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TRANSIENT ANALYSIS OF THE GEO/GEO/1 QUEUE

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Abstract. This paper gives transient distributions for the number of customers in the system in the Geo/Geo/1 queue for both the early arrival and the late arrival models.

1. Introduction

There have been considerable interests in recent years in discrete-time queueing systems. One of the reasons for this is that discrete-time queues fit the slotted nature of telecommunication systems better than the continuous-time counterparts. We refer the readers to the books [1, 6] for more details on discrete-time queues. In discrete-time queues, the time axis is divided into fixed-length intervals, called slots, and customer arrivals and departures can happen simultaneously at a slot boundary. Usually, there are two models in discrete-time: early arrival model and late arrival model (see [1, 6]).

In this paper we consider the Geo/Geo/1 queue where customers arrive at a single server facility according to a Bernoulli process and have service times that are geometrically distributed. Mohanty and Panny [2, 3] considered the Geo/Geo/1 queue and obtained the transient solution for the number of customers in the system by two approaches: analytic and geometric, which are different from the one presented in this paper. In addition, Mohanty et al. [4] obtained in an alternative manner the transient solution for the number of customers in the system and the length of a busy period in the Geo/Geo/1 queue with the early arrival model.

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This paper gives transient distributions for the number of customers in the system in the Geo/Geo/1 queue for both the early arrival and the late arrival models. Although we consider a model which is equivalent to the one in [2, 3, 4], the method for the proof of result in this paper is different from the one in [2, 3, 4].

2. Transient analysis in the early arrival model

We consider the Geo/Geo/1 queue with the early arrival model. Customers arrive at a single facility according to a Bernoulli process and the probability of an arrival during a slot is λ. Service times are geometrically distributed and the probability of completion of a service during a slot is μ.

Assume that the time axis is divided into fixed-length intervals. The time axis is marked by $0, 1, \ldots, n, \ldots$. In the early arrival model, customer arrivals can occur only in $(n, n+)$ and services can be completed only in $(n-, n)$.

Let $X(t)$ denote the number of customers in the system at time t and define $L_n = X(n)$, $n = 0, 1, 2, \ldots$. Then $\{L_n : n = 0, 1, \ldots\}$ is a Markov chain with transition probability matrix $P = (P_{ij})_{i,j\geq 0}$. The (i, j)-component P_{ij} of P is given by

$$P_{ij} = \begin{cases} \lambda(1-\mu) & \text{if } j = i+1, i \geq 0, \\ \mu(1-\lambda) & \text{if } j = i-1, i \geq 1, \\ 1-\lambda(1-\mu) & \text{if } j = i = 0, \\ \lambda\mu + (1-\lambda)(1-\mu) & \text{if } j = i, i \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Now, we define

$$P_{ij}(n) = \mathbb{P}(L_n = j | L_0 = i),\ n = 0, 1, \ldots,$$

and give an explicit formula for this. First we consider a finite stochastic matrix $P^{(N)} = \left(P_{ij}^{(N)}\right)_{0\leq i,j\leq N}$, for any $N \geq 1$, defined by

$$P_{ij}^{(N)} = \begin{cases} \lambda(1-\mu) & \text{if } j = i+1, 0 \leq i \leq N-1, \\ \mu(1-\lambda) & \text{if } j = i-1, 1 \leq i \leq N, \\ 1-\lambda(1-\mu) & \text{if } j = i = 0, \\ \lambda\mu + (1-\lambda)(1-\mu) & \text{if } j = i, 1 \leq i \leq N-1, \\ 1-\mu(1-\lambda) & \text{if } j = i = N, \\ 0 & \text{otherwise.} \end{cases}$$

The following lemma gives an expression for $P_{ij}^{(N)}(n) \equiv \left[\left(P^{(N)}\right)^n\right]_{ij}$. The proof is carried out by the same method as in the proof of Theorem 1 on page 13 of [5].

LEMMA 2.1. *If $\lambda \neq \mu$, then*

$$\begin{aligned}
&P_{ij}^{(N)}(n) \\
&= \frac{1-\frac{\lambda(1-\mu)}{\mu(1-\lambda)}}{1-\left(\frac{\lambda(1-\mu)}{\mu(1-\lambda)}\right)^{N+1}}\left(\frac{\lambda(1-\mu)}{\mu(1-\lambda)}\right)^j + \frac{2}{N+1}\left(\frac{\lambda(1-\mu)}{\mu(1-\lambda)}\right)^{\frac{j-i}{2}} \\
&\quad \times \sum_{k=1}^{N} \frac{\left(\lambda\mu+(1-\lambda)(1-\mu)+2\sqrt{\lambda\mu(1-\lambda)(1-\mu)}\cos(\frac{k\pi}{N+1})\right)^n}{\left[1-2\sqrt{\frac{\lambda(1-\mu)}{\mu(1-\lambda)}}\cos(\frac{k\pi}{N+1})+\frac{\lambda(1-\mu)}{\mu(1-\lambda)}\right]} \\
&\quad \times \left[\sin\left(\frac{ik\pi}{N+1}\right)-\sqrt{\frac{\lambda(1-\mu)}{\mu(1-\lambda)}}\sin\left(\frac{(i+1)k\pi}{N+1}\right)\right] \\
&\quad \times \left[\sin\left(\frac{jk\pi}{N+1}\right)-\sqrt{\frac{\lambda(1-\mu)}{\mu(1-\lambda)}}\sin\left(\frac{(j+1)k\pi}{N+1}\right)\right],
\end{aligned}$$

and if $\lambda = \mu$, then

$$\begin{aligned}
&P_{ij}^{(N)}(n) \\
&= \frac{1}{N+1}+\frac{1}{N+1}\sum_{k=1}^{N}\frac{\left(\lambda^2+(1-\lambda)^2+2\lambda(1-\lambda)\cos(\frac{k\pi}{N+1})\right)^n}{\left[1-\cos(\frac{k\pi}{N+1})\right]} \\
&\quad \times \left[\sin\left(\frac{ik\pi}{N+1}\right)-\sin\left(\frac{(i+1)k\pi}{N+1}\right)\right] \\
&\quad \times \left[\sin\left(\frac{jk\pi}{N+1}\right)-\sin\left(\frac{(j+1)k\pi}{N+1}\right)\right].
\end{aligned}$$

THEOREM 2.2. *The $P_{ij}(n)$ is given by*

$$P_{ij}(n) = \begin{cases} p_{ij}(n)+\left(1-\frac{\lambda(1-\mu)}{\mu(1-\lambda)}\right)\left(\frac{\lambda(1-\mu)}{\mu(1-\lambda)}\right)^j & \text{if } \lambda<\mu, \\ p_{ij}(n) & \text{if } \lambda \geq \mu, \end{cases} \tag{2.1}$$

where

$$p_{ij}(n) = \frac{2}{\pi}\left(\frac{\lambda(1-\mu)}{\mu(1-\lambda)}\right)^{\frac{j-i}{2}} \int_0^{\pi} \frac{\left(\lambda\mu+(1-\lambda)(1-\mu)+2\sqrt{\lambda\mu(1-\lambda)(1-\mu)}\cos y\right)^n}{\left[1-2\sqrt{\frac{\lambda(1-\mu)}{\mu(1-\lambda)}}\cos y+\frac{\lambda(1-\mu)}{\mu(1-\lambda)}\right]}$$

$$\times\left[\sin(iy)-\sqrt{\frac{\lambda(1-\mu)}{\mu(1-\lambda)}}\sin((i+1)y)\right]\left[\sin(jy)-\sqrt{\frac{\lambda(1-\mu)}{\mu(1-\lambda)}}\sin((j+1)y)\right]dy. \tag{2.2}$$

Proof. Since $P_{ij}^{(N)} = P_{ij}$ for $0 \le i, j \le N-1$ and both of the matrices $P^{(N)}$ and P are tridiagonal, it can be shown by induction that

$$P_{ij}^{(N)}(n) = P_{ij}(n) \text{ for } 0 \le i, j \le N-n.$$

Hence, for any $i, j, n = 0, 1, \ldots$, we have

$$P_{ij}(n) = \lim_{N\to\infty} P_{ij}^{(N)}(n).$$

Combining this with Lemma 2.1, we have the desired assertion. □

We now give an alternative expression for $P_{ij}(n)$.

THEOREM 2.3. *The* $P_{ij}(n)$ *is given by*

$$P_{ij}(n) = \left(\frac{\lambda(1-\mu)}{\mu(1-\lambda)}\right)^{\frac{j-i}{2}}\left(g_{j-i}^n+\sqrt{\frac{\mu(1-\lambda)}{\lambda(1-\mu)}}g_{i+j+1}^n\right) + \left(1-\frac{\lambda(1-\mu)}{\mu(1-\lambda)}\right)\sum_{\nu=2}^{n-i-j}\left(\frac{\lambda(1-\mu)}{\mu(1-\lambda)}\right)^{\frac{j-i-\nu}{2}} g_{i+j+\nu}^n,$$

where

$$g_\nu^n = \begin{cases} \sum_{k=0}^{n-|\nu|}\binom{n}{k}\binom{n}{|\nu|+k}(\lambda\mu)^{n-\frac{|\nu|}{2}-k}((1-\lambda)(1-\mu))^{\frac{|\nu|}{2}+k} & \text{if } |\nu|\le n,\\ 0 & \text{if } |\nu|>n. \end{cases} \tag{2.3}$$

Proof. Since the integrand of (2.2) is an even function of y, we can replace the $\int_0^\pi$ by $\frac{1}{2}\int_{-\pi}^{\pi}$ in (2.2). Making the change of variables $z = e^{y\sqrt{-1}}$ yields

$$p_{ij}(n) = \frac{1}{4\pi\sqrt{-1}}\left(\frac{\lambda(1-\mu)}{\mu(1-\lambda)}\right)^{\frac{j-i}{2}}\oint_{|z|=1}\frac{f(z)}{z}dz, \tag{2.4}$$

where

$$f(z) = \frac{-\left(\lambda\mu + (1-\lambda)(1-\mu) + \sqrt{\lambda\mu(1-\lambda)(1-\mu)}\left(z+\frac{1}{z}\right)\right)^n}{\left[1 - \left(z+\frac{1}{z}\right)\sqrt{\frac{\lambda(1-\mu)}{\mu(1-\lambda)}} + \frac{\lambda(1-\mu)}{\mu(1-\lambda)}\right]} \times \left[(z^i - z^{-i}) - \sqrt{\frac{\lambda(1-\mu)}{\mu(1-\lambda)}}(z^{i+1} - z^{-i-1})\right] \times \left[(z^j - z^{-j}) - \sqrt{\frac{\lambda(1-\mu)}{\mu(1-\lambda)}}(z^{j+1} - z^{-j-1})\right]. \tag{2.5}$$

If $\lambda < \mu$, then the integrand in (2.4) has two singular points at $z = 0$ and $z = \sqrt{\frac{\lambda(1-\mu)}{\mu(1-\lambda)}}$ in the unit disk $\{z \in \mathbb{C} : |z| \leq 1\}$. Its residue at $z = \sqrt{\frac{\lambda(1-\mu)}{\mu(1-\lambda)}}$ is $-\left(1 - \frac{\lambda(1-\mu)}{\mu(1-\lambda)}\right)\left(\frac{\lambda(1-\mu)}{\mu(1-\lambda)}\right)^{\frac{i+j+1}{2}}$. If $\lambda = \mu$, then $z = 0$ is a singular point of the integrand in (2.4). If $\lambda > \mu$, then the integrand in (2.4) has two singular points at $z = 0$ and $z = \sqrt{\frac{\mu(1-\lambda)}{\lambda(1-\mu)}}$. Its residue at $z = \sqrt{\frac{\mu(1-\lambda)}{\lambda(1-\mu)}}$ is $\left(1 - \frac{\lambda(1-\mu)}{\mu(1-\lambda)}\right)\left(\frac{\lambda(1-\mu)}{\mu(1-\lambda)}\right)^{\frac{i+j+1}{2}}$. Let f_0 be the constant term in the Laurent expansion of $f(z)$ at $z = 0$. Then its residue at $z = 0$ is f_0. By the residue theorem, we have that

$$p_{ij}(n) = \begin{cases} \frac{1}{2}\left(\frac{\lambda(1-\mu)}{\mu(1-\lambda)}\right)^{\frac{j-i}{2}} f_0 - \frac{1}{2}\left(1 - \frac{\lambda(1-\mu)}{\mu(1-\lambda)}\right)\left(\frac{\lambda(1-\mu)}{\mu(1-\lambda)}\right)^j & \text{if } \lambda < \mu, \\ \frac{1}{2}\left(\frac{\lambda(1-\mu)}{\mu(1-\lambda)}\right)^{\frac{j-i}{2}} f_0 & \text{if } \lambda = \mu, \\ \frac{1}{2}\left(\frac{\lambda(1-\mu)}{\mu(1-\lambda)}\right)^{\frac{j-i}{2}} f_0 + \frac{1}{2}\left(1 - \frac{\lambda(1-\mu)}{\mu(1-\lambda)}\right)\left(\frac{\lambda(1-\mu)}{\mu(1-\lambda)}\right)^j & \text{if } \lambda > \mu. \end{cases}$$

Substituting the above into (2.1) yields

$$P_{ij}(n) = \frac{1}{2}\left(\frac{\lambda(1-\mu)}{\mu(1-\lambda)}\right)^{\frac{j-i}{2}} f_0 + \frac{1}{2}\left(1 - \frac{\lambda(1-\mu)}{\mu(1-\lambda)}\right)\left(\frac{\lambda(1-\mu)}{\mu(1-\lambda)}\right)^j. \tag{2.6}$$

Now we calculate f_0. It follows from (2.5) that

$$\begin{aligned}
f(z) &= \frac{-\left(\sqrt{\lambda\mu}+\sqrt{(1-\lambda)(1-\mu)}z\right)^n\left(\sqrt{\lambda\mu}+\sqrt{(1-\lambda)(1-\mu)}\frac{1}{z}\right)^n}{\left(1-\sqrt{\frac{\lambda(1-\mu)}{\mu(1-\lambda)}}z\right)\left(1-\sqrt{\frac{\lambda(1-\mu)}{\mu(1-\lambda)}}\frac{1}{z}\right)} \\
&\quad\times\left[z^i\left(1-\sqrt{\frac{\lambda(1-\mu)}{\mu(1-\lambda)}}z\right)-z^{-i}\left(1-\sqrt{\frac{\lambda(1-\mu)}{\mu(1-\lambda)}}\frac{1}{z}\right)\right] \\
&\quad\times\left[z^j\left(1-\sqrt{\frac{\lambda(1-\mu)}{\mu(1-\lambda)}}z\right)-z^{-j}\left(1-\sqrt{\frac{\lambda(1-\mu)}{\mu(1-\lambda)}}\frac{1}{z}\right)\right] \\
&= \sum_{\nu=-n}^{n} g_\nu^n z^\nu\left(z^{i-j}+z^{j-i}-z^{i+j}\frac{1-\sqrt{\frac{\lambda(1-\mu)}{\mu(1-\lambda)}}z}{1-\sqrt{\frac{\lambda(1-\mu)}{\mu(1-\lambda)}}\frac{1}{z}}-z^{-i-j}\frac{1-\sqrt{\frac{\lambda(1-\mu)}{\mu(1-\lambda)}}\frac{1}{z}}{1-\sqrt{\frac{\lambda(1-\mu)}{\mu(1-\lambda)}}z}\right),
\end{aligned}$$

where g_ν^n is the coefficient of z^ν in the Laurent expansion of

$$\left(\sqrt{\lambda\mu}+\sqrt{(1-\lambda)(1-\mu)}z\right)^n\left(\sqrt{\lambda\mu}+\sqrt{(1-\lambda)(1-\mu)}\frac{1}{z}\right)^n$$

at $z=0$. Hence, g_ν^n satisfies (2.3). After some arithmetic, we get

$$\begin{aligned}
f(z) &= \sum_{\nu=-n}^{n} g_\nu^n z^\nu\Bigg\{z^{i-j}+z^{j-i}+\sqrt{\frac{\mu(1-\lambda)}{\lambda(1-\mu)}}z^{i+j+1}+\sqrt{\frac{\lambda(1-\mu)}{\mu(1-\lambda)}}z^{-i-j-1} \\
&\quad-\left(1-\frac{\mu(1-\lambda)}{\lambda(1-\mu)}\right)\sum_{\nu=0}^{\infty}\left(\frac{\mu(1-\lambda)}{\lambda(1-\mu)}\right)^{\frac{\nu}{2}}z^{i+j+\nu+2} \\
&\quad-\left(1-\frac{\lambda(1-\mu)}{\mu(1-\lambda)}\right)\sum_{\nu=0}^{\infty}\left(\frac{\lambda(1-\mu)}{\mu(1-\lambda)}\right)^{\frac{\nu}{2}}z^{-i-j+\nu}\Bigg\}.
\end{aligned}$$

Therefore

$$\begin{aligned}
f_0 &= g_{j-i}^n+g_{i-j}^n+\sqrt{\frac{\mu(1-\lambda)}{\lambda(1-\mu)}}g_{-i-j-1}^n+\sqrt{\frac{\lambda(1-\mu)}{\mu(1-\lambda)}}g_{i+j+1}^n \\
&\quad-\left(1-\frac{\mu(1-\lambda)}{\lambda(1-\mu)}\right)\sum_{\nu=0}^{\infty}\left(\frac{\mu(1-\lambda)}{\lambda(1-\mu)}\right)^{\frac{\nu}{2}}g_{-i-j-\nu-2}^n \\
&\quad-\left(1-\frac{\lambda(1-\mu)}{\mu(1-\lambda)}\right)\sum_{\nu=0}^{\infty}\left(\frac{\lambda(1-\mu)}{\mu(1-\lambda)}\right)^{\frac{\nu}{2}}g_{i+j-\nu}^n.
\end{aligned}$$

Noting that $g_\nu^n = g_{-\nu}^n$ for all n and ν and

$$\sum_{\nu=-\infty}^{\infty}\left(\frac{\lambda(1-\mu)}{\mu(1-\lambda)}\right)^{\frac{\nu}{2}}g_\nu^n = 1,$$

we have

$$\begin{aligned}
f_0 &= 2g^n_{j-i} + \left(\sqrt{\frac{\mu(1-\lambda)}{\lambda(1-\mu)}} + \sqrt{\frac{\lambda(1-\mu)}{\mu(1-\lambda)}}\right) g^n_{i+j+1} \\
&\quad - \left(1 - \frac{\mu(1-\lambda)}{\lambda(1-\mu)}\right) \sum_{\nu=0}^{\infty} \left(\frac{\mu(1-\lambda)}{\lambda(1-\mu)}\right)^{\frac{\nu}{2}} g^n_{-i-j-\nu-2} \\
&\quad - \left(1 - \frac{\lambda(1-\mu)}{\mu(1-\lambda)}\right) \sum_{\nu=-\infty}^{\infty} \left(\frac{\lambda(1-\mu)}{\mu(1-\lambda)}\right)^{\frac{\nu}{2}} g^n_{i+j-\nu} \\
&\quad + \left(1 - \frac{\lambda(1-\mu)}{\mu(1-\lambda)}\right) \sum_{\nu=-\infty}^{-1} \left(\frac{\lambda(1-\mu)}{\mu(1-\lambda)}\right)^{\frac{\nu}{2}} g^n_{i+j-\nu} \\
&= 2g^n_{j-i} + \left(\sqrt{\frac{\mu(1-\lambda)}{\lambda(1-\mu)}} + \sqrt{\frac{\lambda(1-\mu)}{\mu(1-\lambda)}}\right) g^n_{i+j+1} \\
&\quad + \left(1 - \frac{\lambda(1-\mu)}{\mu(1-\lambda)}\right) \sum_{\nu=2}^{\infty} \left(\frac{\mu(1-\lambda)}{\lambda(1-\mu)}\right)^{\frac{\nu}{2}} g^n_{-i-j-\nu} \\
&\quad - \left(1 - \frac{\lambda(1-\mu)}{\mu(1-\lambda)}\right) \left(\frac{\lambda(1-\mu)}{\mu(1-\lambda)}\right)^{\frac{i+j}{2}} \\
&\quad + \left(1 - \frac{\lambda(1-\mu)}{\mu(1-\lambda)}\right) \sum_{\nu=1}^{\infty} \left(\frac{\mu(1-\lambda)}{\lambda(1-\mu)}\right)^{\frac{\nu}{2}} g^n_{i+j+\nu}.
\end{aligned}$$

Therefore

$$\begin{aligned}
f_0 &= 2g^n_{j-i} - \left(1 - \frac{\lambda(1-\mu)}{\mu(1-\lambda)}\right) \left(\frac{\lambda(1-\mu)}{\mu(1-\lambda)}\right)^{\frac{i+j}{2}} + 2\sqrt{\frac{\mu(1-\lambda)}{\lambda(1-\mu)}} g^n_{i+j+1} \\
&\quad +2 \left(1 - \frac{\lambda(1-\mu)}{\mu(1-\lambda)}\right) \sum_{\nu=2}^{\infty} \left(\frac{\mu(1-\lambda)}{\lambda(1-\mu)}\right)^{\frac{\nu}{2}} g^n_{i+j+\nu}.
\end{aligned}$$

Finally, substituting the above into (2.6) completes the proof. □

3. Transient analysis in the late arrival model

In this section we deal with the late arrival model. In the late arrival model, customer arrivals can occur only in $(n-, n)$ and services can be completed only in $(n, n+)$. Therefore, arriving customers see a departing customer about to leave, and the departing customer leaves behind the customers that have just arrived.

Let $\widetilde{L}_n$ be the number of customers in the system at the end of the nth slot, i.e., $\widetilde{L}_n = X(n-)$. Then $\{\widetilde{L}_n : n = 0, 1, 2, \ldots\}$ is a Markov chain with transition probability matrix $\widetilde{P} = \left(\widetilde{P}_{ij}\right)_{i,j\geq 0}$. The (i,j)-component $\widetilde{P}_{ij}$ of $\widetilde{P}$ is given by

$$\widetilde{P}_{ij} = \begin{cases} \lambda & \text{if } i = 0, j = 1, \\ \lambda(1-\mu) & \text{if } j = i+1, i \geq 1, \\ \mu(1-\lambda) & \text{if } j = i-1, i \geq 1, \\ 1-\lambda & \text{if } j = i = 0, \\ \lambda\mu + (1-\lambda)(1-\mu) & \text{if } j = i, i \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Now, we define

$$\widetilde{P}_{ij}(n) = \mathbb{P}(\widetilde{L}_n = j | \widetilde{L}_0 = i), \ n = 0, 1, \ldots,$$

and give an explicit formula for this. Let us define two infinite matrices $L = (L_{ij})$ and $U = (U_{ij})$ as follows:

$$L_{ij} = \begin{cases} 1 & \text{if } i = j = 0, \\ 1-\mu & \text{if } i = j \geq 1, \\ \mu & \text{if } j = i-1, i \geq 1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$U_{ij} = \begin{cases} 1-\lambda & \text{if } i = j \geq 0, \\ \lambda & \text{if } j = i+1, i \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$P = UL \quad \text{and} \quad \widetilde{P} = LU,$$

and so

$$\widetilde{P}^n = LP^{n-1}U.$$

Since $\widetilde{P}_{ij}(n) = \left(\widetilde{P}^n\right)_{ij}$ and $P_{ij}(n) = (P^n)_{ij}$, we have the following theorem.

THEOREM 3.1. *For $n \geq 1$, $\widetilde{P}_{ij}(n)$ is given by*

$$\widetilde{P}_{ij}(n) = \begin{cases} (1-\lambda)P_{00}(n-1) & \text{if } i = j = 0, \\ \lambda P_{0,j-1}(n-1) + (1-\lambda)P_{0j}(n-1) & \text{if } i = 0, j \geq 1, \\ \mu(1-\lambda)P_{i-1,0}(n-1) + (1-\mu)(1-\lambda)P_{i0}(n-1) & \text{if } i \geq 1, j = 0, \\ \mu\lambda P_{i-1,j-1}(n-1) + \mu(1-\lambda)P_{i-1,j}(n-1) & \\ +(1-\mu)\lambda P_{i,j-1}(n-1) + (1-\mu)(1-\lambda)P_{ij}(n-1) & \text{if } i \geq 1, j \geq 1. \end{cases}$$

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