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ON THE STABILITY OF A CUBIC FUNCTIONAL EQUATION

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Abstract. In this paper, we prove the stability of the functional equation

$$\sum_{i=0}^{3} {}_3C_i(-1)^{3-i} f(ix+y) - 3!f(x) = 0$$

in the sense of P. Găvruta on the punctured domain. Also, we investigate the superstability of the functional equation.

1. Introduction

Throughout this paper, let V be a vector space, Y a Banach space, and $\mathbb{N}$ the set of positive integers. Let $n \in \mathbb{N}$. For a given mapping $f : V \to Y$, define a mapping $D_n f : V \times V \to Y$ by

$$D_n f(x,y) := \sum_{i=0}^{n} (-1)^{n-i} {}_nC_i f(ix+y) - n!f(x)$$

for all $x, y \in V$, where ${}_nC_i = \frac{n!}{i!(n-i)!}$. A mapping $f : V \to Y$ is called a monomial function of degree $n \in \mathbb{N}$ if f satisfies the functional equation $D_n f(x,y) = 0$. The functional equation $D_n f(x,y) = 0$ is called a monomial functional equation of degree $n \in \mathbb{N}$. In particular, a mapping $f : V \to Y$ is called an additive (quadratic, cubic, respectively) mapping if f satisfies the functional equation $D_1 f = 0$ ($D_2 f = 0$, $D_3 f = 0$, respectively). The functional equation $D_1 f = 0$ ($D_2 f = 0$, $D_3 f = 0$, respectively) is called a Cauchy equation (quadratic functional equation, cubic functional equation, respectively). Let $\mathbb{R}$ be the set of real numbers. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = ax^n$ satisfies the

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functional equation $D_n f = 0$, where a is a real constant.

If we replace a given functional equation by a functional inequality, when can one assert that the solutions of the inequality must be close to the solutions of the given equation? If the answer is affirmative, we would say that a given functional equation is stable.

In 1941, D.H. Hyers [6] proved the stability of Cauchy equation $D_1 f = 0$ and in 1978, Th.M. Rassias [13] gave a significant generalization of the Hyers' result. Th.M. Rassias [14] during the 27th International Symposium on Functional Equations, that took place in Bielsko-Biala, Poland, in 1990, asked the question whether such a theorem can also be proved for a more general setting. Z. Gadja [4] following Th.M. Rassias's approach [13] gave an affirmative solution to the question. Recently, P. Găvruta [5] obtained a further generalization of Rassias' theorem, the so-called generalized Hyers-Ulam stability.

A stability problem for the quadratic functional equation $D_2 f = 0$ was proved by F. Skof [15] for a function $f : X \to W$, where X is a normed space and W is a Banach space. P.W. Cholewa [2] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. S. Czerwik [3] proved the Hyers-Ulam-Rassias stability of the quadratic functional equation.

J.C. Parnami, H.L. Vasudeva [11] and J.M. Rassias [12] investigated the stability of the cubic functional equation $D_3 f = 0$ (see also [7]). A stability problem for the functional equation $D_n f = 0$ was proved by L.Cădariu and V. Radu [1] (see also [8], [10]).

In this paper, we prove the generalized Hyers-Ulam stability and the superstability of the functional equation $D_3 f = 0$ in the sense of P. Găvruta.

2. Stability of a cubic functional equation

Throughout this section, we denote by V and Y a normed space and a Banach space, respectively. The authors [9] obtained the following lemma.

LEMMA 2.1. *Let a be a positive real number and $\Phi : V \setminus \{0\} \to [0, \infty)$ a map. Suppose that the function $f : V \to Y$ satisfies the inequality*

$$\left\| f(x) - \frac{f(2x)}{a} \right\| \le \frac{\Phi(x)}{a} \quad \text{and} \quad f(0) = 0.$$

(i) If $\sum_{l=0}^{\infty} \frac{1}{a^{l+1}} \Phi(2^l x) < \infty$ *for all* $x \in V\setminus\{0\}$*, then there exists a unique function* $F : V \to Y$ *satisfying*

$$\|f(x) - F(x)\| \leq \sum_{l=0}^{\infty} \frac{1}{a^{l+1}} \Phi(2^l x)$$

for all $x \in V\setminus\{0\}$ *and* F *is given by* $F(x) = \lim_{n\to\infty} \frac{f(2^n x)}{a^n}$ *for all* $x \in V$*.*
(ii) If $\sum_{l=0}^{\infty} a^l \Phi(\frac{x}{2^{l+1}}) < \infty$ *for all* $x \in V \setminus \{0\}$*, then there exists a unique function* $F : V \to Y$ *satisfying*

$$\|f(x) - F(x)\| \leq \sum_{l=0}^{\infty} a^l \Phi(\frac{x}{2^{l+1}}) < \infty$$

for all $x \in X \setminus \{0\}$ *and* F *is given by* $F(x) = \lim_{n\to\infty} a^n f(\frac{x}{2^n})$ *for all* $x \in V$*.*

Theorem 2.2. *Let* $\varphi : V \setminus \{0\} \times V \setminus \{0\} \to [0, \infty)$ *be a mapping satisfying the condition*

$$\sum_{i=0}^{\infty} \frac{\varphi(2^i x, 2^i y)}{8^{i+1}} < \infty$$

for all $x, y \in V\setminus\{0\}$*. If a function* $f : V \to Y$ *satisfies the inequality*

$$\|D_3 f(x, y)\| \leq \varphi(x, y) \tag{2.1}$$

for all $x, y \in V\setminus\{0\}$*, then there exists a unique cubic function* $C : V \to Y$ *such that*

$$\|f(x) - C(x)\| \leq \Phi(x) \tag{2.2}$$

for all $x \in V\setminus\{0\}$*, where* Φ *is defined by*

$$\Phi(x) = \sum_{i=0}^{\infty} \frac{\varphi(2^i x, -2^i x) + \varphi(-2^i x, 2^i x)}{2 \cdot 8^{i+1}} + \frac{\varphi(x, -x) + \varphi(-x, 2x)}{12}.$$

In particular, C *is given by*

$$C(x) = \lim_{n\to\infty} \frac{f(2^n x) - f(-2^n x)}{2 \cdot 8^n}$$

for all $x \in V$*.*

Proof. From (2.1), we see that

$$\begin{aligned}\left\| \frac{f(x) - f(-x)}{2} - \frac{f(2x) - f(-2x)}{16} \right\| &= \left\| \frac{D_3 f(-x, x) - D_3 f(x, -x)}{16} \right\| \\ &\leq \frac{\varphi(x, -x) + \varphi(-x, x)}{16}\end{aligned}$$

for all $x \in V\backslash\{0\}$. By Lemma 2.1, there exists

$$C(x) = \lim_{n\to\infty} \frac{f(2^n x) - f(-2^n x)}{2 \cdot 8^n}$$

for all $x \in V$ satisfying

$$\left\| \frac{f(x) - f(-x)}{2} - C(x) \right\| \leq \sum_{i=0}^{\infty} \frac{\varphi(2^i x, -2^i x) + \varphi(-2^i x, 2^i x)}{2 \cdot 8^{i+1}} \tag{2.3}$$

for all $x \in V \setminus \{0\}$. From the definition of C and the inequality

$$\left\| \frac{D_3 f(2^n x, 2^n y) - D_3 f(-2^n x, -2^n y)}{2 \cdot 8^n} \right\| \leq \frac{\varphi(2^n x, 2^n y) + \varphi(-2^n x, -2^n y)}{2 \cdot 8^n}$$

for all $x, y \in V \setminus \{0\}$ and $n \in \mathbb{N}$, we obtain

$$D_3 C(x, y) = 0,$$
$$C(4x) - 3C(3x) + 3C(2x) - 7C(x) = D_3 C(x, x) = 0$$

for all $x, y \in V \setminus \{0\}$. Using $C(2x) = 8C(x)$, $C(0) = 0$ and $C(-x) = -C(x)$, we get $C(3x) = 27C(x)$ for all $x \in V \setminus \{0\}$. Hence the equation

$$D_3 C(x, y) = 0$$

holds for $x = 0$ or $y = 0$ and C is a cubic function.
On the other hand, from (2.1), we have

$$\begin{aligned} \left\| \frac{f(x) + f(-x)}{2} \right\| &= \frac{\|D_3 f(x, -x) + D_3 f(-x, 2x)\|}{12} \\ &\leq \frac{\varphi(x, -x) + \varphi(-x, 2x)}{12} \end{aligned} \tag{2.4}$$

for all $x \in V \setminus \{0\}$. The inequality (2.2) follows from (2.3), (2.4) and the inequality

$$\|f(x) - C(x)\| \leq \left\| \frac{f(x) - f(-x)}{2} - C(x) \right\| + \left\| \frac{f(x) + f(-x)}{2} \right\|$$

for all $x \in V \setminus \{0\}$. Now, let C' be another cubic function satisfying (2.2). Since $C, C' : V \to Y$ are cubic functions, we get

$$\begin{aligned} \|C(x) - C'(x)\| &= \frac{1}{8^n} \|C(2^n x) - C'(2^n x)\| \\ &\leq \frac{1}{8^n} \|f(2^n x) - C(2^n x)\| + \frac{1}{8^n} \|f(2^n x) - C'(2^n x)\| \\ &\leq \frac{2\Phi(2^n x)}{8^n} \end{aligned}$$

for all $x \in V \setminus \{0\}$ and $n \in \mathbb{N}$. As $n \to \infty$, we may conclude that $C(x) = C'(x)$ for all $x, y \in V$. □

LEMMA 2.3. *Let φ and f be mappings as in Theorem 2.2. Then there exists a unique cubic function $C : V \to Y$ such that*

$$\|f(x) - C(x)\| \leq \Psi(x) \tag{2.5}$$

for all $x \in V \setminus \{0\}$, where Ψ is defined by

$$\Psi(x) = \sum_{i=0}^{\infty} \Big(\frac{\varphi(2^i x, -2^i x)}{2 \cdot 8^{i+1}} + \frac{7\varphi(-2^i x, 2^i x)}{6 \cdot 8^{i+1}} + \frac{\varphi(-2^i x, 2^{i+1} x)}{12 \cdot 8^i} + \frac{\varphi(2^{i+1} x, -2^{i+1} x)}{12 \cdot 8^{i+1}} + \frac{\varphi(-2^{i+1} x, 2^{i+2} x)}{12 \cdot 8^{i+1}} \Big).$$

In particular, C is given by $C(x) = \lim_{n\to\infty} \frac{f(2^n x)}{8^n}$ for all $x \in V$.

Proof. From (2.1), we know

$$\begin{aligned}\Big\| f(x) - \frac{f(2x)}{8} \Big\| &= \Big\| \frac{D_3 f(-x, x)}{16} - \frac{7D_3 f(x, -x)}{48} - \frac{D_3 f(-x, 2x)}{12} \\ &\quad + \frac{D_3 f(2x, -2x)}{96} + \frac{D_3 f(-2x, 4x)}{96} \Big\| \\ &\leq \frac{\varphi(-x, x)}{16} + \frac{7\varphi(x, -x)}{48} + \frac{\varphi(-x, 2x)}{12} \\ &\quad + \frac{\varphi(2x, -2x) + \varphi(-2x, 4x)}{96}\end{aligned}$$

for all $x \in V \setminus \{0\}$. By Lemma 2.1 and the similar method used in Theorem 2.2, we obtain this lemma.

□

THEOREM 2.4. *Let φ, f, Φ and Ψ be as in Theorem 2.2 and Lemma 2.3. Then there exists a unique cubic function $C : V \to Y$ such that*

$$\|f(x) - C(x)\| \leq \min\{\Psi(x), \Phi(x)\}$$

for all $x \in V \setminus \{0\}$. In particular, C is given by $C(x) = \lim_{n\to\infty} \frac{f(2^n x)}{8^n}$ for all $x \in V$.

Proof. Let C be a cubic function satisfying (2.2) and let C' be a cubic function satisfying (2.5). Since $C, C' : V \to Y$ are cubic functions, we get

$$\begin{aligned}\|C(x) - C'(x)\| &= \frac{1}{8^n}\|C(2^n x) - C'(2^n x)\| \\ &\le \frac{1}{8^n}\|f(2^n x) - C(2^n x)\| + \frac{1}{8^n}\|f(2^n x) - C'(2^n x)\| \\ &\le \frac{\Phi(2^n x) + \Psi(2^n x)}{8^n}\end{aligned}$$

for all $x \in V \setminus \{0\}$ and $n \in \mathbb{N}$. As $n \to \infty$, we may conclude that $C(x) = C'(x)$ for all $x, y \in V$. □

Theorem 2.5. *Let $\varphi : V\backslash\{0\} \times V\backslash\{0\} \to [0,\infty)$ be a mapping satisfying the condition*

$$\sum_{i=0}^{\infty} 8^i \varphi(\frac{x}{2^{i+1}}, \frac{y}{2^{i+1}}) < \infty$$

for all $x, y \in V\backslash\{0\}$. If a function $f : V \to Y$ satisfies the inequality (2.1) for all $x, y \in V\backslash\{0\}$, then there exists a unique cubic function $C : V \to Y$ such that

$$\begin{aligned}\|f(x) - C(x)\| \le \sum_{i=0}^{\infty} \frac{8^i}{2}[\varphi(\frac{x}{2^{i+1}}, -\frac{x}{2^{i+1}}) + \varphi(-\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}})] \\ + \frac{\varphi(x,-x) + \varphi(-x, 2x)}{12}\end{aligned}$$

for all $x \in V\backslash\{0\}$. In particular, C is given by

$$C(x) = \lim_{n\to\infty} 8^n f(\frac{x}{2^n})$$

for all $x \in V$.

Proof. The proof is similar to those of Theorem 2.2, Lemma 2.3 and Theorem 2.4. □

Corollary 2.6. *Let $p \neq 3$. If a function $f : V \to Y$ satisfies the inequality*

$$\|D_3 f(x,y)\| \le \varepsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in V\backslash\{0\}$, then there exists a unique cubic function $C : V \to Y$ such that

$$\|f(x) - C(x)\| \le \left(\frac{2}{|8-2^p|} + \frac{2^p+3}{12}\right)\varepsilon\|x\|^p$$

for all $x \in V\backslash\{0\}$.

Corollary 2.7. *If a function $f : V \to Y$ satisfies the inequality*

$$\|D_3 f(x,y)\| \le \varepsilon$$

for all $x, y \in V\backslash\{0\}$, then there exists a unique cubic function $C : V \to Y$ such that

$$\|f(x) - C(x)\| \le \frac{13}{42}\varepsilon$$

for all $x \in V\backslash\{0\}$.

3. Superstability of a cubic functional equation

Theorem 3.1. *Let $\varphi : V \setminus \{0\} \times V \setminus \{0\} \to [0,\infty)$ be a mapping satisfying the condition*

$$\lim_{(s,t)\to(\infty,\infty)} \varphi(sx, ty) = 0 \tag{3.1}$$

for all $x, y \in V\backslash\{0\}$ with $(s,t) \in \mathbb{R}\times\mathbb{R}$. If a function $f : V \to Y$ satisfies the inequality (2.1), then f is a cubic function.

Proof. Note that if $\varphi : V\backslash\{0\}\times V\backslash\{0\} \to [0,\infty)$ satisfies the condition (3.1), then φ satisfies the condition in Theorem 2.2. By Theorem 2.2, there exists a unique cubic function $C : V \to Y$ such that the inequality (2.2) holds for all $x \in V \setminus \{0\}$. Hence the inequalities

$$\begin{aligned} 3\|f(x) - C(x)\| &\le \|D_3 f((k+1)x, -kx) - D_3 C((k+1)x, -kx)\| \\ &\quad + \|(f-C)((2k+3)x)\| + 3\|(f-C)((k+2)x)\| \\ &\quad + \|(f-C)(-kx)\| + 6\|(f-C)((k+1)x)\| \\ &\le \varphi((k+1)x, -kx) + \Phi((2k+3)x) \\ &\quad + 3\Phi((k+2)x) + \Phi(-kx) + 6\Phi((k+1)x), \end{aligned}$$

$$\begin{aligned} 3\|f(0) - C(0)\| &\le \|D_3 f(kx, -kx) - D_3 C(kx, -kx)\| \\ &\quad + \|(f-C)(2kx)\| + 9\|(f-C)(kx)\| + \|(f-C)(-kx)\| \\ &\le \varphi(kx, -kx) + \Phi(2kx) + 9\Phi(kx) + \Phi(-kx) \end{aligned}$$

hold for all $x \in V \setminus \{0\}$ and $k \in \mathbb{N}$, where Φ is defined as in Theorem 2.2. Taking as $k \to \infty$, we conclude $f(x) = C(x)$ for all $x \in V$. □

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