1 1$ for any $n \in N - \{1\}$; hence $\to_1$ is not symmetric.

(3) In an indiscrete space $(X, \mathcal{I})$, where X has at least two elements, every elements in X are one-sided near; hence $\rightarrow_1$ is not anti-symmetric.

(4) If $x \rightarrow_1 y$ and $y \rightarrow_1 z$, then any open set G containing z contains y, so G contains x; hence $\rightarrow_1$ is transitive.

In all, $\rightarrow_1$ is a pre-order, but neither an equivalence relation nor a partial order.

REMARK 2.3. (1) Clearly, the relation $\rightarrow_w$ is reflexive.

(2) In an indiscrete space $(X, \mathcal{I})$, where X has at least two elements, every elements in X are wholly near. So the relation $\rightarrow_w$ is symmetric but not anti-symmetric.

(3) The relation $\rightarrow_w$ is transitive.

Thus the relation $\rightarrow_w$ is an equivalence relation but not a partial order.

REMARK 2.4. (1) Clearly, the relation $\rightarrow_G$ is reflexive.

(2) Let $X = \{x, y, z\}$ and $\mathcal{T} = \{X, \emptyset, \{x, y\}, \{y, z\}, \{y\}\}$, then $x \rightarrow_G y$ and $y \rightarrow_G z$, but $x \not\rightarrow_G z$, so $\rightarrow_G$ is not transitive.

(3) In an indiscrete space $(X, \mathcal{I})$, where X has at least two elements, no distinct elements in X are G-near.

By (2), the relation $\rightarrow_G$ is not a pre-order.

REMARK 2.5. Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$, then A is open if and only if for any $a \in A$ there is an open set $G \subseteq A$ such that every point which is G-near to a point a also belongs to A.

In the above remark, the set G is dependent to the set A. So the definition of 'sufficiently nearness' can not be free from the set A.

PROPOSITION 2.6. *Let $(X, \mathcal{T})$ be a topological space. Then the following are equivalent:*

(1) *$(X, \mathcal{T})$ is T_1.*
(2) *$x \rightarrow_1 y$ if and only if $x = y$.*
(3) *$x \not\rightarrow_1 y$ and $y \not\rightarrow_1 x$ for all $x \neq y \in X$.*

Proof. Consider $x \rightarrow_1 y$ if and only if $y \in \mathrm{cl}\{x\}$. Since $y \in \mathrm{cl}\{x\} = \{x\}$ in a T_1-space, we get the result. □

PROPOSITION 2.7. *Let $(X, \mathcal{T})$ be a topological space. Then $(X, \mathcal{T})$ is T_0 if and only if $x \not\rightarrow_1 y$ or $y \not\rightarrow_1 x$ for all $x \neq y \in X$.*

Proof. Let $x \neq y \in X$. Since $(X, \mathcal{T})$ is T_0, there is an open set G in X such that $x \in G, y \notin G$ or $x \notin G, y \in G$. If $x \in G$ and $y \notin G$, then $y \not\rightarrow_1 x$. If $x \notin G$ and $y \in G$, then $x \not\rightarrow_1 y$. So $x \not\rightarrow_1 y$ or $y \not\rightarrow_1 x$.

Conversely, let $x \neq y \in X$. Then $x \not\to_1 y$ or $y \not\to_1 x$. If $x \not\to_1 y$, then there is an open set G in X such that $y \in G$ and $x \notin G$. So $(X, \mathcal{T})$ is T_0. □

REMARK 2.8. (1) If $\{x\}$ is an open set, then $z \not\to_1 x$ for all $z \in X - \{x\}$.

(2) If $z \to_1 x$ for all $z \in X$, then open set containing x is only X.

(3) $[x]_1 = \{y \mid \mathrm{cl}\{y\} = \mathrm{cl}\{x\}\}$.

(4) Let $(X, \mathcal{T})$ be a topological space. Then $\{[x]_1 \mid x \in X\}$ and $\{[x]_W \mid x \in X\}$ can be bases for some topologies on X.

PROPOSITION 2.9. *Let $\to_p$ be a pre-order on a nonempty set X, then $\{< x >_p = \{y \mid y \to_p x\} \mid x \in X\}$ is a base for some topology on X.*

Proof. By reflexivity, $\bigcup\{< x >_p \mid x \in X\} = X$. Let $z \in < x >_p \cap < y >_p$. By transitivity, $< z >_p \subseteq < x >_p \cap < y >_p$. □

PROPOSITION 2.10. *Let $(X, \mathcal{T})$, $(Y, \mathcal{T}^*)$ be topological spaces and $f : (X, \mathcal{T}) \to (Y, \mathcal{T}^*)$ be a function. Then for $x_0 \in X$ we have:*

(1) *If f is continuous at x_0 and $x \to_1 x_0$, then $f(x) \to_1 f(x_0)$.*

(2) *If f is an one-to-one open function and $f(x) \to_1 f(y)$, then $x \to_1 y$.*

Proof. (1) Let H be an open set containing $f(x_0)$. Since f is continuous at x_0, there is an open set G containing x_0 with $G \subseteq f^{-1}(H)$. Since $x \to_1 x_0$, $x \in G$. So $f(x) \in f(G) \subseteq ff^{-1}(H) \subseteq H$ and hence $f(x) \to_1 f(x_0)$.

(2) Let G be an open set containing y. Then $f(G)$ is open and $f(G)$ contains $f(x)$. Since f is one-to-one, $x \in f^{-1}f(G) = G$. □

REMARK 2.11. Consider the function $1_R : (R, \mathcal{C}_C) \to (R, \mathcal{U})$ is a sequentially continuous function from a cocountable space $(R, \mathcal{C}_C)$ with the set of all real numbers R into a usual space $(R, \mathcal{U})$. Since the cocountable space $(R, \mathcal{C}_C)$ is a T_1-space, $x \to_1 x_0$ if and only if $x = x_0$. So $x \to_1 x_0$ in $(R, \mathcal{C}_C)$ implies $f(x) \to_1 f(x_0)$ in $(R, \mathcal{U})$, but f is not continuous at every point $x \in R$.

Now we define the concept of sufficiently nearness. A subset A of $(R, \mathcal{U})$ is open if and only if there is a positive number ϵ with $(a - \epsilon, a + \epsilon) \subseteq A$ for all $a \in A$. So every point $b \in (a - \epsilon, a + \epsilon)$ is sufficiently near(actually, ϵ near) to a.

DEFINITION 2.12. Let $(X, \mathcal{T})$ be a topological space, $A \subseteq X$ and $x, y \in A$. Then x *is said to be sufficiently near* y *in* A, *denoted by* $x \to_s y$, if there is an open set $G (\subseteq A)$ containing x and y.

3. New topology generated by partitions

In this section we study the induced topology by a set of equivalence classes as a base.

The term *extremally disconnected* was introduced by M.H. Stone([6]) as follows:

Definition 3.1. Let $(X, \mathcal{T})$ be a topological space. Then

(1) $(X, \mathcal{T})$ is said to be *extremally disconnected* if the closure of every open subset of X is open in X or equivalently if the interior of every closed subset of X is closed in X.
(2) A subset A of X is said to be *regular-open* if $\mathrm{int}(\mathrm{cl}(A)) = A$. A set A is called *regular-closed* if $\mathrm{cl}(\mathrm{int}(A)) = A$.
(3) A subset A of X is said to be *saturated* if for any $a \in A$, $[a] \subseteq A$, where $[a]$ is the equivalence class of a.

Remark 3.2. Let $(X, \mathcal{T})$ be a topological space and $x \sim y$ if and only if $x \rightarrow_1 y$ and $y \rightarrow_1 x$.

We denoted $[x] = \{y \mid x \sim y\}$. By Proposition 2.9, $\{[x] \mid x \in X\}$ can be a base for some topology $\mathcal{T}^*$ on X. We have the followings:

(1) $[x]$ is not closed with respect to $\mathcal{T}$, for $X = \{1, 2, 3\}$ and $\mathcal{T} = \{X, \emptyset, \{1\}, \{2\}, \{1, 2\}\}$, $[1] = \{1\}$ is not closed.

(2) $[x]$ is not open with respect to $\mathcal{T}$, for $[3] = \{3\}$ is not open.

(3) $A \in \mathcal{T}^*$ if and only if A is saturated with respect to $\mathcal{T}$. To show it, let $A \in \mathcal{T}^*$ and $x \in A$. Then there is $[x]$ with $x \in [x] \subseteq A$. Hence A is saturated. Conversely, let x $\in A$, then $x \in [x] \subseteq A$ for A is saturated. Since $[x]$ belongs to the base for $\mathcal{T}^*$, $A \in \mathcal{T}^*$.

(4) A is saturated if and only if A^c is saturated. If A is saturated, then for any $x \in A^c$, $x \notin A$ Then $[x] \nsubseteq A$. Thus $[x] \subseteq A^c$. (because, if A is saturated and $[x] \nsubseteq A^c$, then there is $y \in [x]$ with $y \in A^c$. But $[x] = [y] \subseteq A$. It's a contradiction.)

(5) A is closed with respect to $\mathcal{T}$ if and only if A is saturated with respect to $\sim$.

(6) $(X, \mathcal{T}^*)$ is extremally disconnected since every open subset of X is closed.

(7) Every open subset of $(X, \mathcal{T}^*)$ is regular open and regular closed.

4. Nearness spaces

In 1974 H. Herrich invented nearness spaces. The concept of nearness spaces enables us to unify topological spaces. Now we introduce a nearness structure([3]).

Let X be a set and $\xi \subseteq \mathcal{P}(\mathcal{P}(X))$, where $\mathcal{P}(\mathcal{P}(X))$ is the power set of the power set of X. For $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(X)$ and $A, B \subseteq X$, we use the following notations.

(1) $\mathcal{A} \vee \mathcal{B} = \{A \cup B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$.
(2) $\mathcal{A}$ corefines $\mathcal{B}$, denoted by $\mathcal{A} < \mathcal{B}$, if for each $A \in \mathcal{A}$, there is $B \in \mathcal{B}$ with $B \subseteq A$.
(3) $\mathcal{A}$ refines $\mathcal{B}$, denoted by $\mathcal{A} \prec \mathcal{B}$, if for each $A \in \mathcal{A}$, there is $B \in \mathcal{B}$ with $A \subseteq B$.

DEFINITION 4.1. Let X be a set and $\xi \subseteq \mathcal{P}(\mathcal{P}(X))$. Then ξ is said to be a *nearness structure on* X if it satisfies the following:

(N1) $\mathcal{A} < \mathcal{B} \in \xi$ implies $\mathcal{A} \in \xi$.
(N2) $\bigcap \mathcal{A} \neq \emptyset$ implies $\mathcal{A} \in \xi$.
(N3) $\emptyset \neq \xi \neq \mathcal{P}(\mathcal{P}(X))$.
(N4) If $\mathcal{A} \vee \mathcal{B} \in \xi$, then $\mathcal{A} \in \xi$ or $\mathcal{B} \in \xi$.
(N5) $Cl_\xi \mathcal{A} = \{Cl_\xi A \mid A \in \mathcal{A}\} \in \xi$ implies $\mathcal{A} \in \xi$, where $Cl_\xi A = \{x \in X \mid \{\{x\}, A\} \in \xi\}$.

In case, the pair (X, ξ) is called a nearness space or shortly an N-space and $\mathcal{A}$ is said to be near if $\mathcal{A} \in \xi$.

ξ is said to be a *quasi-nearness structure or shortly a Q-nearness structure on* X if ξ satisfies (N1), (N2), (N3) and (N4).

Given a nearness space (X, ξ), the operator Cl_ξ is a closure operator on X([3]). Hence there exists a topology associated with each nearness space. This topology is denoted by $\tau(\xi)$. This topology is symmetric. That is, if $x \in \mathrm{cl}\{y\}$, then $y \in \mathrm{cl}\{x\}$.

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