한국지능시스템학회 논문지 2008, Vol. 18, No. 6, pp. 837-841

확률집합의 구간치 용적 범함수에 대한 쇼케이 약 수렴성에 관한 연구

Choquet weak convergence for interval-valued capacity functionals of random sets.

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Abstract

In this paper, we consider interval probability as a unifying concept for uncertainty and Choquet integrals with resect to a capacity functional. By using interval probability, we will define an interval-valued capacity functional and Choquet integral with respect to an interval-valued capacity functional. Furthermore, we investigate Choquet weak convergence of interval-valued capacity functionals of random sets.

Key Words : random sets, interval probability, interval-valued capacity functional, Choquet integrals, Choquet weak convergence.

1. Introduction

D.Feng and H.T. Nguyen ([2]) studied Choquet weak convergence of capacity functionals of random sets. We note that non-additive measures, that is, capacities and their corresponding Choquet integrals are very useful tools in the fields of approximate inference, subjective probability, and decision theory. K Weichselberger([5]) studied the theory of interval probaility as a unifying concept for uncertainty which is motivated by the following goals; Different kinds of uncertainty should be treated by the same concept. This applies to

(i) imprecise probability and uncertain knowledge;
(ii) imprecise data;
(iii) the use of capacities;
(iv) the concept of ambiguity and its employment in decision theory;
(v) belief functions and related concepts;
(vi) interpretation of interval-estimates in classical theory;
(vii) the study of experiments with possibly diverging relative frequencies;
(viii) non-additive measures(fuzzy measures).

In this paper, we consider an interval probability and Choquet integrals with respect to a capacity functional. We remark that an interval probability is an inetrval-valued fuzzy measure but the converse does not hold, in general.

The aims of this paper are to define interval-valued capacity functionals of random sets and to investigate Choquet weak convergence of interval-valued capacity functionals of random sets.

In section 2, we list various definitions and notations of interval probabilities, random sets, and Choquet integrals with respect to a capacity functional. In section 3, by using an interval probability, we define interval-valued capacity functionals and discuss their properies. Furthermore, we investigate Choquet Choquet weak convergence of interval-valued capacity functionals of random sets.

2. Interval probabilities and Choquet integrals

Throughout the paper, R is the set of real numbers and

$$I(R)=\{[a,b]|a,b\in R \text{ and } a\leq b\}.$$

Then a element in $I(R)$ is called an interval number. On the interval number set, we define(see[3,4]); for each pair $[a,b],[c,d]\in I(R)$ and $k\in R$,

접수일자 : 2008년 5월 1일
완료일자 : 2008년 9월 25일

$$[a,b]+[c,d]=[a+c,\ b+d]$$
$$[a,b]\cdot[c,d]=[a\cdot c\wedge a\cdot d\wedge b\cdot c\wedge b\cdot d,\ a\cdot c\vee a\cdot d\vee b\cdot c\vee b\cdot d]$$
$$k[a,b]=\begin{cases}[ka,kb], & k\ge 0\\ [kb,ka], & k<0\end{cases}$$
$$[a,b]\le[c,d] \quad \text{if and only if} \quad a\le c \text{ and } b\le d$$

$$\max_{1\le i\le n}[a_i,b_i]=[\max_{1\le i\le n}a_i,\ \max_{1\le i\le n}b_i]$$

$$\min_{1\le i\le n}[a_i,b_i]=[\min_{1\le i\le n}a_i,\ \min_{1\le i\le n}b_i].$$

We note that $(I(R),d_H)$ is a metric space, where d_H is the Hausdorff metric defined by

$$d_H(A,B)=\max\{\sup_{a\in A}\inf_{b\in B}|a-b|,\ \sup_{b\in B}\inf_{a\in A}|a-b|\}$$

for all $A,B\in I(R)$. Then it is easy to see that for $[a,b],[c,d]\in I(R)$,

$$d_H([a,b],[c,d])=\max\{|a-c|,|b-d|\}.$$

Let X be a locally, compact, second-countable, Hausdorff (LCSCH) space. Let Φ be the class of all closed sets in X, Ψ the class of all compact sets in X, O the class of all open sets in X, $\Psi_0=\Psi-\varnothing$, and $B(X)$ the class of all Borel sets in X. We note that $B(\Phi)$ is the σ-algebra classes a generated by Φ^K, $K\in\Psi$ and Φ_G, $G\in O$, where $\Phi^K=\{F\in\Phi|F\cap K=\varnothing\}$ and $\Phi_G=\{F\in\Phi|F\cap G=\varnothing\}$.

Definition 2.1 ([5]) Let (Ω,Σ,P) be a probability space and $(\Phi,B(\Phi))$ ameasurable space.
(1) A measurable mapping S: $(\Omega,\Sigma,P)\to(\Phi,B(\Phi))$ is called a random closed set.
(2) The probability measure Q induced on $B(\Phi)$ is defined by for each $B\in B(\Phi)$, $Q(B)=P(S^{-1}(B))$.

We note that the distribution of a random set S is uniquely determined by its hitting functional T_S on Ψ such that

$$T_S(K)=P(\{w|S(w)\cap K\neq\varnothing\}),\ K\in\Psi.$$

From this, it is easy to see that T_S satisfies the following properties(see[2]):
(T_1) T_S is upper semi-continuous on Ψ,

i.e., $K_n\searrow K$ in $\Psi \Rightarrow T_S(K_n)\searrow T_S(K)$;

(T_2) $T_S(\varnothing)=0$ and $0\le T_S\le 1$;
(T_3) T_S is monotone increasing on Ψ and for $K_1,K_2,\cdots,K_n\in\Psi$, $n\ge 2$,

$$T_S(\cap_{j=1}^n K_j)\le\sum_{\varnothing\neq J\subset\{1,2,\cdots,n\}}(-1)^{|J|+1}T_S(\cup_{j\in J}K_j).$$

Let Γ denote the class of all the capacity functionals T on Ψ satisfying (T_1)-(T_3) above. Such functionals are called alternating Choquet capacities of infinite order or, for brevity, simply capacity functionals. In the LCSCH space X, the capacity functional T can be extended on $B(X)$ by

$$T(B)=\sup\{T(K)|K\in\Psi,\ K\subset B\}$$

for $B\in B(X)$. Then we obtain
(T_1') T is lower semi-continuous on O, i.e., $G_n\searrow G$ in $O\Rightarrow T_S(G_n)\searrow T_S(G)$.

Theorem 2.2 ([2]) Every probability measure Q on $(\Phi,B(\Phi))$ determine a Choquet capacity functional T on Ψ through the correspondences
(C_1) $T(K)=Q(\Phi_K)$, $\forall K\in\Psi$, and
(C_1) $T(G)=Q(\Phi_G)$, $\forall G\in O$
Conversely, every Choquet capacity functional T on Ψ determines a unique probability measure Q on $(\Phi,B(\Phi))$ that satisfies above condition (C_1) and (C_2).
Let $C_b(X)$ be the class of all bounded real-valued continuous functions on X

Definition 2.3 ([2]) (1) Let $f\in C_b(X)$. The Choquet integral of f with respect to T is defined by

$$(C)\int f\,dT=\int_0^{+\infty}T(f\ge t)dt+\int_{-\infty}^0[T(\{f\ge t\})-T(X)]dt.$$

(2) The sequence of capacity functionals T_n converges in the Choquet weak sense to the capacity T in X if

$$(C)\int f\,dT_n\to(C)\int f\,dT,\quad \forall f\in C_b(X)$$

whenever $T_n\to_{C-W}T$ as $n\to\infty$.
A subset $B\subset X$ is called T-continuous functional closed set if

$$B=(\{x|f(x)\ge a\})=T(\{x|f(x)>a\}).$$

For a given capacity T, a family of capacity functionals $\mathfrak{I}\subset\Gamma$ is said to be T-tight if for any T-continuous functional closed set B and for each $\varepsilon>0$, there exists a compact set $K_B\subset X$ such that

$$\sup_{R\in\mathfrak{I}}[R(B)-R(B\cap K_B)]<\epsilon.$$

Definition 2.4 ([5]) An interval-valued set function $\overline{P}(\cdot)$ on Σ is called an interval probability if

(i) $\overline{P}(A)=[L(A), U(A)]$,
$0 \le L(A) \le U(A) \le 1, \forall A \in \Sigma.$

(ii) $\{A \in \Sigma | L(A) \le P(A) \le U(A)\} \neq \varnothing.$

Definition 2.5 ([5]) The interval probability measure $\overline{Q}$ induced on $B(\Phi)$ is defined by for each $B(\Phi)$, $\overline{Q}(B)=\overline{P}(S^{-1}(B))$.

3. Interval-valued capacity functionals

In this section, we will define interval-valued capacity functionals and Choquet integrals with respect to an interval-valued capacity functional which are different to interval-valued Choquet integrals in [3,4] as follows:

Definition 3.1 (1) For every random closed set S, an nterval-valued mapping $\overline{T}_S$ is said to be an interval-valued capacity functional if there exist two capacity functionals T_S^1 and T_S^2 such that

$$\overline{T}_S=[T_S^1, T_S^2].$$

$\overline{\Gamma}$ denote the class of all interval-valued capacity functionals.

(2) Let $f \in C_b(X)$. The Choquet integral of f with respect to an interval-valued capacity functional $\overline{T}=[T^1, T^2]$ is defined by

$$(C)\int f d\overline{T}=[(C)\int f dT^1, (C)\int f dT^2].$$

(3) The sequence of interval-valued capacity functionals $\overline{T}_n$ d_H-converges in the Choquet weak sense to the interval-valued capacity functional $\overline{T}$ in X, denoted by $\overline{T}_n \to_{d_H-C-W} \overline{T}$ as $n \to \infty$ if

$$(C)\int f d\overline{T}_n \to_{d_H} (C)\int f d\overline{T}, \forall f \in C_b(X)$$

as $n \to \infty$.

From Definition 3.1, Definition 2.5, and Choquet's theorem 2.2, we can obtain the following theorem.

Theorem 3.2 Every interval-valued probability measure $\overline{Q}=[Q_1, Q_2]$ on $(\Phi, B(\Phi))$ determine an interval-valued Choquet capacity functional $\overline{T}=[T_1, T_2]g$ on Ψ through the correspondences

(C_1) $\overline{T}(K)=\overline{Q}(\Phi_K)$, $\forall K \in \Psi$, and

(C_1) $\overline{T}(G)=\overline{Q}(\Phi_G)$, $\forall G \in O$.

Conversely, every Choquet capacity functional $\overline{T}$ on Ψ determines a unique probability measure $\overline{Q}$ on $(\Phi, B(\Phi))$ that satisfies above condition (C_1) and (C_2).

A compact set $K \in \Psi$ is called a continuity set for $\overline{T}$ if and only if

$\overline{T}(K)=\overline{T}(\mathrm{INT}(K))$,

where $\mathrm{INT}(K)$ is the interior of K. Thus a compact set K is $\overline{T}$-continuous if and only if Φ_K is $\overline{Q}$-continuous. Let

$$\mathfrak{I}_{\overline{T}}=\{K \in \Psi | \overline{T}(K)=\overline{T}(\mathrm{INT}(K))\}.$$

Now we investigate some properties of interval-valued Choquet capacity functionals. From Definition 2.1, Definition 3.1 and Lemma 1.2([2]), we obtain the following theorem:

Theorem 3.3 Let X be an LCSCH space. Then the following statements are equivalent:

(i) $\overline{Q_n} \Rightarrow_{d_H-W} \overline{Q}$, that is,

$$\lim_{n\to\infty} d_H(\overline{Q_n}(K), \overline{Q}(K))=0, \quad \forall K \in \Phi.$$

(ii) $d_H-\lim_{n\to\infty} \overline{T_n}(A)=\overline{T}(A), \forall A \in \mathfrak{I}_{\overline{T}}$.

Theorem 3.4 Let $\{\overline{T}_n=[T_n^1, T_n^2]\}$ be a sequence of interval-valued capacity functionals and $\overline{T}=[T^1, T^2]$ an interval-valued capacity functional. Then $\overline{T}_n \to_{d_H-C-W} \overline{T}$ as $n \to \infty$ if and only if $T_n^i \to_{C-W} T^i$ for $i=1,2$ as $n \to \infty$.

Proof. We note that

$$\lim_{n\to\infty} |(C)\int f dT_n^i-(C)\int f dT^i|=0,,$$

$\forall f \in C_b(X)$ if and only if

$$\lim_{n\to\infty} d_H((C)\int f d\overline{T_n}, (C)\int f d\overline{T})$$

$$\lim_{n\to\infty} \max_{i=1,2}\left\{|(C)\int f dT_n^i-(C)\int f dT^i|\right\}$$

$$=0.$$

Thus, the proof of the theorem holds.

From Definition 2.5, we obtain the following theorem:

Theorem 3.5 Let $\{\overline{Q}_n=[Q_n^1, Q_n^2]\}$ be a sequence of interval-valued probability measures and $\overline{Q}=[Q^1, Q^2]$ an interval-valued probability measure. Then

$\overline{Q}_n \rightarrow_{d_H - W} \overline{Q}$ as $n \rightarrow \infty$ if and only if $Q_n^i \rightarrow_W Q^i$ for $i=1,2$ as $n \rightarrow \infty$.

From Theorem 3.3 and Lemma 3.2([2]) and Lemma 3.3([2]), we obtain the following theroems:

Theorem 3.6 Let $\{\overline{T}_n = [T_n^1, T_n^2]\}$ be a sequence of interval-valued capacity functionals and $T=[T^1, T^2]$ an interval-valued capacity functional. If $\overline{T}_n \rightarrow_{d_H - C-W} \overline{T}$ as $n \rightarrow \infty$, then we have

(1) $d_H - \lim_{n\rightarrow\infty} \overline{T}_n(B) \le \overline{T}(B),\ \forall B \in \Phi.$

(2) $d_H - \lim_{n\rightarrow\infty} \overline{T}_n(B) \ge \overline{T}(B),\ \forall B \in O.$

Proof. (1) By Theorem 3.4, we have

$$T_n^i \rightarrow_{C-W} T^i \text{ for } i=1,2 \text{ as } n \rightarrow \infty.$$

Thus, by Lemma 3.2([2]), we obtain

$$d_H - \lim_{n\rightarrow\infty} sup\, T_n^i(B) \le T^i(B),\ \forall B \in \Phi.$$

for $i=1,2$. The definition of the order operation($\le$) imply that

$$\begin{aligned} & d_H - \lim_{n\rightarrow\infty} sup\, \overline{T}_n(B) \\ & \le [\lim_{n\rightarrow\infty} sup\, T_n^1(B), \lim_{n\rightarrow\infty} sup\, T_n^2(B)] \\ & = [T^1, T^2] = \overline{T}(B),\ \forall B \in \Phi. \end{aligned}$$

(2) The proof of (2) is similar to (1).

We also consider the concept of $\overline{T}$-tight as follows:

Definition 3.7 (1) A subset $B \subset X$ is called $\overline{T}$-continuous functional closed set if it is both T^1 and T^2-continuous functional closed set.

(2) For a given capacity $\overline{T}=[T^1, T^2]$, a family of capacity functionals $\overline{\Upsilon} \subset \overline{\Phi}$ is said to be $\overline{T}$-tight if (i) there exist $\Upsilon^i \subset \Gamma\ (i=1,2)$ such that $\overline{\Gamma} = \Upsilon^1 \times \Upsilon^2$ and

$$\Upsilon^i = \{T^i \in \Gamma \mid \overline{T} = [T^1, T^2] \in \overline{\Gamma}\}$$

is T^i-tight for $i=1,2$.

From Definition 3.7 and Theorem 3.4([2]), we obtain the following theorem:

Theorem 3.8 Let $\{\overline{T}_n = [T_n^1, T_n^2]\}$ be a sequence of interval-valued capacity functionals and $T=[T^1, T^2]$ an interval-valued capacity functional. If $\overline{T}_n \rightarrow_{d_H - C-W} \overline{T}$ as $n \rightarrow \infty$, then the sequence $\{\overline{T}_n\}$ is $\overline{T}$-tight.

From Theorem 3.5, Lemma 3.5([2]) and Lemma 3.6([2]), we obtain the following theorem:

Theorem 3.9 Let $\{\overline{Q}_n = [Q_n^1, Q_n^2]\}$ be a sequence of interval-valued probability measures and $\overline{Q} = [Q^1, Q^2]$ an interval-valued probability measure. If $\overline{Q}_n \rightarrow_{d_H - W} \overline{Q}$ as $n \rightarrow \infty$ and the sequence $\{\overline{T}_n\}$ is $\overline{T}$-tight, then we have

(1) $d_H - \lim_{n\rightarrow\infty} inf\ (C)\int d\overline{T}_n \ge (C)\int f d\overline{T},\quad \forall f \in C_b(X).$

(2) $d_H - \lim_{n\rightarrow\infty} sup\ (C)\int d\overline{T}_n \le (C)\int f d\overline{T},\quad \forall f \in C_b(X).$

From Theorem 3.3, Theorem 3.4 and Theorem 3.7([2]), we have the following theorem:

Theorem 3.10 If $\overline{Q}_n \rightarrow_{d_H - W} \overline{Q}$ as $n \rightarrow \infty$ and the sequence $\{\overline{T}_n\}$ is $\overline{T}$-tight, then

$$\overline{T}_n \rightarrow_{d_H - C-W} \overline{T} \text{ as } n \rightarrow \infty.$$

From Theorem 3.3, Theorem 3.4 and Lemma 3.8([2]), the converse of Theorem 3.10 holds.

Theorem 3.7 If $\overline{T}_n \rightarrow_{d_H - C-W} \overline{T}$ as $n \rightarrow \infty$, then we have

$$\overline{Q}_n \rightarrow_{d_H - W} \overline{Q} \text{ as } n \rightarrow \infty.$$

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