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STABILITY OF FUNCTIONAL EQUATIONS WITH RESPECT TO BOUNDED DISTRIBUTIONS

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Abstract. We consider the Hyers-Ulam type stability of the Cauchy, Jensen, Pexider, Pexider-Jensen differences:

$$C(u) := u \circ A - u \circ P_1 - u \circ P_2, \tag{0.1}$$

$$J(u) := 2u \circ \frac{A}{2} - u \circ P_1 - u \circ P_2, \tag{0.2}$$

$$P(u, v, w) := u \circ A - v \circ P_1 - w \circ P_2, \tag{0.3}$$

$$JP(u, v, w) := 2u \circ \frac{A}{2} - v \circ P_1 - w \circ P_2, \tag{0.4}$$

with respect to bounded distributions.

1. Introduction

The classical Hyers-Ulam stability theorem states that if f is a map from a semi group G to a Banach space B satisfying the inequality

$$|f(x+y) - f(x) - f(y)| \le \varepsilon \quad \text{for all} \quad x,\, y \in G, \tag{1.1}$$

there exists an additive map $A : G \to B$(i.e. $A(x+y) = A(x) + A(y),\ x, y \in G$) such that

$$|f(x) - A(x)| \le \varepsilon \quad \text{for all} \quad x \in G.$$

The motivation of the above stability problems goes back to 1940 when S. M. Ulam proposed the following problem[20]:

Let f be a mapping from a group G_1 to a metric group G_2 with metric $d(\cdot,\cdot)$ such that

$$d(f(xy),\, f(x)f(y)) \le \varepsilon.$$

Then do there exist a group homomorphism h and $\delta_\epsilon > 0$ such that

$$d(f(x),\, h(x)) \le \delta_\epsilon$$

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for all $x \in G_1$*?*

In 1978, Th. M. Rassias[18] firstly generalized the result of Hyers and since then, stability problems of many other functional equations have been investigated [12, 14, 15, 17, 18].

Recently, the above stability problem (1.1) and its related equations

$$|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)| \le \varepsilon, \tag{1.2}$$

$$|f(x+y) - g(x) - h(y)| \le \varepsilon, \tag{1.3}$$

$$|2f\left(\frac{x+y}{2}\right) - g(x) - h(y)| \le \varepsilon, \tag{1.4}$$

have been considered in various spaces of generalized functions such as the space $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions of L. Schwartz, the space $\mathcal{F}'(\mathbb{R}^n)$ of Fourier hyperfunctions(see [3, 4, 5]). For example, a distribution version of the inequality (1.1) has been reformulated for generalized functions u as

$$u \circ A - u \circ P_1 - u \circ P_2 \in L^\infty_\epsilon(\mathbb{R}^{2n})$$

where $u \circ A$ is the pullback of u by $A(x,y) = x+y,\ P_1(x,y) = x,\ P_2(x,y) = y,\ x, y \in \mathbb{R}^n$ and $L^\infty_\epsilon(\mathbb{R}^{2n})$ denotes the space of bounded measurable functions ϕ on $\mathbb{R}^{2n}$ such that $\|\phi\|_{L^\infty} \le \epsilon$. Due to L. Schwartz [19] the space L^∞ of bounded measurable functions has been generalized to the space $\mathcal{D}'_{L^\infty}$ of bounded distributions which is a subspace of tempered distributions and later the space $\mathcal{D}'_{L^\infty}$ was further generalized to the space $\mathcal{A}'_{L^\infty}$ of bounded hyperfunctions which is a subspace of Sato hyperfunctions.

In this paper, we generalize the stability problem (1.1)$\sim$(1.4) to the space of distributions and consider the stability problem when the differences (0.1)$\sim$(0.4) belong to the space of bounded distributions, which is a very natural generalization of the classical Hyers-Ulam stability problem to the spaces of distributions. In the space of bounded distributions, however, the validity of the bound $\epsilon > 0$ is deprived. Thus it is worthwhile to consider the stability problems

$$u \circ A - u \circ P_1 - u \circ P_2 \in \mathcal{D}'_{L^\infty}(\mathbb{R}^{2n}) \text{ [resp. } \mathcal{A}'_{L^\infty}(\mathbb{R}^{2n})], \tag{1.5}$$

$$2u \circ \frac{A}{2} - u \circ P_1 - u \circ P_2 \in \mathcal{D}'_{L^\infty}(\mathbb{R}^{2n}) \text{ [resp. } \mathcal{A}'_{L^\infty}(\mathbb{R}^{2n})], \tag{1.6}$$

$$u \circ A - v \circ P_1 - w \circ P_2 \in \mathcal{D}'_{L^\infty}(\mathbb{R}^{2n}) \text{ [resp. } \mathcal{A}'_{L^\infty}(\mathbb{R}^{2n})], \tag{1.7}$$

$$2u \circ \frac{A}{2} - v \circ P_1 - w \circ P_2 \in \mathcal{D}'_{L^\infty}(\mathbb{R}^{2n}) \text{ [resp. } \mathcal{A}'_{L^\infty}(\mathbb{R}^{2n})]. \tag{1.8}$$

As results we prove that all the solutions of the above stability problems (1.5)∼(1.8) are additive functions up to bounded distributions.

2. The heat kernel method in distributions and hyperfunctions

We first introduce the space $\mathcal{F}'$ of hyperfunctions which is a natural generalization of the space $\mathcal{S}'$ of tempered distributions (see [10, 11] for these spaces). We use the notations: $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, for $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}_0^n$, where $\mathbb{N}_0$ is the set of non-negative integers and $\partial_j = \frac{\partial}{\partial x_j}$.

Definition 2.1. [10] *We denote by $\mathcal{F}$ or $\mathcal{F}(\mathbb{R}^n)$ the space of all infinitely differentiable functions φ in $\mathbb{R}^n$ such that*

$$\|\varphi\|_{h,k} = \sup_{x \in \mathbb{R}^n,\, \alpha,\, \beta \in \mathbb{N}_0^n} \frac{|x^\alpha \partial^\beta \varphi(x)|}{h^{|\alpha|} k^{|\beta|} \alpha! \beta!} < \infty$$

for some h, $k > 0$. We say that $\varphi_j \longrightarrow 0$ as $j \to \infty$ if $||\varphi_j||_{h,k} \longrightarrow 0$ as $j \to \infty$
for some h, k, and denote by $\mathcal{G}'$ the dual space of $\mathcal{G}$ and call its elements Fourier hyperfunctions.

Following Schwartz[19] we introduce the space $\mathcal{D}'_{L^\infty}$ of bounded distributions.

Definition 2.2. *We denote by $\mathcal{D}_{L^1}(\mathbb{R}^n)$ the space of smooth functions on $\mathbb{R}^n$ such that $\partial^\alpha \varphi \in L^1(\mathbb{R}^n)$ for all $\alpha \in \mathbb{N}_0^n$ equipped with the topology defined by the countable family of seminorms*

$$\|\varphi\|_m = \sum_{|\alpha| \le m} \|\partial^\alpha \varphi\|_{L^1}, \quad m \in \mathbb{N}_0.$$

We denote by $\mathcal{D}'_{L^\infty}$ the dual space of $\mathcal{D}_{L^1}$ and call its elements bounded distributions.

Generalizing bounded distributions the space $\mathcal{A}'_{L^\infty}$ of bounded hyperfunctions has been introduced [7] as a subspace of hyperfunctions.

Definition 2.3. *We denote by $\mathcal{A}_{L^1}$ the space of smooth functions on $\mathbb{R}^n$ satisfying*

$$\|\varphi\|_h = \sup_\alpha \frac{\|\partial^\alpha \varphi\|_{L^1}}{h^{|\alpha|} \alpha!} < \infty$$

for some constant $h > 0$. We say that $\varphi_j \to 0$ in $\mathcal{A}_{L^\infty}$ as $j \to \infty$ if there is a positive constant h such that

$$\sup_\alpha \frac{\|\partial^\alpha \varphi_j\|_{L^1}}{h^{|\alpha|}\alpha!} \to 0 \quad \text{as } j \to \infty.$$

We denote by $\mathcal{A}'_{L^\infty}$ the dual space of $\mathcal{A}_{L^1}$.

It is well known that the following topological inclusions hold:

$$\mathcal{F} \hookrightarrow \mathcal{S} \hookrightarrow \mathcal{D}_{L^1}, \qquad D'_{L^\infty} \hookrightarrow \mathcal{S}' \hookrightarrow \mathcal{F}'$$
$$\mathcal{F} \hookrightarrow \mathcal{A}_{L^1} \hookrightarrow \mathcal{D}_{L^1}, \quad \mathcal{D}'_{L^\infty} \hookrightarrow \mathcal{A}'_{L^\infty} \hookrightarrow \mathcal{F}'.$$

It is easy to see that the n-dimensional heat kernel $E_t(x)$ given by

$$E_t(x) = (4\pi t)^{-n/2}\exp(-|x|^2/4t),\ t > 0$$

belongs to the space $\mathcal{F}(\mathbb{R}^n)$ for each $t > 0$.

3. Main theorems

The main tool of the proofs of the results is *the heat kernel method* initiated by T. Matsuzawa [16] which represents the generalized functions in some class as the initial values of solutions of the heat equation with appropriate growth conditions [7, 16]. Making use of the heat kernel method we can convert (1.5)∼(1.8) to the following classical Hyers-Ulam type stability problems; there exist $C > 0,\ N > 0$ [resp. for every $\epsilon > 0$ there exists $C_\epsilon > 0$] such that

$$|f(x+y,t+s) - g(x,t) - h(y,s)| \le C\left(\frac{1}{t} + \frac{1}{s}\right)^N \ [\text{resp. } C_\epsilon e^{\epsilon(1/t+1/s)}\,]$$

for all $x,\ y \in \mathbb{R}^n,\ 0 < t, s < 1$, where $f, g, h : \mathbb{R}^n \times (0,\infty) \to \mathbb{C}$ are the corresponding solutions of the heat equation. Thus we first consider the above stability problem in a more general setting: Let G be a group, S an semigroup divisible by 2 and $\psi : S \times S \to [0,\infty)$.

THEOREM 3.1. *Let $f, g, h : G \times S \to \mathbb{C}$ satisfy*

$$|f(x+y,t+s) - g(x,t) - h(y,s)| \le \psi(t,s) \tag{3.1}$$

for all $x,\ y \in G,\ t, s \in S$. Then exists an additive function $A : G \to \mathbb{C}$ such that

$$|f(x,t) - A(x) - g(0,\frac{t}{2}) - h(0,\frac{t}{2})| \le 3\psi(\frac{t}{2},\frac{t}{2}), \tag{3.2}$$

$$|g(x,t) - A(x) - g(0,t)| \le 4\psi(t,t), \tag{3.3}$$

$$|h(x,t) - A(x) - h(0,t)| \le 4\psi(t,t), \tag{3.4}$$

for all $(x,t) \in G \times S$.

Proof. It follows from (3.1) that

$$|f(x,2t) - g(x,t) - h(0,t)| \le \psi(t,t), \tag{3.5}$$

$$|f(y,2s) - h(y,s) - g(0,s)| \le \psi(s,s) \tag{3.6}$$

for all $(x,t) \in G \times S$. Using triangle inequality with (3.1), (3.5) and (3.6) we have

$$\begin{aligned} &|f(x+y,t+s) - f(x,2t) - f(y,2s) - g(0,s) - h(0,t)| \\ &\qquad\qquad \le \psi(t,s) + \psi(t,t) + \psi(s,s), \end{aligned} \tag{3.7}$$

for all $x,\ y \in G,\ t,s \in S$. Putting $y = x,\ s = t$ in (3.7) we have

$$|f(2x,2t) - 2f(x,2t) - g(0,t) - h(0,t)| \le 3\psi(t,t) \tag{3.8}$$

for all $(x,t) \in G \times S$. Fixing $t > 0$ and using the well known induction argument of Hyers-Ulam[13] with respect to x it is easy to see that the mapping $A(x,t) := \lim_{n\to\infty} 2^{-n} f(2^n x, t)$ satisfies

$$A(x+y,t+s) - A(x,2t) - A(y,2s) = 0, \tag{3.9}$$

and

$$|f(x,2t) - A(x,2t) - g(0,t) - h(0,t)| \le 3\psi(t,t). \tag{3.10}$$

It follows from (3.9) that $A(0,2t) = 0$, $A(x,t+s) = A(x,s+t)$ and $A(x,2t) = A(x,t+s) = A(x,s+t) = A(x,2s)$ for all $x \in G$, $t,s \in S$. Since S is divisible by 2, $A(x,t)$ is independent of $t \in S$. If we denote $A(x,t)$ by $A(x)$, A is an additive function on G. Thus the inequality (3.2) follows. Now (3.3) follows from (3.5) and (3.10), and (3.4) follows from (3.6), (3.10). This completes the proof. □

The following structure theorem for bounded distributions and bounded hyperfunctions will be useful.

Lemma 3.2. [7, 19] (i) *Every* $u \in \mathcal{D}'_{L^\infty}(\mathbb{R}^n)$ *can be expressed as*

$$u = \sum_{|\alpha|\le m} \partial^\alpha f_\alpha \tag{3.11}$$

for some $m \in \mathbb{N}_0$ *where* f_α *are bounded continuous functions on* $\mathbb{R}^n$.

(ii) *Every* $u \in \mathcal{A}'_{L^\infty}(\mathbb{R}^n)$ *can be expressed by*

$$u = \left(\sum_{k=0}^{\infty} a_k \, \Delta^k\right) g + h \tag{3.12}$$

where Δ denotes the Laplacian, g, h are bounded continuous functions on $\mathbb{R}^n$ and $a_k,\ k = 0, 1, 2, \ldots$ satisfy the estimates; for every $L > 0$ there exists $C > 0$ such that

$$|a_k| \leq CL^k/k!^2$$

for all $k = 0, 1, 2, \ldots$.

LEMMA 3.3. (i) *Let $u := u(\xi, \eta) \in \mathcal{D}'_{L^\infty}(\mathbb{R}^{2n})$. Then we have the estimate; there exist positive constants C, N and d such that*

(3.13)
$$|[u*(E_t(\xi)E_s(\eta))](x, y)| \leq C\ (1/t + 1/s)^N, \quad \text{for all } x \in \mathbb{R}^n,\ 0 < t,\ s < 1.$$

(ii) *Let $u = u(\xi, \eta) \in \mathcal{A}'_{L^\infty}(\mathbb{R}^{2n})$. Then we have the estimate; for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that*

(3.14)
$$|[u * (E_t(\xi)E_s(\eta))](x, y)| \leq C_\epsilon\, e^{\epsilon(1/t+1/s)}, \quad \text{for all } x \in \mathbb{R}^n,\ 0 < t,\ s < 1.$$

LEMMA 3.4. [7, 16] *The Gauss transform $U(x, t) = (u * E)(x, t)$ of $u \in \mathcal{F}'(\mathbb{R}^n)$ is a smooth solution of the heat equation $(\Delta - \partial/\partial_t)U = 0$ satisfying :*

(i)*For every $\epsilon > 0$ there exists a positive constant C_ϵ such that*

$$(3.15) \qquad |U(x, t)| \leq C_\epsilon \exp(\epsilon(1/t + |x|)) \quad \text{for all } x \in \mathbb{R}^n,\ t \in (0,\ \delta).$$

(ii) *$U(x, t) \to u$ as $t \to 0^+$ in the sense that for every $\varphi \in \mathcal{D}_{L^1}$,*

$$\langle u, \varphi\rangle = \lim_{t\to 0^+} \int U(x, t)\varphi(x)\, dx.$$

Conversely, every smooth solution $U(x, t)$ of the heat equation satisfying the estimate (3.15) can be uniquely expressed as $U(x, t) = (u * E)(x, t)$ for some $u \in \mathcal{F}'(\mathbb{R}^n)$.

Similarly we can represent bounded distributions and bounded hyperfunctions as initial values of solutions of the heat equation. In these cases only the estimate (3.15) is replaced by the followings, respectively;

There exist constants $C > 0$ and $N \geq 0$ such that

$$(3.16) \qquad |U(x, t)| \leq Ct^{-N} \quad \text{for all } x \in \mathbb{R}^n,\ t \in (0,\ \delta);$$

For every $\epsilon > 0$ there exists a positive constant C_ϵ such that

$$(3.17) \qquad |U(x, t)| \leq C_\epsilon \exp(\epsilon/t) \quad \text{for all } x \in \mathbb{R}^n,\ t \in (0,\ \delta).$$

Now we state and prove the main theorems.

THEOREM 3.5. *Let $u, v, w \in \mathcal{F}'(\mathbb{R}^n)$. Then*

$$u \circ A - v \circ P_1 - w \circ P_2 \in \mathcal{D}'_{L^\infty}(\mathbb{R}^{2n}) \ [resp.\ \mathcal{A}'_{L^\infty}(\mathbb{R}^{2n})] \tag{3.18}$$

if and only if

$$u = c \cdot x + u_0,\ v = c \cdot x + v_0,\ w = c \cdot x + w_0, \tag{3.19}$$

where $c \in \mathbb{C}^n$ and $u_0,\ v_0,\ w_0 \in \mathcal{D}'_{L^\infty}(\mathbb{R}^n)$ [resp. $\mathcal{A}'_{L^\infty}(\mathbb{R}^n)$].

Proof. Convolving in (3.18) the tensor product $E_t(x)E_s(y)$ of n- dimensional heat kernels we have

$$\begin{aligned}[(u \circ A) * (E_t(\xi)E_s(\eta))](x,\, y) &= \langle u_\xi,\, \int E_t(x - \xi + \eta)E_s(y - \eta)\, d\eta\rangle \\ &= \langle u_\xi,\, (E_t * E_s)(x + y - \xi)\rangle \\ &= \langle u_\xi,\, (E_{t+s})(x + y - \xi)\rangle \\ &= U(x + y, t + s).\end{aligned}$$

Similarly we have

$$\begin{aligned}&[(v \circ P_1) * (E_t(\xi)E_s(\eta))](x,\, y) = V(x, t),\\ &[(w \circ P_1) * (E_t(\xi)E_s(\eta))](x,\, y) = W(x, t),\end{aligned}$$

where $U(x, t), V(x, t)$ are the Gauss transforms of $u,\, v,$ respectively. Thus by Lemma 3.3 we have the following stability problem

$$|U(x + y, t + s) - V(x, t) - W(y, s)| \le \psi(t, s), \tag{3.20}$$

for all $x, y \in \mathbb{R}^n,\ t, s > 0$, where $\psi(t, s) = C(1/t+1/s)^N$ [resp. $C_\epsilon e^{\epsilon(1/t+1/s)}$]. Now we apply Theorem 3.1. Since $U,\ V,\ W$ are continuous functions we have $A(x) = c \cdot x$ for some $c \in \mathbb{C}^n$. Thus we have

$$|U(x, t) - c \cdot x| \le \Psi_1(t), \tag{3.21}$$

$$|V(x, t) - c \cdot x| \le \Psi_2(t), \tag{3.22}$$

$$|W(x, t) - c \cdot x| \le \Psi_3(t), \tag{3.23}$$

for all $(x, t) \in G \times S$, where

$$\begin{aligned}\Psi_1(t) &= 3\psi(\frac{t}{2}, \frac{t}{2}) + |V(0, \frac{t}{2})| + |W(0, \frac{t}{2})|,\\ \Psi_2(t) &= 4\psi(t, t) + |V(0, t)|,\\ \Psi_3(t) &= 4\psi(t, t) + |W(0, t)|.\end{aligned}$$

Now we consider the growth of $\Psi_j(t),\ j = 1, 2, 3,$ as $t \to 0^+$. Letting $x = y = 0,\ s = 1$ in (3.20) and using the triangle inequality we have for

$0 < t < 1$,

$$|V(0,t)| \le \psi(t,1) + |U(0,t+1)| + W(0,1)|,$$
$$\le \psi(t,1) + M_1.$$

for some $M_1 > 0$, since U is continuous. Similarly we have

$$|W(0,t)| \le \psi(1,t) + |U(0,t+1)| + V(0,1)|,$$
$$\le \psi(1,t) + M_2$$

for some $M_2 > 0$.

Thus for the case when $u \circ A - v \circ P_1 - w \circ P_2 \in \mathcal{D}'_{L^\infty}(\mathbb{R}^{2n})$; there exist C, $N > 0$ such that

$$\Psi_j(t) \le Ct^{-N}, \quad \text{for all} \quad 0 < t < 1, \ j = 1, 2, 3,$$

and for the case when $u \circ A - v \circ P_1 - w \circ P_2 \in \mathcal{A}'_{L^\infty}(\mathbb{R}^{2n})$; for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$\Psi_j(t) \le C_\epsilon \exp(\epsilon/t) \quad \text{for all} \quad 0 < t < 1, \ j = 1, 2, 3.$$

Note that $U(x,t) - c \cdot x$, $V(x,t) - c \cdot x$, $W(x,t) - c \cdot x$ are the Gauss transforms of $u - c \cdot x$, $v - c \cdot x$, $w - c \cdot x$, respectively. Now applying Lemma 3.4 for the inequalities (3.21), (3.22) and (3.23) we have $u - c \cdot x$, $v - c \cdot x$, $w - c \cdot x \in \mathcal{D}'_{L^\infty}(\mathbb{R}^{2n})$ [resp. $\mathcal{A}'_{L^\infty}(\mathbb{R}^{2n})$]. This completes the proof.

□

As a direct consequence of Theorem 3.5 we have the following.

Corollary 3.6. *Let $u \in \mathcal{F}'(\mathbb{R}^n)$. Then*

$$u \circ A - u \circ P_1 - u \circ P_2 \in \mathcal{D}'_{L^\infty}(\mathbb{R}^{2n}) \ [\textit{resp. } \mathcal{A}'_{L^\infty}(\mathbb{R}^{2n})] \tag{3.24}$$

if and only if

$$u = c \cdot x + u_0, \tag{3.25}$$

where $c \in \mathbb{C}^n$ and $u_0 \in \mathcal{D}'_{L^\infty}(\mathbb{R}^n)$ [resp. $\mathcal{A}'_{L^\infty}(\mathbb{R}^n)$].

Theorem 3.7. *Let $u, v, w \in \mathcal{F}'(\mathbb{R}^n)$. Then*

$$2u \circ \frac{A}{2} - v \circ P_1 - w \circ P_2 \in \mathcal{D}'_{L^\infty}(\mathbb{R}^{2n}) \ [\textit{resp. } \mathcal{A}'_{L^\infty}(\mathbb{R}^{2n})] \tag{3.26}$$

if and only if

$$u = c \cdot x + u_0, \ v = c \cdot x + v_0, \ w = c \cdot x + w_0, \tag{3.27}$$

where $c \in \mathbb{C}^n$ and $u_0, \ v_0, \ w_0 \in \mathcal{D}'_{L^\infty}(\mathbb{R}^n)$ [resp. $\mathcal{A}'_{L^\infty}(\mathbb{R}^n)$].

Proof. Convolving in (3.26) the tensor product $E_t(x)E_s(y)$ of n- dimensional heat kernels we have following stability problem

$$\left|2U\left(\frac{x+y}{2}, \frac{t+s}{4}\right) - V(x,t) - W(y,s)\right| \le \psi(t,s), \tag{3.28}$$

for all $x, y \in \mathbb{R}^n$, $t, s > 0$, where U, V, W are the Gauss transforms of u, v, w, respectively. Letting $U_1(x,t) = U(x/2, t/4)$ and applying the proof of Theorem 3.5 we have

$$|U(x,t) - c \cdot x| \le \frac{1}{2}\Psi_1(4t), \tag{3.29}$$

$$|V(x,t) - c \cdot x| \le \Psi_2(t), \tag{3.30}$$

$$|W(x,t) - c \cdot x| \le \Psi_3(t), \tag{3.31}$$

for all $(x,t) \in G \times S$, where $\Psi_j(t)$, $j = 1, 2, 3$ are as in Theorem 3.5. Following the same approach as in Theorem 3.5 we have the result. □

As a direct consequence of Theorem 3.7 we have the following.

Corollary 3.8. *Let $u \in \mathcal{F}'(\mathbb{R}^n)$. Then*

$$2u \circ \frac{A}{2} - u \circ P_1 - u \circ P_2 \in \mathcal{D}'_{L^\infty}(\mathbb{R}^{2n}) \ [resp.\ \mathcal{A}'_{L^\infty}(\mathbb{R}^{2n})] \tag{3.32}$$

if and only if

$$u = c \cdot x + u_0, \tag{3.33}$$

where $c \in \mathbb{C}^n$ and $u_0 \in \mathcal{D}'_{L^\infty}(\mathbb{R}^n)$ [resp. $\mathcal{A}'_{L^\infty}(\mathbb{R}^n)$].

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