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STRONG CONVERGENCE THEOREM OF FIXED POINT FOR RELATIVELY ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

Xiaolong Qin*, Shin Min Kang**, and Sun Young Cho***

Abstract. In this paper, we prove strong convergence theorems of Halpern iteration for relatively asymptotically nonexpansive mappings in the framework of Banach spaces. Our results extend and improve the recent ones announced by [C. Martinez-Yanes, H. K. Xu, Strong convergence of the CQ method for fixed point iteration processes, Nonlinear Anal. 64 (2006), 2400–2411], [X. Qin, Y. Su, Strong convergence theorem for relatively nonexpansive mappings in a Banach space, Nonlinear Anal. 67 (2007), 1958–1965] and many others.

1. Introduction and preliminaries

Let E be a real Banach space, C a nonempty closed convex subset of E, and $T : C \to C$ a mapping. A point $x \in C$ is a *fixed point* of T provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T; that is, $F(T) = \{x \in C : Tx = x\}$.

A iterative process is often used to approximate a fixed point of a nonexpansive mapping, which is introduced by Halpern [9] and is defined as follows: Take an initial guess $x_0 \in C$ arbitrarily and define $\{x_n\}$ recursively by

$$x_{n+1} = t_n x_0 + (1 - t_n)Tx_n, \quad n \geq 0, \tag{1.1}$$

where $\{t_n\}_{n=1}^{\infty}$ is a sequence in the interval [0, 1].

In 1967, Halpern [9] first introduced this iteration scheme (1.1). He pointed out that the conditions $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$ are necessary in the sense that, if the iteration scheme (1.1) converges to a fixed point of T, then these conditions must be satisfied. Ten years

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later, Lions [11] investigated the general case in Hilbert space under the conditions

$$(C1): \lim_{n\to\infty} \alpha_n = 0,\ (C2): \sum_{n=1}^{\infty} \alpha_n = \infty \text{ and } (C3): \lim_{n\to\infty} \frac{\alpha_n - \alpha_{n+1}}{\alpha_{n+1}^2} = 0$$

on the parameters. However, Lions' conditions on the parameters were more restrictive and did not include the natural candidate $\{\alpha_n\} = \frac{1}{n}$. In 1980, Reich [18] studied the iteration scheme (1.1) in the case when E is uniformly smooth and $\alpha_n = n^{-\delta}$ with $0 < \delta < 1$.

In 1992, Wittmann [22] studied the iteration scheme (1.1) in the case when E is a Hilbert space and $\{\alpha_n\}$ satisfies

$$(C1): \lim_{n\to\infty} \alpha_n = 0,\ (C2): \sum_{n=1}^{\infty} \alpha_n = \infty \text{ and } (C4): \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$$

In 1994, Reich [19] obtained a strong convergence of the iteration (1.1) with two necessary and decreasing conditions on parameters for convergence in the case when E is uniformly smooth with a weakly continuous duality mapping. It is well know that process (1.1) is widely believed to have slow convergence because the restriction of condition $C2$. Moreover, Halpern [9] proved that the condition $(C1)$ and $(C2)$ are indeed necessary in the sense that if process (1.1) is strongly convergent for all closed convex subsets C of a Hilbert space H and all nonexpansive mappings T on C, then the sequence $\{\alpha_n\}$ must satisfy the conditions $(C1)$ and $(C2)$. (However, it is unknown whether these two conditions are also sufficient; see [23] for more detail.) Thus to improve the rate of convergence of process (1.1), one cannot rely only on the process itself.

Recently, Martinez-Yanes and Xu [14] adapted the iteration (1.1) in Hilbert spaces as follows.

(1.2)

$$\begin{cases} x_0 \in C \quad \textit{arbitrarily chosen}, \\ y_n = \alpha_n x_0 + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + \alpha_n(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle)\}, \\ Q_n = \{z \in C : \langle x_0 - x_n, x_n - z\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0. \end{cases}$$

To be more precisely, they proved the following theorem.

Theorem 1.1. *Let H be a real Hilbert space, C a closed convex subset of H and $T: C \to C$ a nonexpansive mapping such that $F(T) \neq$*

$\emptyset$. *Assume that* $\{\alpha_n\} \subset (0,1)$ *is such that* $\lim_{n\to\infty} \alpha_n = 0$. *Then the sequence* $\{x_n\}$ *defined by* (1.2) *converges strongly to* $P_{F(T)}x_0$.

Very recently, Qin and Su [15] modified (1.1) in the framework of Banach spaces to have strong convergence theorem for relatively nonexpansive mappings. More precisely, they proved the following theorem.

THEOREM 1.2. *Let* E *be a uniformly convex and uniformly smooth Banach space, let* C *be a nonempty closed convex subset of* E, *let* $T : C \to C$ *be a relatively nonexpansive mapping. Assume that* $\{\alpha_n\}_{n=0}^{\infty}$ *is a sequence in* $(0,1)$ *such that* $\lim_{n\to\infty} \alpha_n = 0$. *Define a sequence* $\{x_n\}$ *in* C *by the following algorithm*

$$\begin{cases} x_0 \in C \quad \textit{chosen arbitrarily}, \\ y_n = j^{-1}(\alpha_n j x_0 + (1-\alpha_n) jTx_n), \\ C_n = \{v \in C : \phi(v, y_n) \le \alpha_n \phi(v, x_0) + (1-\alpha_n)\phi(v, x_n), \\ Q_n = \{v \in C : \langle Jx_0 - Jx_n, x_n - v\rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \end{cases}$$

where j *is the single-valued duality mapping on* E. *If* $F(T)$ *is nonempty, then* $\{x_n\}$ *converges to* $\Pi_{F(T)}x_0$.

The purpose of this paper is in the framework of Banach spaces to obtain a strong convergence theorems for relatively asymptotically nonexpansive mappings which was first introduced by Su and Qin [20]. We obtain strong convergence theorems only under the condition $(C1)$. Our results also improve Martinez-Yanes and Xu [14] from Hilbert spaces to Banach spaces and also extend Qin and Su [15] from relatively nonexpansive mappings to relatively asymptotically nonexpansive mappings.

Let E be a Banach space with dual E^*. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \{f^* \in E^* : \langle x, f^*\rangle = \|x\|^2 = \|f^*\|^2\}, \tag{1.3}$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It well known that if E^* is uniformly convex, then J is uniformly continuous on bounded subsets of E. We shall denote the single-valued duality mapping by j.

As we all known that if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. In this connection, Alber [1] recently introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection in Hilbert spaces.

Next, we assume that E is a smooth Banach space. Consider the functional defined as [1, 2] by

$$\phi(x,y)=\|x\|^2-2\langle x,j(y)\rangle+\|y\|^2,\quad \forall x,y\in E. \tag{1.4}$$

Observe that, in a Hilbert space H, (1.4) reduces to

$$\phi(x,y)=\|x-y\|^2,\quad \forall x,y\in H.$$

The generalized projection $\Pi_C : E\to C$ is a map that assigns to an arbitrary point $x\in E$ the minimum point of the functional $\phi(x,y)$, that is, $\Pi_C x=\bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x},x)=\min_{y\in C}\phi(y,x). \tag{1.5}$$

existence and uniqueness of the operator Π_C follow from the properties of the functional $\phi(x,y)$ and strict monotonicity of the mapping J (see, for example, [3]). In Hilbert spaces, $\Pi_C=P_C$. It is obvious from the definition of function ϕ that

$$(\|y\|-\|x\|)^2\le\phi(y,x)\le(\|y\|+\|x\|)^2,\quad \forall x,y\in E. \tag{1.6}$$

Remark 1.1. If E is a reflexsive strictly convex and smooth Banach space, then for $x,y\in E$, $\phi(x,y)=0$ if and only if $x=y$. It is sufficient to show that if $\phi(x,y)=0$ then $x=y$. From (1.6), we have $\|x\|=\|y\|$. This implies $\langle x,jy\rangle=\|x\|^2=\|jy\|^2$. From the definitions of j, we have $jx=jy$. That is, $x=y$; see [6, 21] for more details.

Let C be a closed convex subset of E, and let T be a mapping from C into itself. A point of p in C is said to be an *asymptotic fixed point* of T [16] if C contains a sequence $\{x_n\}$ which converges weakly to p such that the strong $\lim_{n\to\infty}\|Tx_n-x_n\|=0$. The set of asymptotic fixed points of T will be denoted by $\widehat{F}(T)$. A mapping T from C into itself is called *nonexpansive* if $\|Tx-Ty\|\le\|x-y\|$ for all $x,y\in C$ and *relatively nonexpansive* [4, 5, 7] if $\widehat{F}(T)=F(T)$ and $\phi(p,Tx)\le\phi(p,x)$ for all $x\in C$ and $p\in F(T)$. A mapping T from C into itself is called *asymptotically nonexpansive* [8] if there exists a sequence $\{k_n\}$ of positive real numbers with $\lim_{n\to\infty}k_n=1$ and such that $\|T^nx-T^ny\|\le k_n\|x-y\|$ for all $n\ge1$ and $x,y\in C$ and *relatively asymptotically nonexpansive* if $\widehat{F}(T)=F(T)$ and $\phi(p,T^nx)\le k_n^2\phi(p,x)$ for all $x\in C$ and $p\in F(T)$.

A Banach space E is said to be *strictly convex* if $\|\frac{x+y}{2}\|<1$ for all $x,y\in E$ with $\|x\|=\|y\|=1$ and $x\ne y$. It is said to be *uniformly convex* if $\lim_{n\to\infty}\|x_n-y_n\|=0$ for any two sequences $\{x_n\}$, $\{y_n\}$ in E such that $\|x_n\|=\|y_n\|=1$ and $\lim_{n\to\infty}\|\frac{x_n+y_n}{2}\|=1$. Let $U=\{x\in E:\|x\|=1\}$ be the unit sphere of E. Then the Banach space E is said to be *smooth* provided $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for each $x, y \in U$. It is also said to be *uniformly smooth* if the limit is attained uniformly for $x, y \in E$. It is well known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E. A Banach space is said to have the Kadec-Klee property if a sequence $\{x_n\} \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, then $x_n \to x$. It is known that if E is uniformly convex, then E has the Kadec-Klee property; see [6, 21] for more details.

We need the following lemmas for the proof of our main results.

LEMMA 1.3 ([10]). *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$, $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $x_n - y_n \to 0$.*

LEMMA 1.4 ([1]). *Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then, $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, jx - jx_0\rangle \geq 0, \quad \forall y \in C.$$

LEMMA 1.5 ([1]). *Let E be a reflexive, strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C.$$

LEMMA 1.6 ([12]). *Let T be an asymptotically nonexpansive mapping defined on a bounded closed convex subset C of a Hilbert space H. Assume that $\{x_n\}$ is a sequence in C with the properties $x_n \rightharpoonup p$ and $Tx_n - x_n \to 0$. Then $p \in F(T)$.*

LEMMA 1.7 ([20]). *Let E be a uniformly convex and uniformly smooth Banach space, let C be a closed convex subset of E, and let T be a relatively asymptotically nonexpansive mapping from C into itself. If T is continuous, then $F(T)$ is closed and convex.*

2. Main results

THEOREM 2.1. *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty bounded closed convex subset of E and let $T : C \to C$ be a relatively asymptotically nonexpansive mapping with sequence $\{k_n\}$ such that $k_n \to 1$ as $n \to \infty$ and $F(T) \neq \emptyset$. Assume that $\{\alpha_n\}_{n=0}^{\infty}$ is sequence in $[0, 1]$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $k_n^2(1 -$*

$\alpha_n) \le 1$ for all $n \ge 0$. Define a sequence $\{x_n\}$ in C by the following algorithm:

$$
\begin{cases} x_0 \in C \quad \textit{arbitrarily chosen}, \\ y_n = j^{-1}(\alpha_n jx_0 + (1-\alpha_n)jT^n x_n), \\ C_n = \{v \in C : \phi(v, y_n) \le \phi(v, x_n) + \alpha_n M\}, \\ Q_n = \{v \in C : \langle jx_0 - jx_n, x_n - v\rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \end{cases} \tag{2.1}
$$

where M is an appropriate constant such that $M > \phi(v,\ x_0)$ for each $v \in C$ and j is the single-valued duality mapping on E. If T is uniformly equi-continuous, then $\{x_n\}$ converges to $q = \Pi_{F(T)} x_0$.

Proof. First, we show that C_n and Q_n are closed and convex for each $n \ge 0$. It is obvious that C_n is closed and Q_n is closed and convex for each $n \ge 0$. Next, we prove that C_n is convex. Since

$$\phi(v, y_n) \le \phi(v, x_n) + \alpha_n M$$

is equivalent to

$$2\langle v, jx_n - jy_n\rangle \le \|x_n\|^2 - \|y_n\|^2 + \alpha_n M,$$

we obtain that C_n is convex. Next, we show that $F(T) \subset C_n$ for all n. Indeed, from condition $k_n^2(1-\alpha_n) \le 1$, we have

$$
\begin{aligned}
&\phi(p, y_n) \\
&= \phi(p, j^{-1}(\alpha_n jx_0 + (1-\alpha_n)jT^n x_n)) \\
&= \|p\|^2 - 2\langle p, \alpha_n jx_0 + (1-\alpha_n)jT^n x_n\rangle + \|\alpha_n jx_0 + (1-\alpha_n)jT^n x_n)\|^2 \\
&\le \|p\|^2 - 2\alpha_n\langle p, jx_0\rangle - 2(1-\alpha_n)\langle p, jT^n x_n\rangle \\
&\quad + \alpha_n\|x_0\|^2 + (1-\alpha_n)\|T^n x_n\|^2 \\
&\le \alpha_n\phi(p, x_0) + (1-\alpha_n)\phi(p, T^n x_n) \\
&\le \alpha_n\phi(p, x_0) + k_n^2(1-\alpha_n)\phi(p, x_n) \\
&= \phi(p, x_n) - (1 - k_n^2(1-\alpha_n))\phi(p, x_n) + \alpha_n\phi(p, x_0) \\
&\le \phi(p, x_n) + \alpha_n M,
\end{aligned}
$$

for each $p \in F(T)$. So $p \in C_n$ for all n, which implies that $F(T) \subset C_n$. Next we show that

$$F(T) \subset Q_n, \quad \forall n \ge 0. \tag{2.2}$$

We prove this by induction. For $n = 0$, we have $F(T) \subset C = Q_0$. Assume that $F(T) \subset Q_n$. Since x_{n+1} is the projection of x_0 onto $C_n \cap Q_n$, by

Lemma 1.4 we have

$$\langle jx_0 - jx_{n+1}, x_{n+1} - z\rangle \geq 0, \quad \forall z \in C_n \cap Q_n.$$

As $F(T) \subset C_n \cap Q_n$ by the induction assumptions, the last inequality holds, in particular, for all $z \in F(T)$. From the definition of Q_{n+1}, we have that $F(T) \subset Q_{n+1}$. Hence (2.2) holds for all $n \geq 0$. This implies that $\{x_n\}$ is well defined. Since $x_{n+1} = \Pi_{C_n \cap Q_n} x_0 \in Q_n$, we have

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0), \quad \forall n \geq 0.$$

Therefore $\{\phi(x_n, x_0)\}$ is nondecreasing. Since C is bounded, we have $\phi(x_n, x_0)$ is bounded. Moreover, from (1.6), we have that $\{x_n\}$ is bounded. Therefore, we obtain that the limit of $\{\phi(x_n, x_0)\}$ exists. From Lemma 1.5, we have

$$\begin{aligned}\phi(x_{n+1}, x_n) = \phi(x_{n+1}, \Pi_{C_n} x_0) &\leq \phi(x_{n+1}, x_0) - \phi(\Pi_{C_n} x_0, x_0)\\ &= \phi(x_{n+1}, x_0) - \phi(x_n, x_0)\end{aligned}$$

for all $n \geq 0$. This implies that

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{2.3}$$

By using Lemma 1.3, one arrives at

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{2.4}$$

Since $x_{n+1} = \Pi_{C_n \cap Q_n} x_0 \in C_n$ and from the definition of C_n, we also have

$$\phi(x_{n+1}, y_n) \leq \phi(x_{n+1}, x_n) + \alpha_n M. \tag{2.5}$$

It follows from (2.3) and $\lim_{n\to\infty} \alpha_n = 0$ that

$$\lim_{n\to\infty} \phi(x_{n+1}, y_n) = 0.$$

From Lemma 1.3, one has

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0. \tag{2.6}$$

Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|jx_{n+1} - jy_n\| = \lim_{n\to\infty} \|jx_{n+1} - jx_n\| = 0. \tag{2.7}$$

Noticing

$$\begin{aligned}&\|jx_{n+1}-jy_n\|\\&=\|jx_{n+1}-(\alpha_nJx_0+(1-\alpha_n)jT^nx_n)\|\\&=\|\alpha_n(jx_{n+1}-jx_0)+(1-\alpha_n)(jx_{n+1}-jT^nx_n)\|\\&=\|(1-\alpha_n)(jx_{n+1}-jT^nx_n)-\alpha_n(jx_0-jx_{n+1})\|\\&\geq(1-\alpha_n)\|jx_{n+1}-jT^nx_n\|-\alpha_n\|jx_0-jx_{n+1}\|,\end{aligned}$$

we have

$$\|jx_{n+1}-jT^nx_n\|\leq\frac{1}{1-\alpha_n}(\|jx_{n+1}-jy_n\|+\alpha_n\|jx_0-jx_{n+1}\|).$$

From (2.7) and $\lim_{n\to\infty}\alpha_n=0$, we obtain

$$\lim_{n\to\infty}\|jx_{n+1}-jT^nx_n\|=0.$$

Since j^{-1} is also uniformly norm-to-norm continuous on bounded sets, we obtain

$$\lim_{n\to\infty}\|x_{n+1}-T^nx_n\|=0. \tag{2.8}$$

On the other hand, we have

$$\|x_n-T^nx_n\|\leq\|x_{n+1}-x_n\|+\|x_{n+1}-T^nx_n\|.$$

It follows from (2.4) and (2.8) that $\lim_{n\to\infty}\|T^nx_n-x_n\|=0$. Putting $L=\sup\{k_n:n\geq1\}<\infty$, we obtain

$$\begin{aligned}\|Tx_n-x_n\|\leq&\|Tx_n-T^{n+1}x_n\|+\|T^{n+1}x_n-T^{n+1}x_{n+1}\|\\&+\|T^{n+1}x_{n+1}-x_{n+1}\|+\|x_{n+1}-x_n\|.\end{aligned}$$

Since T is uniformly equi-continuous, we have

$$\|Tx_n-x_n\|\to0\quad\text{as } n\to\infty.$$

Finally, we prove that $x_n\to q=\Pi_{F(T)}x_0$. Assume that a $\{x_{n_i}\}$ is a subsequence of $\{x_n\}$ such that $\{x_{n_i}\}\rightharpoonup q\in C$. It follows that $q=\widehat{F}(T)=F(T)$. Next we show that $q=\Pi_{F(T)}x_0$ and convergence is strong. Put $q'=\Pi_{F(T)}x_0$. From $x_{n+1}=\Pi_{C_n\cap Q_n}x_0$ and $q'=F(T)\subset C_n\cap Q_n$, we have $\phi(x_{n+1},x_0)\leq\phi(q',x_0)$. On the other hand, from weakly lower semi-continuity of the norm, we obtain

$$\begin{aligned}\phi(q, x_0) &= \|q\|^2 - 2\langle q, jx_0\rangle + \|x_0\|^2\\ &\le \liminf_{i\to\infty}(\|x_{n_i}\|^2 - \langle x_{n_i}, jx_0\rangle + \|x_0\|^2)\\ &\le \liminf_{i\to\infty} \phi(x_{n_i}, x_0)\\ &\le \limsup_{i\to\infty} \phi(x_{n_i}, x_0) \le \phi(q', x_0).\end{aligned}$$

It follows from definition of $\Pi_{F(T)}x_0$, we obtain $q = \Pi_{F(T)}x_0$. It follows that

$$\lim_{i\to\infty} \phi(x_{n_i}, x_0) = \phi(q', x_0) = \phi(q, x_0).$$

Therefore, we obtain $\lim_{i\to\infty} \|x_{n_i}\| = \|q\|$. Using the Kadec-Klee property of E, we obtain that $\{x_{n_i}\}$ converges strongly to $q = P_{F(T)}x_0$. Since $\{x_{n_i}\}$ is an arbitrarily weakly convergent sequence of $\{x_n\}$, we can conclude that $\{x_n\}$ converges strongly to $q = \Pi_{F(T)}x_0$. This completes the proof. □

As a application of Theorem 2.1, we have the following results.

Corollary 2.2. *Let H be a Hilbert space, C a nonempty bounded closed convex subset of E and $T : C \to C$ an asymptotically nonexpansive mapping with the sequence $\{k_n\}$ such that $k_n \to 1$ as $n \to \infty$. Assume that $\{\alpha_n\}_{n=0}^{\infty}$ is sequence in $[0, 1]$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $k_n^2(1 - \alpha_n) - 1 \le 0$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 \in C \quad \text{arbitrarily chosen},\\ y_n = \alpha_n x_0 + (1 - \alpha_n)T^n x_n,\\ C_n = \{v \in C : \|y_n - v\|^2 \le \|x_n - v\|^2 + \alpha_n M\},\\ Q_n = \{v \in C : \langle x_0 - x_n, x_n - v\rangle \ge 0\},\\ x_{n+1} = \Pi_{C_n\cap Q_n}x_0,\end{cases}$$

where M is an appropriate constant such that $M > \|x_0 - v\|^2$ for each $v \in C$. Then $\{x_n\}$ converges to some $q = \Pi_{F(T)}x_0$.

Proof. From [8], we know $F(T) \ne \emptyset$. The key is to show that T is asymptotically nonexpansive, then T is relatively asymptotically nonexpansive. Taking $p \in \widehat{F}(T)$, we have that there exists a sequence $\{x_n\} \subset C$ such that $x_n \rightharpoonup p$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Observing that T is asymptotically nonexpansive mapping and from Lemma 1.5, we arrive at $p \in F(T)$. On the other hand, we have $F(T) \subset \widehat{F}(T)$. In Hilbert spaces, we know (1.4) reduces to $\phi(x, y) = \|x - y\|^2,\ x, y \in H$.

Therefore, T is also relatively asymptotically nonexpansive. It is easy to obtain the desired conclusion from the proof of Theorem 2.1. This completes the proof. □

REMARK 2.1. We improve Martinez-Yanes and Xu [14]'s results from two distinct directions. One one hand, we extend the framework of spaces from Hilbert spaces to Banach spaces. On the other hand, we extend mappings from nonexpansive mappings to relatively asymptotically nonexpansive mappings. The results presented in this paper also improve Qin and Su [15] from relatively nonexpansive mappings to relatively asymptotically nonexpansive mappings in the framework of Banach spaces.

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Department of Mathematics
Gyeongsang National University
Chinju 660-701, Republic of Korea
E-mail: ljjhqxl@yahoo.com.cn;qxlxajh@163.com

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Department of Mathematics and the RINS
Gyeongsang National University
Chinju 660-701, Republic of Korea
E-mail: smkang@nongae.gsnu.ac.kr

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Department of Mathematics
Gyeongsang National University
Chinju 660-701, Republic of Korea
E-mail: ooly61@yahoo.co.kr