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CHARACTERISTIC MULTIFRACTAL IN A SELF-SIMILAR CANTOR SET

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Abstract. We study essentially disjoint one dimensionally indexed classes whose members are distribution sets of a self-similar Cantor set. The Hausdorff dimension of the union of distribution sets in a same class does not increases the Hausdorff dimension of the characteristic distribution set in the class. Further we study the Hausdorff dimension of some uncountable union of distribution sets.

1. Introduction

Recently distribution sets of a self-similar set were investigated in [2, 5]. We can apply these results to a self-similar Cantor set. There are uncountably many disjoint distribution sets $F[r_1, r_2]$ where $0 \leq r_1 \leq r_2 \leq 1$, which means that there are two dimensionally indexed distribution sets. Every distribution set $F[r_1, r_2]$ has its own Hausdorff dimension according to its lower distribution r_1 and upper distribution r_2 of the digits in a self-similar Cantor set. It is well-known that the Hausdorff dimension of a countable union of subsets is the supremum of their Hausdorff dimensions. We do not have any general formula to get the Hausdorff dimension of an uncountable union of subsets. In some cases in a self-similar Cantor set, we can compute easily the Hausdorff dimension of an uncountable union of subsets.

In this paper, we classify the two-dimensionally many distribution sets into one-dimensionally many classes which have their own characteristic distribution sets in their classes. For this purpose, we use our recent results([1]) about a complete decomposition of a self-similar Cantor set using the relation between the distribution sets and local dimension sets of a self-similar measure on the self-similar Cantor set.

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The distribution set in [1] is of the form $\cup_{0\le r_1\le r_2} F[r_1, r_2] \equiv \overline{F}(r_2)$ or $\cup_{r_1\le r_2\le 1} F[r_1, r_2] \equiv \underline{F}(r_1)$ where $0 \le r_1 \le r_2 \le 1$. $\overline{F}(r_2)$ are one dimensionally indexed classes of distribution sets. So are $\underline{F}(r_1)$. We will consider another one dimensionally many classes $H(r_1, r_2) = \underline{F}(r_1) \cup \overline{F}(r_2)$ where (r_1, r_2) are from a dimension formula $\delta(r_1) = \delta(r_2) \in [0, s]$ where $a^s + b^s = 1$ in a self-similar Cantor set whose contraction ratios are a and b. These classes $H(r_1, r_2)$ are essentially disjoint in the sense that the intersection of any two classes $H(r_1, r_2)$ has distribution sets of zero dimensional indices or only two distribution sets in two-dimensionally many distribution sets. Every class $H(r_1, r_2)$ has its characteristic distribution set as $F[r_1, r_2] = \underline{F}(r_1) \cap \overline{F}(r_2)$ whose Hausdorff dimension is the Hausdorff dimension of the union of the members of $H(r_1, r_2)$. Further we compute the Hausdorff dimension of some uncountable union of distribution sets related to a coordinate (r_1, r_2) where $F[r_1, r_2]$ is a characteristic distribution set.

2. Preliminaries

We denote F a self-similar Cantor set, which is the attractor of the similarities $f_1(x) = ax$ and $f_2(x) = bx + (1-b)$ on $I = [0, 1]$ with $a > 0$, $b > 0$ and $1 - (a + b) > 0$. Let $I_{i_1,\cdots,i_k} = f_{i_1} \circ \cdots \circ f_{i_k}(I)$ where $i_j \in \{1, 2\}$ and $1 \le j \le k$. We note that if $x \in F$, then there is $\sigma \in \{1, 2\}^{\mathbb{N}}$ such that $\bigcap_{k=1}^{\infty} I_{\sigma|k} = \{x\}$ (Here $\sigma|k = i_1, i_2, \cdots, i_k$ where $\sigma = i_1, i_2, \cdots, i_k, i_{k+1}, \cdots$). Let $p \in (0, 1)$ and we denote γ_p a self-similar Borel probability measure on F satisfying $\gamma_p(I_1) = p$(cf. [4]). $\dim(E)$ denotes the Hausdorff dimension of E([4]). We denote $n_1(x|k)$ the number of times the digit 1 occurs in the first k places of $x = \sigma$(cf. [1]).

For $r \in [0, 1]$, we define the lower(upper) distribution set $\underline{F}(r)(\overline{F}(r))$ containing the digit 1 in proportion r by

$$\underline{F}(r) = \{x \in F : \liminf_{k\to\infty} \frac{n_1(x|k)}{k} = r\},$$

$$\overline{F}(r) = \{x \in F : \limsup_{k\to\infty} \frac{n_1(x|k)}{k} = r\}.$$

We call $\{\underline{F}(r) : 0 \le r \le 1\}$ the lower distribution class and $\{\overline{F}(r) : 0 \le r \le 1\}$ the upper distribution class.

Similarly for $r_1, r_2 \in [0,1]$ with $r_1 \le r_2$, we define a distribution set $F[r_1, r_2]$ by

$$F[r_1, r_2] = \underline{F}(r_1) \cap \overline{F}(r_2).$$

We write $\underline{E}_\alpha^{(p)}$ ($\overline{E}_\alpha^{(p)}$) for the set of points at which the lower(upper) local dimension of γ_p on F is exactly α, so that

$$\underline{E}_\alpha^{(p)} = \{x : \liminf_{r \to 0} \frac{\log \gamma_p(B_r(x))}{\log r} = \alpha\},$$

$$\overline{E}_\alpha^{(p)} = \{x : \limsup_{r \to 0} \frac{\log \gamma_p(B_r(x))}{\log r} = \alpha\}.$$

Similarly for $\alpha_1, \alpha_2 \in [0,1]$ with $\alpha_1 \le \alpha_2$, we define a subset $E_{[\alpha_1,\alpha_2]}^{(p)}$

$$E_{[\alpha_1,\alpha_2]}^{(p)} = \underline{E}_{\alpha_1}^{(p)} \cap \overline{E}_{\alpha_2}^{(p)}.$$

In this paper, we assume that $0 \log 0 = 0$ for convenience. We define for $r \in [0,1]$

$$g(r,p) = \frac{r \log p + (1-r)\log(1-p)}{r \log a + (1-r) \log b}.$$

From now on we will use $g(r,p)$ as the above definition.

3. Main results

Proposition 3.1. ([1]) *Let s be a real number satisfying $a^s + b^s = 1$. Then*
(1) $\underline{F}(r) = \underline{E}_{g(r,p)}^{(p)}$ *if* $0 < p < a^s$,
(2) $\underline{F}(r) = \overline{E}_{g(r,p)}^{(p)}$ *if* $a^s < p < 1$,
(3) $\overline{F}(r) = \overline{E}_{g(r,p)}^{(p)}$ *if* $0 < p < a^s$,
(4) $\overline{F}(r) = \underline{E}_{g(r,p)}^{(p)}$ *if* $a^s < p < 1$.
Further if we put $\delta(r) = g(r,r)$, *then*
(5) $\underline{F}(r) = \underline{E}_{\delta(r)}^{(r)}$ *if* $0 < r < a^s$,
(6) $\overline{F}(r) = \underline{E}_{\delta(r)}^{(r)}$ *if* $a^s < r < 1$.

From now on, s is a real number satisfying $a^s + b^s = 1$ and $\delta(r) = g(r,r)$ where $0 \le r \le 1$.

Proposition 3.2. ([1]) *We have*
(1) $\dim(\underline{F}(r)) = \delta(r)$ *if* $0 \le r \le 1$,
(2) $\dim(\overline{F}(r)) = \delta(r)$ *if* $0 \le r \le 1$.

Proposition 3.3. ([3, 5]) *For* $0 \le r_1 \le r_2 \le 1$,

$$\dim(F[r_1, r_2]) = \inf_{r_1 \le r \le r_2} \delta(r).$$

Theorem 3.4. *The function δ is a unimodal function on $[0,1]$ satisfying $\delta(0) = 0 = \delta(1)$.*

Proof. It is not difficult to show that $\delta'(r) > 0$ for $0 \le r < a^s$, $\delta'(a^s) = 0$ and $\delta'(r) < 0$ for $a^s < r \le 1$. It follows that δ is a unimodal function on $[0,1]$. We easily see that $\delta(0) = 0 = \delta(1)$. □

Corollary 3.5. *For* $0 \le r_1 \le r_2 \le 1$,

$$\dim(F[r_1, r_2]) = \min\{\delta(r_1), \delta(r_2)\}.$$

Proof. From the above Theorem, δ is a unimodal function on $[0,1]$. It is immediate from the graph of δ and Proposition 3.3. □

Remark 3.6. In Proposition 3.3, the upper bound of the Hausdorff dimension of $F[r_1, r_2]$ follows essentially from Proposition 3.1 and the Frostman's density theorem([4]). In view of the above Corollary, Proposition 3.3 means that the upper bound is a sharp upper bound.

Corollary 3.7. *For any $d \in [0, s)$, there exist r_1 and r_2 such that $0 \le r_1 < r_2 \le 1$ and $\delta(r_1) = \delta(r_2) = d$. Further $\delta(a^s) = s$.*

Proof. From the above Theorem, δ is a unimodal function on $[0,1]$. It is immediate from the graph of δ. □

Using the above Corollary, we define *a characteristic coordinate set*

$$\Delta = \{(r_1, r_2) \in [0,1] \times [0,1] : \delta(r_1) = \delta(r_2) \in [0, s), r_1 < r_2\} \cup \{(a^s, a^s)\}$$

and also define *characteristic classes*

$$G(d) = \underline{F}(r_1) \cup \overline{F}(r_2)$$

where $(r_1, r_2) \in \Delta$ with $\delta(r_1) = \delta(r_2) = d$. We sometimes write $H(r_1, r_2)$ for $G(d)$ where $(r_1, r_2) \in \Delta$ with $\delta(r_1) = \delta(r_2) = d$. We call the distribution set $F[r_1, r_2]$ in $H(r_1, r_2)$ *a characteristic distribution set.*

Theorem 3.8. *For the characteristic classes $G(d)$ where $(r_1, r_2) \in \Delta$ with $\delta(r_1) = \delta(r_2) = d \in [0, s]$,*

$$\dim(G(d)) = d = \dim(F[r_1, r_2]).$$

Proof. Let $(r_1, r_2) \in \Delta$ with $\delta(r_1) = \delta(r_2) = d \in [0, s]$. From Proposition 3.3 and Corollary 3.7, $\dim(\underline{F}(r_1) \cap \overline{F}(r_2)) = \delta(r_1) = \delta(r_2)$ since $(r_1, r_2) \in \Delta$. It follows from Proposition 3.2. □

REMARK 3.9. The above Theorem shows that the characteristic distribution set $F[r_1, r_2]$ in $G(d)(= H(r_1, r_2))$ represents the class $H(r_1, r_2)$ in dimensional sense. That is, distribution sets in in $H(r_1, r_2)$ cannot increase the Hausdorff dimension of the union of themselves and the characteristic distribution set $F[r_1, r_2]$ in $H(r_1, r_2)$.

REMARK 3.10. From Proposition 3.1, the characteristic distribution set $F[r_1, r_2]$ in $H(r_1, r_2)(= G(d))$ is $\underline{E}^{(r_1)}_{\delta(r_1)} \cap \underline{E}^{(r_2)}_{\delta(r_2)} = \underline{E}^{(r_1)}_{d} \cap \underline{E}^{(r_2)}_{d}$ for self-similar measures γ_{r_1} and γ_{r_2} on F where $0 \le r_1 \le a^s \le r_2 \le 1$. It can be represented by $E^{(p)}_{[g(r_1,p),g(r_2,p)]}$ for a self-similar measure γ_p on F where $0 < p < a^s$ from Proposition 3.1. Similarly it can be also represented by $E^{(p)}_{[g(r_2,p),g(r_1,p)]}$ for a self-similar measure γ_p on F where $a^s < p < 1$ from Proposition 3.1.

THEOREM 3.11.

$$\cup_{d\in[0,s]} G(d) = F.$$

Proof. It is immediate from the definition of $G(d)$. □

THEOREM 3.12. *Let* $A \subset \{(r_1, r_2) : 0 \le r_1 \le r_2 \le 1\}$. *Assume that* $\cup_{(r_1,r_2)\in A} F[r_1, r_2] \subset \cup_{d\in D_A} G(d)$ *and* D_A *is a countable set. Then* $\dim(\cup_{(r_1,r_2)\in A} F[r_1, r_2]) \le \sup_{d\in D_A} \dim(G(d))$.

Proof. Since Hausdorff dimension is countably stable([4]), it follows from Theorem 3.8. □

EXAMPLE 3.13. *Consider an uncountable set*

$$A = \{(r, \frac{n+1}{n+2}) : n \in \mathbb{N}, r \in \mathbb{Q}^c\} \cup \{(\frac{1}{n+2}, r) : n \in \mathbb{N}, r \in \mathbb{Q}^c\},$$

where $\mathbb{Q}^c$ *is the set of irrational numbers. Then* $\dim(\cup_{(r_1,r_2)\in A} F[r_1, r_2]) = \max\{\delta(\frac{1}{3}), \delta(\frac{2}{3})\}$ *from the above Theorem if* $\frac{1}{3} < a^s < \frac{2}{3}$.

The followings are the second part of our main results. We only consider $F[x, y]$ where $0 \le x \le y \le 1$ since $F[x, y] = \phi$ if $x > y$.

THEOREM 3.14. *Let* $0 \le r_2 < a^s$ *and* $A \subset [0, r_2]$. *Then*

$$\dim(\cup_{x\in A} F[x, r_2]) = \sup_{x\in A} \delta(x) = \delta(\sup A).$$

Proof. We note that $\delta(x)$ is an increasing function on $[0, a^s]$. Since $0 \le x \le r_2 < a^s$, $\delta(x) \le \delta(r_2)$. By Corollary 3.5, $\dim(F[x, r_2]) = \delta(x)$. Hence $\dim(\cup_{x\in A} F[x, r_2]) \ge \sup_{x\in A} \delta(x)$.

We note that $\underline{F}(r) = \underline{E}^{(p)}_{g(r,p)}$ if $0 < p < a^s$ from Proposition 3.1. So $F[x, r_2] \subset \underline{F}(x) = \underline{E}^{(p)}_{g(x,p)}$ if $0 < p < a^s$. Since $\sup A < a^s$ from $A \subset [0, r_2]$ and $0 \le r_2 < a^s$, $F[x, r_2] \subset \underline{F}(x) = \underline{E}^{(x_2)}_{g(x,x_2)}$ where $x_2 = \sup A$. It is not difficult to show that $g(x, x_2)$ is an increasing continuous function for x since $0 < x_2 < a^s$(cf. [1]). We also note that $g(x, x_2) \le g(x_2, x_2) = \delta(x_2) = \sup_{x \in A} \delta(x)$ since $g(x, x_2)$ is an increasing continuous function for x. By the proposition 2.3 of [4] which is an essential result of Frostman's density theorem, $\dim(\cup_{x\in A} F[x, r_2]) \le \sup_{x\in A} \delta(x)$. □

THEOREM 3.15. *Let $a^s < r_1 \le 1$ and $B \subset [r_1, 1]$. Then*

$$\dim(\cup_{y\in B} F[r_1, y]) = \sup_{y\in B} \delta(y) = \delta(\inf B).$$

Proof. It follows from the dual arguments of the proof of the above Theorem. □

THEOREM 3.16. *Let $(r_1, r_2) \in \Delta$ which is the characteristic coordinate set. Then we have*
(1) $\dim(\cup_{x\in A} F[x, r_2]) = \sup_{x\in A\cap[0,r_2]} \delta(x) = \delta(\sup(A \cap [0, r_2]))$ *if* $A \cap [r_1, r_2] = \phi$,
(2) $\dim(\cup_{x\in A} F[x, r_2]) = \delta(r_2)$ *if* $A \cap [r_1, r_2] \ne \phi$,
(3) $\dim(\cup_{y\in B} F[r_1, y]) = \sup_{y\in B\cap[r_1,1]} \delta(y) = \delta(\inf(B \cap [r_1, 1]))$ *if* $B \cap [r_1, r_2] = \phi$,
(4) $\dim(\cup_{y\in B} F[r_1, y]) = \delta(r_1)$ *if* $B \cap [r_1, r_2] \ne \phi$.

Proof. (1) and (3) follow from the same arguments of the proofs of the above two Theorems. Noting that δ is a unimodal function on $[0, 1]$, we easily see that (2) and (4) follow from Corollary 3.5 and Theorem 3.8. □

REMARK 3.17. We easily get $\dim(\cup_{x\in A} F[x, r_2])$ and $\dim(\cup_{y\in B} F[r_1, y])$ if A and B are countable sets since Hausdorff dimension is countable stable. But it is not easy to compute $\dim(\cup_{x\in A} F[x, r_2])$ or $\dim(\cup_{y\in B} F[r_1, y])$ if A and B are uncountable sets. In these cases, we apply our Theorems above to the computation of their Hausdorff dimensions. If $r_1 \le a^s$ or $r_2 \ge a^s$, then we apply the above Theorem to the computation of its Hausdorff dimension. Precisely, if $r_1 \le a^s$, then we easily find the counterpart r_2 such that $(r_1, r_2) \in \Delta$. Similarly if $r_2 \ge a^s$, then we easily find the counterpart r_1 such that $(r_1, r_2) \in \Delta$. If not, we apply Theorems 3.14 and 3.15 to the computation of their Hausdorff dimensions.

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