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A TWO-FUNCTION MINIMAX THEOREM

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Abstract. In this note, using the separation theorem for convex sets, we will give a two functions version generalization of Fan's minimax theorem by relaxing the convexlike assumption to the weak convexlike condition.

1. Introduction

In 1928, von Neumann found his celebrated minimax theorem [8] and, since then, several extensions of von Neumann's minimax theorem were established. Among them, in 1952, Kneser [5] proved the generalization of von Neumann's minimax theorem by weakening the compactness, linearity and continuity assumptions, and it has been very useful in many applications in convex analysis and the theory of games. In 1953, Fan [2] proved the abstract minimax theorem using general convexity assumptions on f without assuming the linear structures on X and Y. Till now, there have been numerous minimax theorems in abstract settings and two functions version of minimax theorems which generalize von Neumann's minimax theorem, e.g., see [1-7].

In this note, using the weak convexlike condition in [4], we will give a two functions version minimax theorem which generalizes Fan's minimax theorem by applying the separation theorem for convex sets. Next we give an example which the previous minimax theorems in [1-6] can not be applied but is suitable for our theorem.

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2. Preliminaries

Let X be a non-empty convex subset of a vector space E and let $f : X \to \mathbb{R}$. We say that f is *quasi-convex* if for each $t \in \mathbb{R}$, $\{x \in X \mid f(x) \leq t\}$ is convex; and that f is *quasi-concave* if $-f$ is quasi-convex. When X and Y are any non-empty sets without linear structures, recall that $f : X \times Y \to \mathbb{R}$ is *convexlike* [2] on X if for any $x_1, x_2 \in X$ and $\lambda \in [0,1]$, there exists an $x_0 \in X$ such that

$$f(x_0, y) \leq \lambda f(x_1, y) + (1-\lambda) f(x_2, y) \quad \text{for all } \ y \in Y;$$

and that f is *concavelike* if $-f$ is convexlike.

In [4], the authors introduce the weak convexlike condition which relax the convexlike condition into a finite subset of Y as follows:

Definition 2.1. Let X and Y be any non-empty sets and $f : X \times Y \to \mathbb{R}$ be a real-valued function on $X \times Y$. Then f is called *weak convexlike* on X if for every $n \geq 2$, whenever $\{x_1, \cdots, x_n\} \subseteq X$ is given and for any finite subset $\{y_1, \cdots, y_m\}$ of Y and $\lambda_i \in [0,1]$, $i = 1, \ldots, n$, with $\sum_{i=1}^n \lambda_i = 1$, there exists a point $x_0 \in X$ such that

$$f(x_0, y) \leq \lambda_1 f(x_1, y) + \cdots + \lambda_n f(x_n, y) \quad \text{for all} \quad y \in \{y_1, \ldots, y_m\};$$

and that f is *weak concavelike* on X if $-f$ is weak convexlike on X.

In the Definition, if Y is a finite set, then the weak convexlike condition is actually the same as the convexlike condition due to Fan [2].

3. A two-function minimax theorem

Using the separation theorem for convex sets, we now prove a two-function minimax theorem which generalizes Fan's minimax theorem by relaxing the concavelike condition as follows:

Theorem 3.1. *Let X be a non-empty compact topological space, and Y be a non-empty discrete set. Let $f, g : X \times Y \to \mathbb{R}$ be two functions satisfying the following conditions:*

(1) $f(x,y) \geq g(x,y)$ for all $(x,y) \in X \times Y$;

(2) for each $y \in Y$, the function $x \mapsto g(x,y)$ is lower semicontinuous and weak convexlike on X;

(3) for each $x \in X$, the function $y \mapsto f(x,y)$ is concavelike on Y.

Then we have

$$\inf_{x\in X}\sup_{y\in Y} g(x,y) \le \sup_{y\in Y}\min_{x\in X} f(x,y).$$

Proof. First, we shall prove that one of the following holds:

(I) there exists $\bar{x}\in X$ such that $\quad g(\bar{x},y)\le 0 \quad$ for all $\quad y\in Y$;

(II) there exists $\bar{y}\in Y$ such that $\quad f(x,\bar{y})\ge 0 \quad$ for all $\quad x\in X$.

Suppose (I) were false. Then for each $x\in X$, there exists $y\in Y$ such that $g(x,y)>0$. Since $x\mapsto g(x,y)$ is lower semicontinuous, the set $U_y := \{x\in X \mid g(x,y)>0\}$ is open for each $y\in Y$. Since X is compact and $X\subseteq \cup_{y\in Y}U_y$, there exists a finite subset $\{y_1,\cdots,y_n\}\subset Y$ such that $X\subseteq \cup_{i=1}^n U_{y_i}$. Therefore, for each $x\in X$, there exists $j\in\{1,\cdots,n\}$ with $x\in U_{y_j}$. Hence, we have

$$\max_{1\le i\le n} g(x,y_i)>0 \quad \text{for each} \quad x\in X.$$

Now, we let

$$C_1 := co\{\big(g(x,y_1),\cdots,g(x,y_n)\big)\in\mathbb{R}^n \mid x\in X\};$$
$$C_2 := \{(z_1,\cdots,z_n)\in\mathbb{R}^n \mid z_i < 0,\ i=1,\cdots,n\}.$$

Then, it is clear that C_1 is a non-empty convex subset of $\mathbb{R}^n$, and C_2 is a non-empty open convex subset of $\mathbb{R}^n$. Now we claim that $C_1\cap C_2=\emptyset$. Indeed, suppose that there exists $(z_1,\cdots,z_n)\in C_1\cap C_2$. Then, there exist $\{x_1,\cdots,x_k\}\subset X$ and $\lambda_i\in(0,1),\ i=1,\dots,k,$ with $\sum_{i=1}^k\lambda_i=1$, such that

$$(z_1,\cdots,z_n)=\Big(\sum_{j=1}^k\lambda_j g(x_j,y_1),\cdots,\sum_{j=1}^k\lambda_j g(x_j,y_n)\Big).$$

Since $x\mapsto g(x,y)$ is weak convexlike, for the given sets $\{x_1,\cdots,x_k\}$ and $\{y_1,\cdots,y_n\}$, and given $\lambda_j\in(0,1),\ j=1,\dots,k,$ with $\sum_{j=1}^k\lambda_j=1$, there exists $x_0\in X$ such that

$$\sum_{j=1}^k\lambda_j g(x_j,y)\ge g(x_0,y) \quad \text{for all} \quad y\in\{y_1,\cdots,y_n\}.$$

Therefore, for each $i\in\{1,\dots,n\}$, we have

$$0 > z_i=\sum_{j=1}^k\lambda_j g(x_j,y_i) \ \ge\ g(x_0,y_i).$$

Since $x_0\in U_{y_j}$ for some $j\in\{1,\cdots,n\}$, we must have $g(x_0,y_j)>0$ which is a contradiction. Therefore, $C_1\cap C_2=\emptyset$. By the separation theorem for convex sets, there exists $(u_1, \cdots, u_n) \in \mathbb{R}^n \setminus \{\mathbb{O}\}$ such that for all $x \in X$ and for all $(z_1, \cdots, z_n) \in C_2$,

$$\sum_{i=1}^{n} u_i \cdot g(x, y_i) \ > \ \sum_{i=1}^{n} u_i \cdot z_i.$$

If we let $z_i \to -\infty$, we have $u_i \geq 0$ for each $i \in \{1, \cdots, n\}$. Therefore, we may assume that $u_i \in [0, 1],\ i = 1, \ldots, n,$ with $\sum_{i=1}^{n} u_i = 1$. For any $\varepsilon > 0$, if we choose $(z_1, \cdots, z_n) = (-\varepsilon, \cdots, -\varepsilon) \in C_2$, then $\sum_{i=1}^{n} u_i \cdot z_i = -\varepsilon$ so that $\sum_{i=1}^{n} u_i \cdot g(x, y_i) > -\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have $\sum_{i=1}^{n} u_i \cdot g(x, y_i) \geq 0$ for all $x \in X$. By the assumption (3), the function $y \mapsto f(x, y)$ is concavelike on Y so that there exists a point $y_0 \in Y$ such that $f(x, y_0) \geq \sum_{i=1}^{n} u_i \cdot f(x, y_i)$ for all $x \in X$. Therefore, by the assumption (1),

$$f(x, y_0) \geq \sum_{i=1}^{n} u_i \cdot f(x, y_i) \geq \sum_{i=1}^{n} u_i \cdot g(x, y_i) \geq 0 \quad \text{for all } \ x \in X,$$

which proves (II). Now, we shall prove the main inequality. If we suppose $\inf_{x \in X} \sup_{y \in Y} g(x, y) = -\infty$, then the conclusion holds since

$$-\infty = \inf_{x \in X} \sup_{y \in Y} g(x, y) \leq \sup_{y \in Y} \min_{x \in X} f(x, y).$$

Next, for any constant $c \in \mathbb{R}$, we can repeat the previous argument to the two functions $f(x, y) - c$ and $g(x, y) - c$. If (I) holds, then there exists $\bar{x} \in X$ such that $\sup_{y \in Y} g(\bar{x}, y) \leq c$ so that we have

$$\inf_{x \in X} \sup_{y \in Y} g(x, y) \leq c. \tag{†}$$

On the other hand, if (II) holds, then there exists $\bar{y} \in Y$ such that $\min_{x \in X} f(x, \bar{y}) \geq c$ so that we have

$$\sup_{y \in Y} \min_{x \in X} f(x, y) \geq c. \tag{‡}$$

For any $\varepsilon > 0$, if we choose $c := \sup_{y \in Y} \min_{x \in X} f(x, y) + \varepsilon$, then the inequality (‡) can not be true so that, from (†), we can obtain that

$$\inf_{x \in X} \sup_{y \in Y} g(x, y) \leq \sup_{y \in Y} \min_{x \in X} f(x, y) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\inf_{x \in X} \sup_{y \in Y} g(x, y) \leq \sup_{y \in Y} \min_{x \in X} f(x, y),$$

which completes the proof. □

REMARK 3.2. When $f = g$, we can obtain a generalization of Fan's minimax theorem in [2] by relaxing the convexlike condition to the weak convexlike condition.

Next, we give an example where Theorem 3.1 can be applied but the previous minimax theorems due to von Neumann, Nikaido, Kneser are not available.

EXAMPLE 3.3. Let $X := [0, 1]$ and $Y := (0, 3]$ be convex sets and the function $f, g : X \times Y \to \mathbb{R}$ be defined by

$$f(x,y) := \begin{cases} 0, & \text{if } \sqrt{x} \le y \le x+1,\ (x,y) \in X \times Y; \\ 1, & \text{otherwise.} \end{cases}$$

$$g(x,y) := \begin{cases} 0, & \text{if } x \le y \le 2,\ (x,y) \in X \times Y; \\ 1, & \text{otherwise.} \end{cases}$$

Then, $f(x, y) \ge g(x, y)$ for all $(x, y) \in X \times Y$. For each $y \in Y$, it is easy to see that $x \mapsto g(x, y)$ is lower semicontinuous and quasi-convex and convexlike on X. Indeed, for any $x_1, x_2 \in X$ and each $\lambda \in [0, 1]$, there exists an $x_0 = 0 \in X$ such that

$$g(0, y) \le \lambda g(x_1, y) + (1-\lambda) g(x_2, y) \quad \text{for each } y \in Y$$

so that $x \mapsto g(x, y)$ is convexlike on X. And, for each $x \in X$, $y \mapsto f(x, y)$ is concavelike but not quasi-concave on Y. Indeed, we can see that for any $y_1, y_2 \in Y$ and each $\lambda \in [0, 1]$, there exists an $y_o = 3 \in Y$ such that

$$1 = f(x, 3) \ge \lambda f(x, y_1) + (1-\lambda) f(x, y_2) \quad \text{for each } x \in X;$$

and the set $\{y \in Y \mid f(\frac{1}{2}, y) \ge \frac{1}{2}\} = (0, \frac{1}{\sqrt{2}}) \cup (\frac{3}{2}, 3]$ is not convex in Y. Therefore, all the hypotheses of Theorem 3.1 are satisfied so that we have

$$1 = \inf_{x \in X} \sup_{y \in Y} g(x, y) \le \sup_{y \in Y} \min_{x \in X} f(x, y) = 1.$$

Note that since the domain of f is not compact and the map $y \mapsto f(x, y)$ is not quasi-concave on Y, the previous minimax theorems due in [2,3,5-8] can not be applied for this function f.

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