JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **21**, No. 3, September 2008

ON h-STABILITY OF LINEAR DYNAMIC EQUATIONS ON TIME SCALES VIA u_∞-SIMLARITY

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Abstract. In this paper we investigate h-stability for linear dynamic equations on time scales under u_∞-similarity.

1. Introduction

The concept of t_∞-similarity by Conti [14] is an effective tool to study the stability for differential systems. Conti [14] showed that t_∞-similarity is an equivalence relation preserving strict, uniform and exponential stability of linear homogeneous differential systems. Hewer [17] studied the variational stability of nonlinear differential systems using the notion of t_∞-similarity. Choi et al. [4] investigated h-stability for the nonlinear differential systems using the notions of t_∞-similarity and Liapunov functions.

Trench [20] introduced summable similarity as a discrete analog of Conti's definition of t_∞-similarity. Choi and Koo [5] studied the variational stability for nonlinear difference systems by using n_∞-similarity. Furthermore, Choi and Koo [13] introduced u_∞-similarity in order to unify t_∞-similarity and n_∞-similarity and then studied the strong stability for linear dynamic equations on time scales by using the concept of u_∞-similarity and Gronwall's inequality.

In this paper we investigate the h-stability for linear dynamic equations on time scales by using the concept of u_∞-similarity and Gronwall's inequality.

Received July 30, 2008; Accepted August 20, 2008.

2000 Mathematics Subject Classification: 34A30, 34D23, 39A11, 39A13.

Key words and phrases: linear dynamic equation, time scale, Gronwall's inequality, u_∞-similarity, h-stability.

This work was supported by the Korea Research Foundation Grant founded by the Korea Government (MOEHRD) (KRF-2005-070-C00015).

2. Preliminaries

For an introductory text and an article on the calculus on time scales we refer to Bohner and Peterson [3] and Hilger [16], respectively. The symbol $|\cdot|$ will be used to denote any convenient vector norm on $\mathbb{R}^n$ or $\mathbb{R}^{n^2}$.

A *time scale* $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$, and the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) = \inf\{s \in \mathbb{T} \ : \ s > t\}$, while the *graininess* $\mu : \mathbb{T} \to \mathbb{R}_+$ is given by $\mu(t) = \sigma(t) - t$. Assume throughout that $\mathbb{T}$ is unbounded above and the graininess μ is bounded.

A function $f : \mathbb{T} \to \mathbb{R}$ is called *differentiable* (in a point $t \in \mathbb{T}$), if there exists a unique derivative $f^\Delta(t) \in \mathbb{R}$, such that for any $\varepsilon > 0$ the estimate

$$|f^\sigma(t) - f(s) - f^\Delta(t)(\sigma(t) - s)| \le \varepsilon|\sigma(t) - s| \text{ for all } s \in U,$$

where $f^\sigma = f \circ \sigma$, holds in a $\mathbb{T}$-neighborhood U of t.

EXAMPLE 2.1. *Let δ be a positive constant and $\delta\mathbb{Z} = \{0, \pm\delta, \pm 2\delta, \cdots,\}$. The derivative $f^\Delta(t) \in \mathbb{R}$ of a function $f : \mathbb{T} \to \mathbb{R}$ reads as*

$$f^\Delta(t) = f'(t) \text{ if } \mathbb{T} = \mathbb{R}, \ f^\Delta(t) = \frac{f(t+\delta) - f(t)}{\delta} \text{ if } \mathbb{T} = \delta\mathbb{Z}.$$

A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* (denoted by $f \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$) if

(i) f is continuous at every right-dense point $t \in \mathbb{T}$,
(ii) $\lim_{s \to t^-} f(s)$ exists and is finite at every left-dense point $t \in \mathbb{T}$.

A function $g : \mathbb{T} \to \mathbb{R}$ is called an *antiderivative* of f on $\mathbb{T}$ if it is differentiable on $\mathbb{T}$ and satisfies $g^\Delta(t) = f(t)$ for $t \in \mathbb{T}$. In this case, we define the *Cauchy integral* of f as $\int_a^t f(s)\Delta s = g(t) - g(a)$ for $t, a \in \mathbb{T}$.

Let $M_n(\mathbb{R})$ be the set of all $n \times n$ matrices over $\mathbb{R}$ and $\mathfrak{M}_n(\mathbb{R})$ the set of all $n \times n$ invertible matrices over $\mathbb{R}$.

An operator $A : \mathbb{T} \to M_n(\mathbb{R})$ is called *regressive* if for each $t \in \mathbb{T}$ the $n \times n$ matrix $I + \mu(t)A(t)$ is invertible. The set of all rd-continuous and regressive operators from $\mathbb{T}$ to $M_n(\mathbb{R})$ is denoted by $C_{\mathrm{rd}}\mathcal{R}(\mathbb{T}, M_n(\mathbb{R}))$.

We consider linear dynamic systems

$$x^\Delta(t) = A(t)x(t), \ A \in C_{\mathrm{rd}}R(\mathbb{T}, M_n(\mathbb{R})), \tag{2.1}$$

and

$$y^\Delta(t) = B(t)y(t), \ B \in C_{\mathrm{rd}}R(\mathbb{T}, M_n(\mathbb{R})). \tag{2.2}$$

We recall the notion of u_∞-similarity on time scales in order to unify continuous similarity and discrete similarity in [13].

Definition 2.2. Let $A, B \in C_{\rm rd}\mathcal{R}(\mathbb{T}, M_n(\mathbb{R}))$ and $t_0 \in \mathbb{T}$. An operator A is u_∞-*similar* to an operator B if there exists an absolutely integrable operator $F \in C_{\rm rd}(\mathbb{T}, M_n(\mathbb{R}))$, i.e., $\int_{t_0}^{\infty} |F(t)|\Delta t < \infty$, such that

$$S^{\Delta}(t) + S^{\sigma}(t)B(t) - A(t)S(t) = F(t),\ t \in \mathbb{T}, \tag{2.3}$$

for both bounded operators S and $S^{-1} \in C^1_{\rm rd}(\mathbb{T}, \mathfrak{M}_n(\mathbb{R}))$.

We say that if A is u_∞-similar to B, then system (2.1) is u_∞-*similar* to system (2.2). We note that u_∞-similarity is an equivalence relation.

Remark 2.3. If $\mathbb{T} = \mathbb{R}$, then u_∞-similarity becomes t_∞-similarity [4, 17] and if $\mathbb{T} = \mathbb{Z}$, then u_∞-similarity becomes n_∞-similarity [5, 6, 8, 20]. Also if A and B are u_∞-similar with $F = 0$ in (2.3), then they are *kinematically similar* [2].

Lemma 2.4. [13]*Assume that A and B are u_∞-similar. Then*

$$\begin{aligned}\Phi_B(t, t_0) &= S^{-1}(t)[\Phi_A(t, t_0)S(t_0) \\ &+ \int_{t_0}^{t} \Phi_A(t, \sigma(s))F(s)\Phi_B(s, t_0)\Delta s],\ t, t_0 \in \mathbb{T},\end{aligned}$$

where $\Phi_A(t, t_0)$ and $\Phi_B(t, t_0)$ are the matrix exponential functions of (2.1) and (2.2), respectively.

We can obtain the following results as special cases of Lemma 2.4.

Remark 2.5. Assume that A and B are u_∞-similar on $\mathbb{T}$.

(i) If $\mathbb{T} = \mathbb{R}$, then u_∞-similarity becomes t_∞-similarity and we obtain

$$\Phi_B(t, t_0) = S^{-1}(t)[\Phi_A(t, t_0)S(t_0) + \int_{t_0}^{t} \Phi_A(t, s)F(s)\Phi_B(s, t_0)ds].$$

(ii) If $\mathbb{T} = \mathbb{Z}$, then u_∞-similarity becomes n_∞-similarity and we obtain

$$\Phi_B(t, t_0) = S^{-1}(t)[\Phi_A(t, t_0)S(t_0) + \sum_{s=t_0}^{t-1} \Phi_A(t, s+1)F(s)\Phi_B(s, t_0)].$$

3. Main results

We consider the dynamic system

$$x^{\Delta} = F(t, x),\ x(t_0) = x_0, \tag{3.1}$$

where $F \in C_{\rm rd}(\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n)$ with $F(t,0)=0$ and x^Δ is the delta derivative of $x : \mathbb{T} \to \mathbb{R}^n$ with respect to $t \in \mathbb{T}$. We assume that the solutions x of (3.1) exist and are unique for $t \geq t_0$.

Pinto introduced the notion of h-stability which is an extension of the notions of exponential stability and uniform Lipschitz stability of differential equations in [19] and difference equations in [18].

We recall the notion of h-stability of dynamic equations on time scales in [7].

Definition 3.1. System (3.1) is said to be

(i) an *h-system* if there exist a positive function $h : \mathbb{T} \to \mathbb{R}$ and a constant $c \geq 1$ such that

$$|x(t,t_0,x_0)| \leq c|x_0|h(t)h(t_0)^{-1},\ t \geq t_0$$

for $|x_0|$ small enough (here $h(t)^{-1} = \frac{1}{h(t)}$).

(ii) *h-stable* if system (3.1) is an h-system and h is bounded.

Choi et al. [7, 9, 10] and DaCunha [15] gave the characterizations of the various types of stability for solutions of dynamic systems on time scales.

Lemma 3.2. [15] *Let $\Phi_A(t,t_0)$ be a matrix exponential function of (2.1). Then system (2.1) is*

(i) *uniformly stable if and only if there exists a positive constant γ such that*

$$|\Phi_A(t,t_0)| \leq \gamma,\ t \geq t_0 \in \mathbb{T}.$$

(ii) *uniformly exponential stable if and only if there exist positive constants γ, λ with $-\lambda \in \mathcal{R}^+$ such that*

$$|\Phi_A(t,t_0)| \leq \gamma e_{-\lambda}(t,t_0),\ t \geq t_0 \in \mathbb{T}.$$

(iii) *uniformly exponential stable if and only if it is uniformly asymptotically stable.*

Lemma 3.3. [7, Lemma 2.3] *System (2.1) is an h-system if and only if there exist a positive function h defined on $\mathbb{T}$ and a constant $c \geq 1$ such that*

$$|\Phi_A(t,t_0)| \leq ch(t)h(t_0)^{-1},\ t \geq t_0 \in \mathbb{T}.$$

Theorem 3.4. *Assume that systems (2.1) and (2.2) are u_∞-similar with $\int_{t_0}^\infty \frac{h(s)}{h(\sigma(s))}|F(s)|\Delta s < \infty$ for each $t_0 \in \mathbb{T}$. Then (2.1) is an h-system if and only if (2.2) is also an h-system.*

Proof. Suppose (2.1) is an h-system. Then, there exist a positive function h defined on $\mathbb{T}$ and a constant $c \geq 1$ such that

$$|\Phi_A(t,t_0)| \leq ch(t)h(t_0)^{-1},\ t \geq t_0 \in \mathbb{T}.$$

From Lemma 2.4 and by virtue of the boundedness of $S(t)$ and $S^{-1}(t)$, there are positive constants c_1 and c_2 such that

$$\begin{aligned} |\Phi_B(t,\ t_0)| &\leq |S^{-1}(t)|[|\Phi_A(t,\ t_0)||S(t_0)| \\ &+ \int_{t_0}^{t} |\Phi_A(t,\ \sigma(s))||F(s)||\Phi_B(s,\ t_0)|\Delta s] \\ &\leq c_1c_2\frac{h(t)}{h(t_0)} + c_1c_2\int_{t_0}^{t}\frac{h(t)}{h(\sigma(s))}|F(s)||\Phi_B(s,\ t_0)|\Delta s, \end{aligned}$$

where $\Phi_B(t,t_0)$ is a matrix exponential function for (2.2). Dividing by $h(t)$ on both sides, we have

$$\frac{|\Phi_B(t,\ t_0)|}{h(t)} \leq \frac{c_1c_2}{h(t_0)} + c_1c_2\int_{t_0}^{t}\frac{h(s)}{h(\sigma(s))}|F(s)|\frac{|\Phi_B(s,\ t_0)|}{h(s)}\Delta s.$$

In view of the Gronwall's inequality on time scale in [3], we have

$$\begin{aligned} \frac{|\Phi_B(t,\ t_0)|}{h(t)} &\leq \frac{c_1c_2}{h(t_0)}e_{p(t)}(t,\ t_0) = \frac{c_1c_2}{h(t_0)}exp\left(\int_{t_0}^{t}\xi_{\mu(s)}(p(s))\Delta s\right) \\ &= \begin{cases} \frac{c_1c_2}{h(t_0)}\exp(\int_{t_0}^{t}\frac{1}{\mu(s)}Log(1+\mu(s)p(s))\Delta s) & \text{if } \mu \neq 0 \\ \\ \frac{c_1c_2}{h(t_0)}\exp(\int_{t_0}^{t}p(s)\Delta s) & \text{if } \mu = 0 \end{cases} \\ &\leq \frac{c_1c_2}{h(t_0)}exp\left(\int_{t_0}^{t}p(s)\Delta s\right) \leq \frac{c_1c_2}{h(t_0)}exp\left(\int_{t_0}^{\infty}p(s)\Delta s\right), \end{aligned}$$

where $p(t) = c_1c_2\frac{h(t)}{h(\sigma(t))}|F(t)|$ and the cylinder transformation $\xi_\mu(z)$ is given by

$$\xi_\mu(z) = \begin{cases} \frac{1}{\mu}Log(1+\mu z) & \text{if } \mu \neq 0 \\ z & \text{if } \mu = 0. \end{cases}$$

Thus we have

$$|\Phi_B(t,\ t_0)| \leq dh(t)h(t_0)^{-1},\ t \geq t_0,$$

where $d \geq c_1c_2\exp(\int_{t_0}^{\infty}p(s)\Delta s)$ is a positive constant. Hence (2.2) is an h-system by Lemma 3.3.

The converse holds by the same method. This completes the proof. □

Remark 3.5. If $h(t)$ is a positive bounded function on $\mathbb{T}$, then $\frac{h(t)}{h(\sigma(t))}$ is not bounded in general. For example, see [6, Remark 3.1] for $\mathbb{T} = \mathbb{Z}_+$.

Corollary 3.6. *Suppose that systems (2.1) and (2.2) are u_∞-similar with bounded function $\frac{h(t)}{h(\sigma(t))}$ on $\mathbb{T}$.*

(i) *Then (2.1) is h-stable if and only if (2.2) is h-stable.*
(ii) *When (2.1) is h-stable with $h(t) = e_{-\lambda}(t, a_0)$ for some positive constant λ with $-\lambda \in \mathcal{R}^+$ in Theorem 3.4, then (2.1) is uniformly exponentially stable if and only if (2.2) is uniformly exponentially stable.*
(iii) *When (2.1) is h-stable with a constant function h, then (2.1) is uniformly stable if and only if (2.2) is also uniformly stable.*

We can obtain the following results as special cases of Theorem 3.4.

Remark 3.7. Suppose that systems (2.1) and (2.2) are u_∞-similar with $\int_{t_0}^{\infty} \frac{h(s)}{h(\sigma(s))} |F(s)| \Delta s < \infty$ for each $t_0 \in \mathbb{T}$.

(i) When $\mathbb{T} = \mathbb{R}$, then (2.1) is h-stable if and only if (2.2) is also h-stable in [4, Theorem 2.2].
(ii) If $\mathbb{T} = \mathbb{Z}$, then $\int_{t_0}^{\infty} \frac{h(s)}{h(\sigma(s))} |F(s)| \Delta s < \infty$ becomes $\sum_{s=t_0}^{\infty} \frac{h(s)}{h(s+1)} |F(s)| < \infty$ for each $t_0 \in \mathbb{Z}$. Thus (2.1) is h-stable if and only if (2.2) is also h-stable in [5, Theorem 3] and [8, Lemma 3.3].

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