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EXISTENCE OF THE POSITIVE SOLUTION FOR THE NONLINEAR SYSTEM OF SUSPENSION BRIDGE EQUATIONS

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Abstract. We prove the existence of the positive solution for the nonlinear system of suspension bridge equations with Dirichlet boundary condition and periodic condition

$$\begin{cases} u_{tt} + u_{xxxx} & +av^+ = 1 + \epsilon_1 h_1(x,t) \quad \text{in } (-\frac{\pi}{2}, \frac{\pi}{2}) \times R, \\ v_{tt} + v_{xxxx} & +bu^+ = 1 + \epsilon_2 h_2(x,t) \quad \text{in } (-\frac{\pi}{2}, \frac{\pi}{2}) \times R, \end{cases}$$

where $u^+ = \max\{u, 0\}$, ϵ_1, ϵ_2 are small numbers and $h_1(x,t)$, $h_2(x,t)$ are bounded , π-periodic in t and even in x and t and $\|h_1\| = \|h_2\| = 1$.

1. Introduction

In this paper we investigate the existence of the solutions for the nonlinear system of suspension bridge equations with Dirichlet boundary condition and periodic condition

$$\begin{cases} u_{tt} + u_{xxxx} & +av^+ = 1 + \epsilon_1 h_1(x,t) \quad \text{in } (-\frac{\pi}{2}, \frac{\pi}{2}) \times R, \\ v_{tt} + v_{xxxx} & +bu^+ = 1 + \epsilon_2 h_2(x,t) \quad \text{in } (-\frac{\pi}{2}, \frac{\pi}{2}) \times R, \\ u(\pm\frac{\pi}{2}, t) & = u_{xx}(\pm\frac{\pi}{2}, t) = v(\pm\frac{\pi}{2}, t) = v_{xx}(\pm\frac{\pi}{2}, t) = 0, \\ u(x, t+\pi) & = u(x,t) = u(-x,t) = u(x,-t), \\ v(x, t+\pi) & = v(x,t) = v(-x,t) = v(x,-t), \end{cases} \tag{1.1}$$

where $u^+ = \max\{u, 0\}$, ϵ_1, ϵ_2 are small numbers and $h_1(x,t)$, $h_2(x,t)$ are bounded, π-periodic in t, even in x and t and $\|h_1\| = \|h_2\| = 1$. The nonlinear system (1.1) of suspension bridge equations with Dirichlet boundary condition is considered as a model of the cross of the two nonlinear oscillations in differential equation. For the case of the single

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suspension bridge equation McKenna and Walter ([6]), Choi and Jung ([3], [4] and [5]) etc., investigate the multiplicity of the solutions via the degree theory or the critical point theory or the variational reduction method. In this paper we improve the multiplicity results of the single suspension bridge equation to the case of the system of the nonlinear suspension bridge equations. The system (1.1) can be rewritten by

$$\begin{cases} U_{tt}+U_{xxxx} & +AU^{+}=\binom{1+\epsilon_1 h_1(x,t)}{1+\epsilon_2 h_2(x,t)}, \\ U(\pm\frac{\pi}{2},t) & =U_{xx}(\pm\frac{\pi}{2},t)=\binom{0}{0}, \\ U(x,t+\pi) & =U(x,t)=U(-x,t)=U(x,-t), \end{cases} \tag{1.2}$$

where $U=\binom{u}{v}$, $U^{+}=\binom{u^{+}}{v^{+}}$, $U_{tt}+U_{xxxx}=\binom{u_{tt}+u_{xxxx}}{v_{tt}+v_{xxxx}}$, $A=\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}\in M_{2\times 2}(R)$. Let us define the Hilbert space spanned by eigenfunctions as follows:
The eigenvalue problem for $u(x,t)$,

$$u_{tt}+u_{xxxx}=\lambda u \qquad \text{in } \ (-\frac{\pi}{2},\frac{\pi}{2})\times R,$$

$$u(\pm\frac{\pi}{2},t)=u_{xx}(\pm\frac{\pi}{2},t)=0,$$

$$u(x,t+\pi)=u(x,t)=u(-x,t)=u(x,-t)$$

has infinitely many eigenvalues

$$\lambda_{mn}=(2n+1)^4-4m^2 \qquad (m,n=0,1,2,\ldots)$$

and corresponding normalized eigenfunctions ϕ_{mn} $(m,n\geq 0)$ given by

$$\phi_{0n}=\frac{\sqrt{2}}{\pi}\cos(2n+1)x \qquad \text{for } \ n\geq 0,$$

$$\phi_{mn}=\frac{2}{\pi}\cos 2mt\cdot\cos(2n+1)x \qquad \text{for } \ m>0, n\geq 0.$$

We can check easily that the eigenvalues in the interval (-19,45) are given by

$$\lambda_{20}=-15<\lambda_{10}=-3<\lambda_{00}=1<\lambda_{41}=17$$

Let Q be the square $[-\frac{\pi}{2},\frac{\pi}{2}]\times[-\frac{\pi}{2},\frac{\pi}{2}]$ and H_0 the Hilbert space defined by

$$H_0=\{u\in L^2(Q)|\ u \text{ is even in } x \text{ and } t\}.$$

The set of functions $\{\phi_{mn}\}$ is an orthonormal basis in H_0. Let us denote an element u, in H_0, by

$$u=\sum h_{mn}\phi_{mn}.$$

We define a Hilbert space H as follows

$$H = \{u \in H_0 : \ \sum |\lambda_{mn}| h_{mn}^2 < \infty\}.$$

Then this space is a Banach space with norm

$$\|u\|^2 = [\sum |\lambda_{mn}| h_{mn}^2]^{\frac{1}{2}}.$$

Let us set $E = H \times H$. We endow the Hilbert E the norm

$$\|(u,v)\|_E^2 = \|u\|^2 + \|v\|^2 \qquad \forall (u,v) \in E.$$

We are looking for the weak solutions of (1.1) in E, that is, (u,v) satisfying the equation

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (u_{tt} + u_{xxxx}) z + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (v_{tt} + v_{xxxx}) w + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (AU^+, (z,w))$$

$$- \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + \epsilon_1 h_1(x,t)) z - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + \epsilon_2 h_2(x,t)) w = 0, \qquad \forall (z,w) \in E,$$

where $u = \sum c_{mn}\phi_{mn}$, $v = \sum d_{mn}\phi_{mn}$ with $u_{tt} + u_{xxxx} = \sum \lambda_{mn} c_{mn}\phi_{mn} \in H$, $v_{tt} + v_{xxxx} = \sum \lambda_{mn} d_{mn}\phi_{mn} \in H$ i.e., with $\sum c_{mn}^2 \lambda_{mn}^2 < \infty$, $\sum d_{mn}^2 \lambda_{mn}^2 < \infty$, which implies u, $v \in H$. Now we state the main result :

THEOREM 1.1. *(Existence of a positive solution)*
Assume that

$$\lambda_{mn}^2 - ab \neq 0, \qquad \text{for all } \ m, n \tag{1.3}$$

$$a < 0, \qquad b < 0 \qquad \text{and} \qquad 1 - ab > 0. \tag{1.4}$$

Then, for each $h_1(x,t)$, $h_2(x,t) \in H$ *with* $\|h_1(x,t)\| = 1$, $\|h_2(x,t)\| = 1$, *there exist small numbers* ϵ_1 *and* ϵ_2 *such that system (1.1) has a positive solution.*

2. Proof of Theorem 1.1

We have some properties. Since $|\lambda_{mn}| \geq 1$ for all m, n, we have that

LEMMA 2.1. *(i)* $\|u\| \geq \|u\|_{L^2(Q)}$, *where* $\|u\|_{L^2(Q)}$ *denotes the* L^2 *norm of* u.
(ii) $\|u\| = 0$ *if and only if* $\|u\|_{L^2(Q)} = 0$.
(iii) $u_{tt} + u_{xxxx} \in H$ *implies* $u \in H$.

LEMMA 2.2. *Suppose that* c *is not an eigenvalue of* L, $Lu = u_{tt} + u_{xxxx}$, *and let* $f \in H_0$. *Then we have* $(L-c)^{-1} f \in H$.

Proof. When n is fixed, we define

$$\lambda_n^+ = \inf_m\{\lambda_{mn} : \ \lambda_{mn} > 0\} = 8n^2 + 8n + 1,$$

$$\lambda_n^- = \sup_m\{\lambda_{mn} : \lambda_{mn} < 0\} = -8n^2 - 8n - 3.$$

We see that $\lambda_n^+ \to +\infty$ and $\lambda_n^- \to -\infty$ as $n \to \infty$. Hence the number of elements in the set $\{\lambda_{mn} : \ |\lambda_{mn}| < |c|\}$ is finite, where λ_{mn} is an eigenvalue of L. Let

$$f = \sum h_{mn}\phi_{mn}.$$

Then

$$(L-c)^{-1}f = \sum \frac{1}{\lambda_{mn} - c} h_{mn}\phi_{mn}.$$

Hence we have the inequality

$$\|(L-c)^{-1}f\| = \sum |\lambda_{mn}| \frac{1}{(\lambda_{mn} - c)^2} h_{mn}^2 \leq C \sum h_{mn}^2$$

for some C, which means that

$$\|(L-c)^{-1}f\| \leq C_1 \|f\|_{L^2(Q)}, \qquad C_1 = \sqrt{C}.$$

□

Lemma 2.3. *For all a, b with $a < 0$, $b < 0$ and $1 - ab > 0$, the system of the boundary value problems*

$$\begin{cases} y^{(4)} + az &= 1 \quad \text{in } (-\frac{\pi}{2}, \frac{\pi}{2}), \\ z^{(4)} + by &= 1 \quad \text{in } (-\frac{\pi}{2}, \frac{\pi}{2}) \\ y(\pm\frac{\pi}{2}) &= y''(\pm\frac{\pi}{2}) = z(\pm\frac{\pi}{2}) = z''(\pm\frac{\pi}{2}) = 0 \\ y(x) &= y(-x), \qquad z(x) = z(-x) \end{cases} \tag{2.1}$$

has a unique positive solution $(y_*, z_*) \in E$.

Proof. Putting the first equation in (2.1)

$$z = \frac{1}{a}(-y^{(4)} + 1)$$

into the second equation in (2.1), we have

$$(D^{(4)} - \sqrt{ab})(D^{(4)} + \sqrt{ab})y = -a \qquad \text{in } (-\frac{\pi}{2}, \frac{\pi}{2}). \tag{2.2}$$

$$y(\pm\frac{\pi}{2}) = y''(\pm\frac{\pi}{2}) = z(\pm\frac{\pi}{2}) = z''(\pm\frac{\pi}{2}) = 0$$

$$y(x) = y(-x), \qquad z(x) = z(-x).$$

Since $a < 0$, $\sqrt{ab} < 1$ and $-\sqrt{ab} < 1$, by Lemma 4 in [8], (2.2) has a unique positive solution (y_*, z_*). □

LEMMA 2.4. *Assume that conditions (1.3) and (1.4) hold. Then the system*

$$\begin{cases} u_{tt} + u_{xxxx} + av^+ = 1 & \text{in } (-\frac{\pi}{2}, \frac{\pi}{2}), \\ v_{tt} + v_{xxxx} + bu^+ = 1 & \text{in } (-\frac{\pi}{2}, \frac{\pi}{2}), \\ u(\pm\frac{\pi}{2}, t) = u_{xx}(\pm\frac{\pi}{2}, t) = v(\pm\frac{\pi}{2}, t) = v_{xx}(\pm\frac{\pi}{2}, t) = 0, \\ u(x, t+\pi) = u(x,t) = u(-x,t) = u(x,-t), \\ v(x, t+\pi) = v(x,t) = v(-x,t) = v(x,-t) \end{cases} \tag{2.3}$$

has a positive solution $(y_*, z_*) \in E$.

Proof. The solution (y_*, z_*) of (2.1) is positive and hence become a positive solution of (2.2). □

LEMMA 2.5. *Assume that the conditions (1.3) and (1.4) hold. Then the system*

$$U_{tt} + U_{xxxx} + AU = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad U = \begin{pmatrix} u \\ v \end{pmatrix} \in E, \tag{2.4}$$

$$U(\pm\frac{\pi}{2}, t) = U_{xx}(\pm\frac{\pi}{2}, t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$U(x, t+\pi) = U(x,t) = U(-x,t) = U(x,-t)$$

has only a trivial solution $U(x,t) = \binom{0}{0}$.

Proof. We assume that there exists a nontrivial solution $U = (u, v) \in E$ of (2.4) of the form $u = \phi_{mn}$ and $v = \phi_{m'n'}$. The equation

$$L\begin{pmatrix} \phi_{mn} \\ \phi_{m'n'} \end{pmatrix} + A\begin{pmatrix} \phi_{mn} \\ \phi_{m'n'} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is equivalent to the equation

$$\begin{pmatrix} \lambda_{mn}\phi_{mn} \\ \lambda_{m'n'}\phi_{m'n'} \end{pmatrix} + \begin{pmatrix} a\phi_{m'n'} \\ b\phi_{mn} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus when $mn \neq m'n'$, we have a contradiction since ϕ_{mn} and $\phi_{m'n'}$ are linearly independent. When $mn = m'n'$, we have $\lambda_{mn} + a = 0$ and $\lambda_{mn} + b = 0$, which means that $\lambda_{mn}^2 - ab = 0$. These contradict to the assumption (1.3). □

LEMMA 2.6. *Let* ϵ_1 *and* ϵ_2 *be any given numbers. Assume that conditions (1.3) and (1.4) hold and* $h_1(x,t)$, $h_2(x,t) \in H$ *with* $\|h_1\| = 1$,

$\|h_2\| = 1$. *Then the system*

$$\begin{cases} u_{tt} + u_{xxxx} + bv = \epsilon_1 h_1(x,t), \\ v_{tt} + v_{xxxx} + au = \epsilon_2 h_2(x,t), \\ u(\pm\frac{\pi}{2},t) = u_{xx}(\pm\frac{\pi}{2},t) = v(\pm\frac{\pi}{2},t) = v_{xx}(\pm\frac{\pi}{2},t) = 0, \\ u(x,t+\pi) = u(x,t) = u(-x,t) = u(x,-t), \\ v(x,t+\pi) = v(x,t) = v(-x,t) = v(x,-t) \end{cases} \tag{2.5}$$

has a unique solution $(u_{\epsilon_1\epsilon_2}, v_{\epsilon_1\epsilon_2}) \in E$.

Proof. Let $\delta > 0$ and $\delta > \max\{-a,-b\}$. Let us consider the modified system

$$\begin{cases} u_{tt} + u_{xxxx} & +av + \lambda_{00}u + \delta u = \epsilon_1 h_1(x,t), \\ v_{tt} + v_{xxxx} & +bu + \lambda_{00}v + \delta v = \epsilon_2 h_2(x,t), \\ u(\pm\frac{\pi}{2},t) & = u_{xx}(\pm\frac{\pi}{2},t) = v(\pm\frac{\pi}{2},t) = v_{xx}(\pm\frac{\pi}{2},t) = 0, \\ u(x,t+\pi) & = u(x,t) = u(-x,t) = u(x,-t), \\ v(x,t+\pi) & = v(x,t) = v(-x,t) = v(x,-t). \end{cases} \tag{2.6}$$

Let us set

$$L_\delta U = U_{tt} + U_{xxxx} + AU + \lambda_{00}U + \delta U, \qquad U = \begin{pmatrix} u \\ v \end{pmatrix}.$$

The system (2.6) is invertible. Thus there exists an inverse operator $L_\delta^{-1} : L^2(Q) \times L^2(Q) \to E$ which is a linear and compact operator such that $(u,v) = L_\delta^{-1}(\epsilon_1 h_1(x,t), \epsilon_2 h_2(x,t))$. Thus we have that if (u,v) is a solution of (2.5) if and only if

$$(u,v) = L_\delta^{-1}((\epsilon_1 h_1(x,t), \epsilon_2 h_2(x,t)) + \lambda_{00}(u,v) + \delta(u,v)). \tag{2.7}$$

Thus we have

$$(I - (\lambda_{00} + \delta)L_\delta^{-1})((\epsilon_1 h_1(x,t), \epsilon_2 h_2(x,t)) + \lambda_{00}(u,v) + \delta(u,v))$$

$$= (\epsilon_1 h_1(x,t), \epsilon_2 h_2(x,t)). \tag{2.8}$$

By conditions (1.3) and (1.4), $\frac{1}{\lambda_{00}+\delta} \notin \sigma(L_\delta^{-1})$. Since L_δ^{-1} is a compact operator, the system (2.8) has a unique solution, thus the system (2.5) has a unique solution. □

PROOF OF THEOREM 1.1

By Lemma 2.4 and Lemma 2.6, $(y_* + u_{\epsilon_1\epsilon_2}, z_* + v_{\epsilon_1\epsilon_2})$ is a solution of the system

$$\begin{cases} u_{tt} + u_{xxxx} + av = 1 + \epsilon_1 h_1(x,t), \\ v_{tt} + v_{xxxx} + bu = 1 + \epsilon_2 h_2(x,t), \\ u(\pm\frac{\pi}{2}, t) = u_{xx}(\pm\frac{\pi}{2}) = v(\pm\frac{\pi}{2}, t) = v_{xx}(\pm\frac{\pi}{2}) = 0, \\ u(x, t+\pi) = u(x,t) = u(-x,t) = u(x,-t), \\ v(x, t+\pi) = v(x,t) = v(-x,t) = v(x,-t), \end{cases} \tag{2.9}$$

where $y_* > 0$ and $z_* > 0$. Therefore we can choose small numbers ϵ_1 and ϵ_2 such that $y_* + u_{\epsilon_1\epsilon_2} > 0$ and $z_* + v_{\epsilon_1\epsilon_2} > 0$.

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