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공통의 납기 구간을 가지는 MSD 문제에서의 납기 결정

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Due Date Determination on the MSD Problem with a Common Due Date Window

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JIT 생산시스템이 세계적으로 많은 회사에 적용됨에 따라 earliness와 tardiness의 페널티를 동시에 최소화하는 문제에 대한 많은 연구가 진행되어 왔다. 이 연구에서는 한정된 완료시간의 편차에 대해서는 페널티를 부과하지 않는, 즉 허용오차가 존재할 때, 공통의 납기로부터 평균제곱편차(MSD : Mean Squared Deviation)를 최소화하는 단일기계 문제를 다룬다. 허용오차가 존재하는 MSD 문제에서 최적의 공통 납기를 결정하는 방법을 개발한다. 스케쥴과 허용오차가 주어질 때, 최적의 납기를 찾는 두 개의 선형시간이 소요되는 알고리즘을 제시한다. 주어진 허용오차 중 하나는 가장 짧은 가공시간을 가지는 작업의 절반보다 작은 경우이며 다른 하나는 허용오차가 임의인 경우이다.

Keywords : Earliness, Tardiness, Mean Squared Deviation, Common Due Date Window

1. Introduction

After the introduction of the just-in-time concept in manufacturing, there has been vast interest in machine scheduling with earliness-tardiness penalties. The just-in-time manufacturing is based on the fact that jobs completed early must be held in finished goods inventory until their due date, while jobs completed late may cause customers to wait. Jobs completed early or tardy from the due date are penalized in earliness-tardiness scheduling problems, which reflects the premise of just-in-time manufacturing ; for a survey, see Baker and Scudder [2]. Examples of earliness-tardiness scheduling are file organization problem, PERT/CPM project, production of perishable goods, final assembly production, presented in Merten and Muller [18], Sidney [20] and Kanet [14].

Earliness-tardiness scheduling problems use various objectives to measure the goodness of a schedule [2]. Generally, there are two main penalty functions : the mean squared deviation (MSD) and the mean absolute deviation (MAD). In most practical situations, large deviations from the due date are highly undesirable, and it might be appropriate to use squared deviations from the due date as the performance

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measure [9, 11, 19].

Determination of due date in the scheduling problem has been studied for several decades because of its practical importance, for example, zero inventory philosophy in advanced manufacturing system, planned order release and resource requirement planning [5, 6]. The jobs have to be met the due date in order to avoid penalty or the loss of customer's goodwill. Recently, the concept of due date window has been studied in the earliness/tardiness scheduling problem. The introduction of due date window is suitable for the real life situation and the problem with due date window has been studied much in the literature. As mentioned in Soloman and Desrosiers [21], the concept of time window has been introduced in vehicle routing and scheduling and this time window is sometimes called as due date tolerance in scheduling problem [2]. The applications of time window are business organizations which work on fixed time schedules, industries that implement JIT concept still allow for some amount of inventory, time bucket in an MRP system, factory automation and production maintenance, presented in Ventura and Weng [22], Liman et al. [17], Baker and Scudder [2], Cheng and Gupta [5], respectivelv.

Cheng [4] introduced the MAD problem with tolerance where jobs are not penalized if they are completed close enough to the common due date with assumption that at most one job can avoid penalty.

Dickman et al. [8] modified the algorithm in Cheng [4] and provided alternative optimal due dates and schedules. Weng and Ventura [23] and Wilamowsky et al. [24] extended the MAD problem with tolerance by relaxing the assumption on the range of tolerance. Ventura and Weng [22] studied the problem to minimize the MAD of job completion times. They proposed an algorithm based on Lagrangian relaxation and two heuristic algorithms. Computational results showed that the obtained solutions by proposed heuristic algorithms are near-optimal solutions.

For the MSD problem, pseudo-polynomial time algorithm was developed by De et al. [6, 7]. The algorithm based on dynamic programming is presented when the earliness and tardiness are both weighted and unweighted [7]. Kahlbacher [13] proposed the polynomial time algorithm for generally powered objection function. Refer [10] for general due date assignment problems.

When there exists a common due date window, Krämer and Lee [15] studied the minimization of weighted earliness and tardiness. A very simple algorithm to determine the location of the due date window was presented. Liman et al. [17] addressed the problem to minimize the weighted sum of earliness, tardiness and penalty of due date window location. An O(nlogn) algorithm was proposed with two special cases. The minimization of weighted earliness and weighted number of tardy jobs was investigated by Liman and Ramaswamy [16]. Algorithm based on dynamic programming was provided.

Yeung et al. [25] investigated the simple machine scheduling problem to minimize the sum of weighted earliness/tardiness, weighted number of early and tardy jobs, common small due date window location and flow time penalties. They proposed the dynamic programming algorithm to solve the problem and several special cases solved in polynomial time are presented. Recently, researches on due windows for several machines have been performed with different objective functions [12, 26].

In this paper, the determination of the optimal common due date in the MSD problem with tolerance is investigated since small buffer stocks exist in most companies which adopted JIT principle [3]. It turned out that the optimal due date of the MSD problem may not be optimal for the MSD problem with tolerance. In Section 2, the problem is mathematically formulated. Section 3 presents two linear time algorithms to find the optimal due date of a given schedule and tolerance : one for the size of tolerance less than half of the processing time of the shortest job, and the other for arbitrary size of tolerance. Finally Section 4 summarizes and concludes the paper.

2. The MSD problem with tolerance

There are n jobs ($n \geq 2$) to be scheduled on a single-machine, which share a common due date d with tolerance t. The machine is continuously available from time zero onwards and can process no more than one job at a time. Each job i ($i = 1, 2, \cdots, n$) requires an uninterrupted and deterministic processing time p_i and ideally should be completed at a common due date d with tolerance t. An arbitrary schedule, σ, specifies a completion time without job preemption for each job i. If there is no ambiguity about the schedule under consideration, C_i is used instead of $C_i(\sigma)$. Given a schedule σ, $\sigma(i)$ and $\overline{C}$ denote the i^{th} job and the mean completion time of jobs in σ, respectively. Let $\overline{C_i}$ be the mean completion time without considering the completion time of job i, i.e., $\overline{C_i} = (\sum_{j=1}^{n} C_j - C_i)/(n-1)$. A job is said to be covered at d if its completion time belongs to the interval $[d-t, d+t]$.

The *earliness* and *tardiness* of job i are denoted by E_i and T_i respectively and are defined by $E_i = \max\{0, d-C_i\}$ and $T_i = \max\{0, C_i - d\}$. For schedule σ and a common due date d, we let $MSD(d)$ be the mean squared deviation, and let $MSD_i(d)$ be the mean squared deviation without penalizing job i. Note that n is constant. Thus, unless otherwise specified, $MSD(d)$ means $\sum_{i=1}^{n}(C_i - d)^2 = \sum_{i=1}^{n}(E_i^2 + T_i^2)$ throughout this paper. Likewise, $MSD_i(d) = MSD(d) - (C_i - d)^2$. And here, $MSD_i(d)$ is the mean squared deviation when a job i is covered at d. Therefore, the total penalty of schedule σ is measured by $f(d, \sigma)$, defined as $f(d,\sigma) = \sum_{i=1}^{n}(E_i^2 U(E_i) + T_i^2 U(T_i))$, where U is a unit step function such that

$$U(x-t) = \begin{cases} 0 & \text{if } x \le t \\ 1 & \text{if } x > t\,. \end{cases}$$

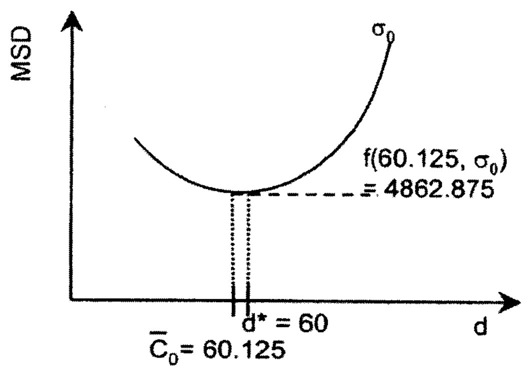
Namely, a job is penalized by the square of the difference between its completion time and the common due date unless its completion time belongs to the interval $[d-t, d+t]$, which is called the due window as in Liman et al. (1996). For the optimal due date d^*, $[d^*-t, d^*+t]$ is said to be the optimal due window. The objective in this paper is to determine the optimal common due date d^* which minimizes the objective function f for any given schedule and tolerance. It is easily observed that no optimal schedule permits idle time between the execution of the jobs, and then we assume that any given schedule permits no idle time from the first job to the last job processed.

3. The optimal due date

Bagchi et al. [1] have shown that for a given schedule, the optimal due date is a point, which is a mean completion time of the schedule.

Lemma 1 : (Bagchi, Sullivan and Chang [1]) For a given schedule, *MSD(d)* is minimized at $d = \overline{C}$.

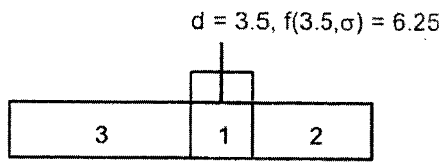
The following example shows that in general Lemma 1 does not hold in the MSD problem with tolerance. That is, the optimal due date of the MSD problem may not be optimal for the MSD problem with tolerance. The example is quoted from Wilamowsky et al. [24], problem #5 in <Table 1>. Consider the 10-job problem with the processing times 19, 18, 16, 13, 10, 9, 8, 5, 2, 1. The optimal schedule of the MSD problem is $\sigma_0 = (1, 3, 4, 6, 9, 10, 8, 7, 5, 2)$ with the optimal due date $d^* = 60.0$. With $t = 1.875$, the optimal due date for schedule σ_0 is 60.125 and $f(60.125, \sigma_0) = 4862.875$, as shown in <Figure 1>.



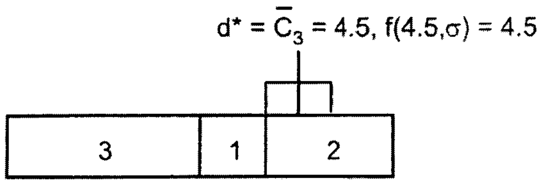
<Figure 1> Schedule σ_0

In the MAD problem with tolerance, the optimal due window always covers as many jobs as possible [23]. In general, this fact does not hold for the MSD problem with tolerance, as illustrated in the following example.

Consider a set of jobs *with* $p_1 = 3, p_2 = 2, p_3 = 1$ and $t = 0.5$. For schedule $\sigma = (1, 3, 2)$, we have $f(3.5, \sigma) = 6.25$ as shown in <Figure 2>. But the optimal due date for the MSD problem with tolerance is 4.5 such that $f(4.5, \sigma) = 4.5$ as shown in <Figure 3>. Note that due window [3, 4] in <Figure 2> covers two jobs, which is the maximum number of jobs that can be covered, while due window [4, 5] in <Figure 3> covers only one job. This implies that the optimal due date cannot be determined simply by covering as many jobs as possible such as in the MAD problem with tolerance.



<Figure 2> d covers C_1 and C_3



<Figure 3> d covers C_3

3.1 When tolerance is small

We start with the special case that the size of tolerance is less than half of the processing time of the shortest job, which is the assumption taken in Cheng [4] and Dickman et al. [8] for the MAD problem with tolerance, so that at most one job can avoid penalty. In this case, the optimal due date can be obtained much faster than that of the general case.

Theorem 1 : Let $t < \frac{1}{2} min_i\, p_i$, and let $\sigma(i)$ be the job such that $\sigma(i) \in \left\{ k : \begin{matrix} \min \\ k \end{matrix} |C_k - \overline{C}| \right\}$. Suppose that there is no job covered at $d = \overline{C}$. Then, the optimal due date is as follows.

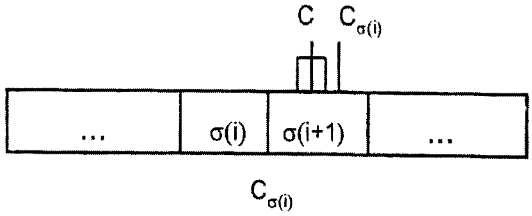
(*i*) if $C_{\sigma(i)} < \overline{C}$, *then* d^*
$\in \{ d : \min(MSD(\overline{C}), MSD_{\sigma(i)}(C_{\sigma(i)} + t) \}$ or

(*ii*) if $C_{\sigma(i)} > \overline{C}$, *then* d^*
$\in \{ d : \min(MSD(\overline{C}), MSD_{\sigma(i)}(C_{\sigma(i)} - t) \}$

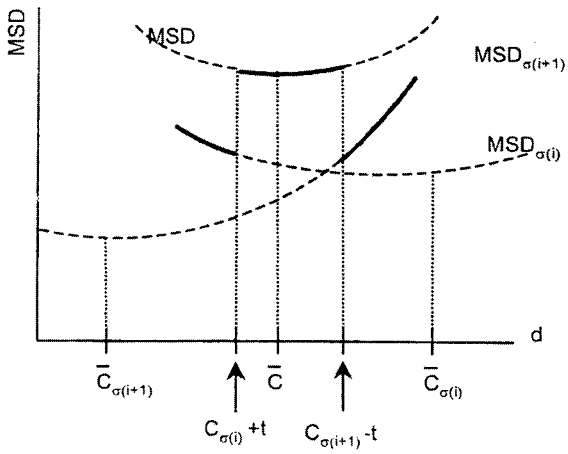
Proof : For $C_{\sigma(i)} \le d \le C_{\sigma(i)} + t$, job $\sigma(i)$ is not penalized and the objective function is $MSD_{\sigma(i)}(d)$. In this interval $[C_{\sigma(i)}, C_{\sigma(i)} + t]$, $MSD_{\sigma(i)}(d)$ is strictly decreasing since $C_{\sigma(i)} + t < \overline{C} < \overline{C}_{\sigma(i)}$. For $C_{\sigma(i+1)} - t \le d \le C_{\sigma(i+1)}$, the objective function is $MSD_{\sigma(i+1)}(d)$ and it is strictly increasing since $C_{\sigma(i+1)} < \overline{C} < C_{\sigma(i+1)} - t$. Since $\overline{C} - (C_{\sigma(i)} + t) \le (C_{\sigma(i+1)} - t) - \overline{C}$ by the way we select job $\sigma(i)$, it is clear that $MSD(C_{\sigma(i)} + t) \le MSD(C_{\sigma(i+1)} - t)$. Therefore,

$$
\begin{aligned}
& MSD_{\sigma(i)}(C_{\sigma(i)} + t) \\
&= MSD(C_{\sigma(i)} + t) - (C_{\sigma(i)} - (C_{\sigma(i)} + t))^2 \\
&\le MSD(C_{\sigma(i+1)} - t) - (C_{\sigma(i+1)} - (C_{\sigma(i+1)} - t))^2 \\
&= MSD_{\sigma(i+1)}(C_{\sigma(i+1)} - t).
\end{aligned}
$$

When $C_{\sigma(i)} \le d \le C_{\sigma(i)} + t$ and $C_{\sigma(i+1)} - t \le d \le C_{\sigma(i+1)}$, the objective function is minimized at $d = C_{\sigma(i)} + t$. For $C_{\sigma(i)} + t < d < C_{\sigma(i+1)} - t$, no job is covered and the objective function is $MSD(d)$, which has a minimum at $\overline{C}$ by Lemma 1. Now, we obtain that $d^* \in \{ d : \min(MSD(\overline{C}), MSD_{\sigma(i)}(C_{\sigma(i)} + t)) \}$ as the required. This proves (i). The proof of (ii) is similar to that of (i).



<Figure 4> The case when $C_{\sigma(i)} < \overline{C}$



<Figure 5> The MSD when $C_{\sigma(i)} < \overline{C}$

Lemma 2 : Let t and $\sigma(i)$ be as stated in Theorem 1. Suppose that job $\sigma(i)$ is covered by $d = \overline{C}$. Then we have $C_{\sigma(i-1)} < \overline{C_{\sigma(i)}} < C_{\sigma(i+1)}$.

Proof : Let $x = \overline{C} - C_{\sigma(i)}$ and $y = \overline{C}_{\sigma(i)} - \overline{C}$. Then $y = \frac{x}{(n-1)}$. Note that

$$C_{\sigma(i-1)} < \overline{C} - t \le C_{\sigma(i)} \le \overline{C} + t < C_{\sigma(i+1)} \qquad (1)$$

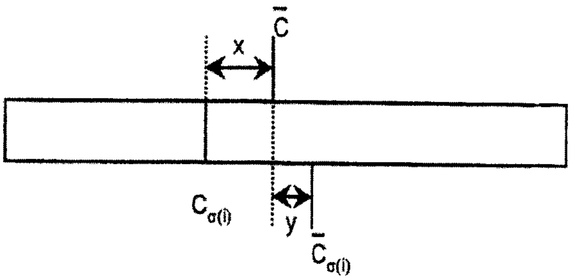
Equation (1) implies that $-t \le x \le t$. Thus, this proof will be done only by checking the signs of $(y-t)$ and $(y+t)$.

Now, we consider :

(case1) if $x = 0$, then $y = 0$;

(case2) if $0 < x \le t$, then $0 < y \le t$;
(case3) if $-t \le x < 0$, then $-t \le y < 0$

In any cases, we know that $y - t = 0$ and $y + t \ge 0$, that is, $\overline{C} - t \le \overline{C_{\sigma(i)}} \le \overline{C} + t$. Thus, by equation (1), we obtain $C_{\sigma(i-1)} < \overline{C_{\sigma(i)}} < C_{\sigma(i+1)}$, completing the proof. □



<Figure 6> The case when job $\sigma(i)$ is covered and $C_{\sigma(i)} < \overline{C}$

Theorem 2 : Let t and $\sigma(i)$ be as stated in Theorem 1. Suppose that job $\sigma(i)$ is covered at $d = \overline{C}$. Then, the optimal due date is as follows.

(i) $d^* = \overline{C}_{\sigma(i)}$ if interval $[\overline{C}_{\sigma(i)} - t, \overline{C}_{\sigma(i)} + t]$ covers job $\sigma(i)$,

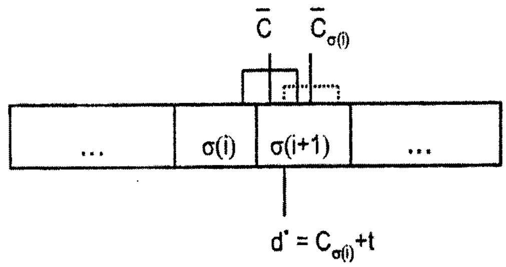
(ii) $d^* = C_{\sigma(i)} + t$ if interval $[\overline{C}_{\sigma(i)} - t, \overline{C}_{\sigma(i)} + t]$ does not cover job $\sigma(i)$ and $C_{\sigma(i)} < \overline{C}$, and

(iii) $d^* = C_{\sigma(i)} - t$ if interval $[\overline{C}_{\sigma(i)} - t, \overline{C}_{\sigma(i)} + t]$ does not cover job $\sigma(i)$ and $C_{\sigma(i)} > \overline{C}$.

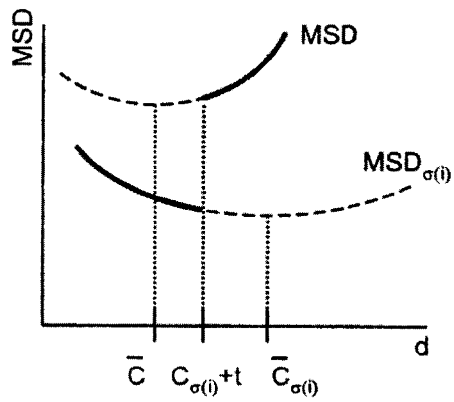
Proof : If interval $[\overline{C}_{\sigma(i)} - t, \overline{C}_{\sigma(i)} + t]$ covers job $\sigma(i)$, it is not penalized and $MSD_{\sigma(i)}(d)$ is minimized at $\overline{C}_{\sigma(i)}$ by Property 1. This proves (i).

To prove (ii), we consider the following two cases. Likewise, the proof of (iii) can be done analogously, omitting here.

Case 1 : This is the case when job $\sigma(i+1)$ is not covered at $d = \overline{C}_{\sigma(i)}$, that is, $C_{\sigma(i+1)} > \overline{C}_{\sigma(i)} + t$. It is easy to see that $\overline{C} \le C_{\sigma(i)} + t < \overline{C}_{\sigma(i)}$. For $\overline{C} \le d \le C_{\sigma(i)} + t$, the objective function is $MSD_{\sigma(i)}(d)$ and it is strictly decreasing since $d < \overline{C}_{\sigma(i)}$. For $C_{\sigma(i)} + t < d \le \overline{C}_{\sigma(i)}$, the objective function is $MSD(d)$ and it is strictly increasing since $\overline{C} < d$ (refer to <Figure 7> and <Figure 8>). It is always true that $MSD_{\sigma(i)}(d) \le MSD(d)$ with equality if and only if $d = C_{\sigma(i)}$, so that the optimal due date is $C_{\sigma(i)} + t$.



<Figure 7> Schedule of Case 1



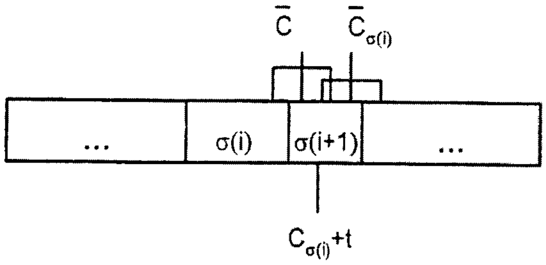
<Figure 8> The MSD of Case 1

Case 2 : This is the case when interval $[\overline{C}_{\sigma(i)} - t, \overline{C}_{\sigma(i)} + t]$ covers job $\sigma(i+1)$ in the schedule. By Lemma 2, $\overline{C}_{\sigma(i)} < C_{\sigma(i+1)}$. It is easy to observe that $\overline{C} \le C_{\sigma(i)} + t < C_{\sigma(i+1)} - t \le \overline{C}_{\sigma(i)}$. For $\overline{C} \le d \le C_{\sigma(i)} + t$, the objective function is $MSD_{\sigma(i)}(d)$ and it is strictly decreasing since $d < \overline{C}_{\sigma(i)}$. For $C_{\sigma(i)} + t < d < C_{\sigma(i+1)} - t$, the objective function is $MSD(d)$ and it is strictly increasing since $\overline{C} < d$. Finally, for $C_{\sigma(i+1)} - t \le d \le C_{\sigma(i+1)}$, the objective function is $MSD_{\sigma(i+1)}(d)$ and it is strictly increasing since $\overline{C}_{\sigma(i+1)} < \overline{C} < d$. So, the optimal due date is $C_{\sigma(i)} + t$ or $C_{\sigma(i+1)} - t$. Since $|\overline{C} - (C_{\sigma(i)} + t)| < |\overline{C} - (C_{\sigma(i+1)} - t)|$, $MSD(C_{\sigma(i)} + t) < MSD(C_{\sigma(i+1)} - t)$ and hence

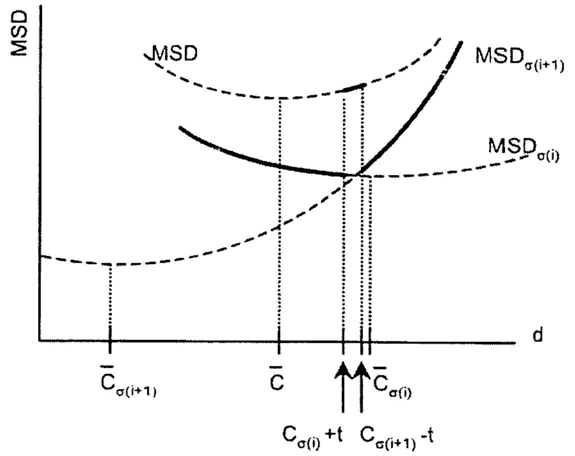
$$
\begin{aligned}
& MSD_{\sigma(i)}(C_{\sigma(i)} + t) \\
& \quad = MSD(C_{\sigma(i)} + t) - (C_{\sigma(i)} - (C_{\sigma(i)} + t))^2 \\
& \quad < MSD(C_{\sigma(i+1)} - t) - (C_{\sigma(i+1)} - (C_{\sigma(i+1)} - t))^2
\end{aligned}
$$

$$= MSD_{\sigma(i+1)}(C_{\sigma(i+1)} - t),$$

as depicted in <Figure 9> and <Figure 10>. Therefore, we obtain that $d^* = C_{\sigma(i)} + t$. □



<Figure 9> Schedule of Case 2



<Figure 10> The MSD of Case 2

Now, we describe an algorithm to decide the optimal due date when the size of tolerance is less than half of the processing time of the shortest job. The correctness of the algorithm immediately follows from Theorem 1 and Theorem 2.

<Algorithm 1> Finding an Optimal Due Date When the Size of Tolerance is Small

Input : Schedule σ of n jobs and tolerance $t < \frac{1}{2} min_j p_j$
Output : Optimal due date for schedule σ

Begin {Computation}
Calculate $\overline{C}$;
Let $\sigma(i)$ be the job such that $\sigma(i) \in \left\{k : \min_k |C_k - \overline{C}|\right\}$;
Find $C_{\sigma(i)}$ and calculate $\overline{C_{\sigma(i)}}$.
If there is no job covered by $[\overline{C}-t, \overline{C}+t]$ then
  begin
    {Determine the optimal due date based on Theorem 2.}
    if $(C_{\sigma(i)} < \overline{C})$ then
      if $(MSD(\overline{C}) < MSD_{\sigma(i)}(C_{\sigma(i)} + t))$
      then $d^* := \overline{C}$;
      else $d^* := C_{\sigma(i)} + t$
      end if
    else
      if $(MSD(\overline{C}) < MSD_{\sigma(i)}(C_{\sigma(i)} - t))$
      then $d^* := \overline{C}$;
      else $d^* := C_{\sigma(i)} - t$
      end if
    end if
  end
  else
  {There exists a job covered at $d = \overline{C}$, that is, job $\sigma(i)$. Determine the optimal due date based on Theorem 2.}
  begin
    If $\overline{C}_{\sigma(i)}$ covers $\sigma(i)$ then $d^* := \overline{C}_{\sigma(i)}$
    else if $C_{\sigma(i)} < \overline{C}_{\sigma(i)} - t$
    then $d^* := \overline{C}_{\sigma(i)} + t$
    else $d^* := \overline{C}_{\sigma(i)} - t$
    end if
  end.
  Output d^*
End {Computation}

It is clear that the time complexity of Algorithm 1 is in $O(n)$. We may refine Algorithm 1 more efficiently by replacing the step of calculating function values stated in Theorem 1 by the following step :

A function $MSD(d)$ can be rewritten as

$$MSD(d) = n(d - \overline{C})^2 + y0 \tag{2}$$

where $y0 = \sum_{i=1}^{n}(C_i - \overline{C})^2$. Similarly, $MSD_i(d)$ can be represented as

$$MSD_{\sigma(i)}(d) = MSD(d) - (C_{\sigma(i)} - d)^2 \tag{3}$$

Using Equation (2) and Equation (3), we have

$$\begin{aligned} & MSD(\overline{C}) - MSD_{\sigma(i)}(C_{\sigma(i)} + t) \\ &= y0 - \{MSD(C_{\sigma(i)} + t) - t^2\} \\ &= y0 - \{n(C_{\sigma(i)} + t - \overline{C})^2 + y0 - t^2\} \\ &= -n(C_{\sigma(i)} + t - \overline{C})^2 + t^2 \end{aligned} \tag{4}$$

Straight forward, we get

$$MSD(\overline{C}) - MSD_{\sigma(i)}(C_{\sigma(i)} - t) \\ = -n(C_{\sigma(i)} - t - \overline{C})^2 + t^2 \quad (5)$$

Therefore, an optimal due date can be readily determined only verifying the sign of Equation (4) or Equation (5).

3.2 When tolerance is arbitrary

In this subsection, we consider the problem determining the optimal due date when the size of tolerance is arbitrary (general case). For schedule σ, let I be a set of consecutive jobs. Then $\overline{C_I}$ denotes the mean completion time without considering the completion times of jobs in I, i.e., $\overline{C_I} = \left(\sum_{i=1}^{n} C_i - \sum_{i \in I} C_i\right) / (n - \| I \|)$, where $\| I \|$ is the number of jobs in I. Without loss of generality, we let $MSD_I(d)$ be the mean square deviation without penalizing jobs in I, i.e., $MSD_I(d) = MSD(d) - \sum_{i \in I}(C_i - d)^2$, where $d - t \le C_i \le d + t$ for all $i \in I$, and $(C_i < d - t$ or $C_i > d + t)$ for all $i \notin I$.

Theorem 3 : For schedule σ, let I be a set of consecutive jobs in σ. If the optimal due window covers exactly set I, then the MSD is minimized at $C_i + t$, $C_j - t$, or $\overline{C_I}$, where i and j are the first and the last covered job in σ.

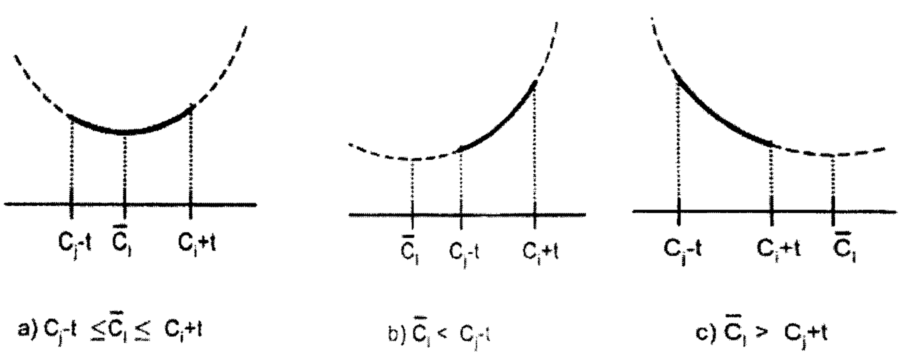
Proof : To cover set I, the optimal due date should be inside $[C_j - t, C_i + t]$. In this interval, the objective function is $MSD_I(d)$ which is minimized at $\overline{C_I}$. If $\overline{C_I} < C_j - t$, the MSD is minimized at $d = C_j - t$. If $\overline{C_I} > C_i + t$, the MSD is minimized at $d = C_j + t$. Otherwise, the MSD is minimized at $\overline{C_I}$. Each case is shown in <Figure 11>.

In the following, we present an algorithm finding the optimal due date for an arbitrary schedule σ and an arbitrary size of tolerance t. Starting from the left-end of schedule σ, the algorithm slides the due window to the right side to find all possible sets of covered jobs. For each set of covered jobs I, it finds a due date that covers I and minimizes the MSD based on Theorem 3. Among such due dates, it finally decides the optimal due date. In the algorithm, i and j denote the i^{th} and the j^{th} jobs in schedule σ, respectively. The variable d^* keeps track of the current optimal due date.

<Algorithm 2> Finding an Optimal Due Date when the Size of Tolerance is Arbitrary

Input : Schedule σ of n jobs and tolerance t.
Output : Optimal due date for schedule σ.

Begin {Computation}
  if t is small, use Algorithm 1
  else
  $I := \varnothing$;
  $d^* := \overline{C_I} := \overline{C}$;
  $i := 1$;
  $j := 0$;
  while $(j \le n$ and $i \le n)$ do
  begin
    {Determine next set of covered jobs.}
    if $(j < n$ and $C_{j+1} \le C_j + 2t)$ then
      $j := i + 1$
    else
      $i := i + 1$
    end if
    Update $\overline{C_I}$ and penalty function $MSD_I()$
    if $(i \le j)$



<Figure 11> Three cases of the optimal due date covering set I

begin
{Determine an candidate due date based on Theorem 3
if $C_i + t \leq \overline{C_I}$ then
$d := C_j + t$
else if $C_j - t \geq \overline{C_I}$ then
$d := C_j - t$
else
$d := \overline{C_I}$;
end if
{Update optimal due date.}
if $(f(d, \sigma) < f(d^*, \sigma)$ then
$d^* := d$
end if
end
end if
end
Output d^*
End {Computation}

In <Algorithm 2>, for each execution of the loop, one job is added into set I or deleted from set I. When a job i is added into set I, new value of $\overline{C_I}$ is calculated as

$$\overline{C_I} = \left(\overline{C_I}(n - \| I \|) - C_i\right) / (n - \| I \| - 1).$$

Since $MSD_I(d)$ is of the form

$$MSD_I(d) = (n - \| I \|)\left(d - \overline{C_I}\right)^2 + y0,$$
$$y0 = \sum_{i=1}^{n} C_i^2 - \sum_{i \in I} C_i^2 - (n - \| I \|) \times \overline{C_I^2},$$

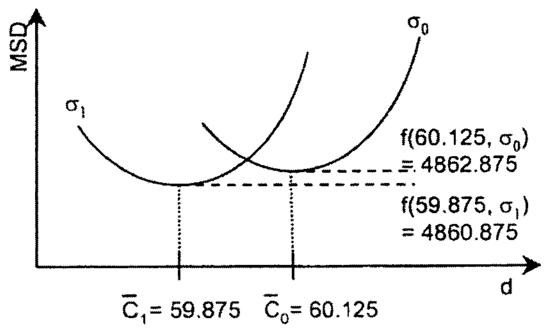
updating $\overline{C_I}$ and $MSD_I()$ can be done in constant time. The maximum number of possible covered sets I is $2n$ and hence <Algorithm 2> runs in linear time.

4. Conclusions

In this paper, a methodology to determine the optimal common due date in the MSD problem with tolerance is presented for the first time. Two linear time algorithms to find the optimal due date of a given schedule and tolerance have been presented : one for the size of tolerance less than half of the processing time of the shortest job, and the other for the arbitrary size of tolerance.

Further research on the MSD problem with tolerance may be to develop an algorithm to find the optimal schedule for the MSD problem with tolerance. When the methodology to assign the due date is developed, the next step is to find optimal sequence. The optimal schedule for the MSD problem may not be the optimal schedule for the MSD problem with tolerance. The following example shows this fact.

Again consider the example in Section 3, which is the 10-job problem with the processing times 19, 18, 16, 13, 10, 9, 8, 5, 2, 1. The optimal schedule of the MSD problem is σ_0 = (1, 3, 4, 6, 9, 10, 8, 7, 5, 2) with the optimal due date d^* = 60.0. With t =1.875, the optimal due date for schedule σ_0 is 60.125 and $f(60.125, \sigma_0) = 4862.875$. But for schedule σ_1 = (1, 3, 4, 7, 9, 10, 8, 6, 5, 2), the optimal due date is 59.875 and $f(59.875, \sigma_1) = 4860.875$ (refer to <Figure 1>). So schedule σ_0 cannot be the optimal schedule for the MSD problem with tolerance t, as shown in <Figure 12>.



<Figure 12> Schedules σ_0 and σ_1

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