Backstepping and Partial Asymptotic Stabilization: Applications to Partial Attitude Control

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Abstract: In this paper, the problem of partial asymptotic stabilization of nonlinear control cascaded systems with integrators is considered. Unfortunately, many controllable control systems present an anomaly, which is the non complete stabilization via continuous pure-state feedback. This is due to Brockett necessary condition. In order to cope with this difficulty we propose in this work the partial asymptotic stabilization. For a given motion of a dynamical system, say $x(t,x_0,t_0)=(y(t,y_0,t_0),z(t,z_0,t_0))$, the partial stabilization is the qualitative behavior of the y-component of the motion (i.e., the asymptotic stabilization of the motion with respect to y) and the z-component converges, relative to the initial vector $x(t_0)=x_0=(y_0,z_0)$. In this work we present new results for the adding integrators for partial asymptotic stabilization. Two applications are given to illustrate our theoretical result. The first problem treated is the partial attitude control of the rigid spacecraft with two controls. The second problem treated is the partial orientation of the underactuated ship.

Keywords: Attitude stabilization, backstepping, linearization, Lyapunov-Malkin theorem, partial asymptotic stabilization, rigid spacecraft, ship.

1. INTRODUCTION

Control problems involving cascaded systems have recently attracted considerable attention in the control community. Mobile robots with steering wheels (unicycle), the rigid spacecraft, the ship and the submarine are examples of cascaded systems. Such systems have fewer actuators than the system degree of freedom and this class is also an example of underactuated systems. For such systems, the tools from linear control theory are not sufficient, and stabilization techniques need to be reconsidered, both at the control objective level and the control design techniques level. The stabilization problem of this class of systems is widely studied and remains one of the most challenging features as long as no special structure is assumed for the system to be stabilized.

In the late eighties, works on the Brockett's necessary condition [3] underlined the fact that a regular feedback may fail to stabilize regular systems. Based on this obstruction, a great effort has been provided for the design of time-varying control laws, [1,5,6,8,9,20,23,25,26,29,30], or discontinuous /discontinuous time-varying feedback laws [11,31]. In order to overcome the limitation imposed by Brockett's necessary condition, the conception of time-varying feedback laws is an important method for solving the stabilization problem. Nevertheless, the fact of introducing the time in these feedback laws produces "undesirable" oscillations of the system around of his equilibrium point, [1,5,20,23,25,26,29, 30]. In addition, and in many situations, the stabilizing feedback laws are continuous in the equilibrium point, however, they are not differentiable at this point. For instance, we can see the stabilization method by homogenous and average feedback laws [20,21,23,24]. This seems to be the major drawback.

To solve the stabilization problem of all controllable systems that do not satisfy the Brockett's condition, and to overcome the drawback of the time-varying-periodic feedback laws, we propose the partial asymptotic stabilization method. The partial asymptotic stabilization considered in this paper is the stabilization with respect to the major components of the system and the rest converges to some position which depends on the initial conditions.

This theory is a natural extension of the classical concept of stabilizability in Lyapunov sense. Stability (respectively, stabilizability) with respect to part of the state also called "partial stability (respectively, partial stabilizability)" has been intensively studied within

Manuscript received June 9, 2007; revised February 23, 2008; accepted June 16, 2008. Recommended by Editorial Board member Gang Tao under the direction of Editor Hyun Seok Yang. This work is supported by the LIM Laboratory of Polytechnic School of Tunisia. The author would like to acknowledge Pr. A. Abichou and Pr. J.-M. Coron for several interesting discussions and helpful comments. We also thanks the reviewers for useful comments and suggestions.

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last 50 years. Basic results in this field belong to Rumyantsev, [13,14,28,33], the founder of the theory of partial stability for systems of ordinary differential equations with continuous right side, and have demonstrated the applicability of this result to problems of stability of more general models of distribute-parameter systems. Subsequently, a large number of researchers have contributed to the development of theory and methods for studying partial stability and stabilization and resolved several important problems; see, for instance, [33].

Our partial asymptotic stability (respectively, stabilizability) definition is different from the definitions given and used in [13,14,33,37] in which the authors have been occupied with the part of the system and suppose that the rest is bounded. The definition used in this paper takes into consideration the complete stability of the system, the asymptotic stability with respect to part of the sate while the rest converges to some position, which depends on the initial conditions. The latter property is interesting since it removes the possible oscillations of the system. This partial asymptotic stability is applied in many engineering fields as the stabilization problems of rigid spacecraft with two controls, the ship [15], the underwater vehicle [16] and the nonholonomic systems. The aim of this paper is to extend the well known backstepping theorem to the case of partial stabilizability of nonlinear control systems. We have shown that if the original system is partially stabilizable then the cascade systems with integrators inherit the same property. To this end, we have developed the inversion Lyapunov theorem for the partial asymptotic stabilizability. The theoretical result is applied to solve two problems. The first is the partial stabilization of the rigid spacecraft with two controls, where we have improved Zuyev's [37] result that the velocity ω_3 of the third axes converges by using smooth state feedback laws. The second problem treated is the attitude of an underactuated ship. We have constructed two smooth feedback laws that stabilize asymptotically five components and the sixth converges.

The paper is structured as follows: The next section deals with some mathematical preliminaries. In particular, the inversion of the Lyapunov theorem of the partial asymptotic stabilizability is proved. The backstepping techniques and partial stabilizability is treated in Section 3. In Section 4, we give two applications for the backstepping result. Numerical simulations are given to validate our results. The conclusion is presented in Section 5.

2. MATHEMATICAL PRELIMINARIES

In this section, the concept of partial asymptotic stability and partial asymptotic stabilizability and some of their results will be reviewed in order to build the mathematical background for the stability proofs.

Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}^n$ denote the set $n\times 1$ real column vectors, $|.|$ denote the Euclidean vector norm and $'$ is the symbol of transposition.

$\mathcal{K}$ the Hahn space defined by $\alpha:[0,+\infty)\to[0,+\infty)$, α continuous, strictly increasing and $\alpha(0)=0$.

Consider the following system:

$$\dot{x}_1 = f_1(x_1,x_2),\ \ \dot{x}_2 = f_2(x_1,x_2), \tag{1}$$

where $f=(f_1,f_2)$ is supposed to be class $C^\infty(\mathbb{R}^n)$, $x_1\in\mathbb{R}^p$, $x_2\in\mathbb{R}^{n-p}$ and p integer such that $0<p\le n$.

We assume that

$$f_1(0,x_2)=0 \text{ and } f_2(0,x_2)=0,\ \forall x_2\in\mathbb{R}^{n-p}. \tag{2}$$

Definition 1 (Partial asymptotic stability): The system (1) is said to be partially asymptotically stable if the following proprieties are satisfied:

a) The equilibrium $0\in\mathbb{R}^n$ of (1) is Lyapunov stable.

b) The system (1) is asymptotically stable with respect to x_1 and x_2 converges:

$$\exists r>0:(|x_1(0)|+|x_2(0)|\le r)\Rightarrow\begin{cases}\lim\limits_{t\to+\infty} x_1(t)=0,\\ \lim\limits_{t\to+\infty} x_2(t)=a,\end{cases} \tag{3}$$

where a is a constant vector depending on the initial conditions.

Now, we consider the nonlinear finite-dimensional control systems of the following form

$$\dot{x}_1 = f_1(x_1,x_2,u),\ \ \dot{x}_2 = f_2(x_1,x_2,u), \tag{4}$$

where $x=(x_1,x_2)\in\mathbb{R}^p\times\mathbb{R}^{n-p}$ is the state, and $u(t)\in\mathbb{R}^m$ is the control.

Definition 2 (Partial asymptotic stabilizability): The system (4) is said to be partially asymptotic stabilizable if there exists a continuous function $\phi:\mathbb{R}^p\times\mathbb{R}^{n-p}\to\mathbb{R}^m$, $\phi(0,x_2)=0$ such that the system in the closed-loop:

$$\dot{x}_1 = f_1(x_1,x_2,\phi(x_1,x_2)),\ \ \dot{x}_2 = f_2(x_1,x_2,\phi(x_1,x_2)), \tag{5}$$

is partially asymptotically stable in the sense of Definition 1.

Having introduced the concept of partial asymptotic stabilizability, it is now possible to state some important results of stability with respect to part.

Thanks to recent contribution of [19], we announce the following proposition, which gives a converse

Lyapunov theorem for the stabilization with respect to part of variables. This result extends the Kurzweil's theorem [7,18]. This result is also due to Rumyantsev and Oziraner cited in [38].

Proposition 1: If the system (4) is partially asymptotically stabilizable, then there exists a continuous feedback law $u(x_1,x_2)$ such that $u(0,x_2)=0$ and a C^1 function $V:\mathbb{R}^p\times\mathbb{R}^{n-p}\to\mathbb{R}$ such that

1. $V(x_1,x_2)$ is positive definite with respect to x_1.
2. $\dot{V}(x_1,x_2)$ is definite negative with respect to x_1.

Proof: Assuming that (4) is partially asymptotically stabilizable in the sense of Definition 2, then there exists a continuous feedback law $u(x_1,x_2)$ such that $u(0,x_2)=0$ and the system (4) in closed loop is partially asymptotically stable. Thus, the system is asymptotically stable with respect to x_1 and x_2 converges. Then by a result due to Rumyantsev and Oziraner [38], the system (4) admits a C^1 Lyapunov function V positive definite with respect to x_1 such that $\dot{V}$ is negative definite with respect to x_1.

3. BACKSTEPPING AND PARTIAL ASYMPTOTIC STABILIZABILITY

In this section, we give an extension of the backstepping techniques of Coron-Praly [10] to partial stabilizability theory.

Theorem 1: We suppose that

$$\begin{cases}\dot{x}_1 &= f_1(x_1,x_2,u)\\ \dot{x}_2 &= f_2(x_1,x_2,u)\end{cases} \tag{6}$$

is partially stabilizable by static state feedback of C^r, $r\geq 1$. Then the augmented cascaded systems with integrators

$$\begin{cases}\dot{x}_1 &= f_1(x_1,x_2,y)\\ \dot{x}_2 &= f_2(x_1,x_2,y)\\ \dot{y} &= u\end{cases} \tag{7}$$

is asymptotic stabilizable with respect to (x_1,y) by static stationary feedback of class C^{r-1}.

Proof: We assume that the system (6) is partially asymptotically stabilizable by state feedback of C^r, then from Definition 2 there exists a C^r map $\phi:\mathbb{R}^p\times\mathbb{R}^{n-p}\to\mathbb{R}^m$, $\phi(0,x_2)=0$, $\forall x_2\in\mathbb{R}^{n-p}$ such that, on closed-loop the system

$$\dot{x}_1=f_1(x_1,x_2,\phi(x_1,x_2)), \dot{x}_2=f_2(x_1,x_2,\phi(x_1,x_2)), \tag{8}$$

is asymptotically stable with respect to x_1.

By using the Proposition 1, there exists a C^1 Lyapunov function V and three $\mathcal{K}$-functions α_1,α_2 and α_3 such that

$$\alpha_2(|x_1|)\leq V(x_1,x_2)\leq\alpha_1(|x_1|), \tag{9}$$

and

$$\dot{V}(x_1,x_2)-\alpha_3(|x_1|). \tag{10}$$

Let the following Lyapunov candidate function

$$W(x_1,x_2,y)=V(x_1,x_2)+\frac{1}{2}|y-\phi(x_1,x_2)|^2. \tag{11}$$

We derive W along the trajectory of system (7), we obtain

$$\begin{aligned}\dot{W}(x,y)&=\frac{\partial V}{\partial x_1}f_1(x,y)+\frac{\partial V}{\partial x_2}f_2(x,y)\\ &+<y-\phi(x),u-\frac{\partial\phi}{\partial x_1}f_1(x,y)-\frac{\partial\phi}{\partial x_2}f_2(x,y)>.\end{aligned} \tag{12}$$

Since $f\in C^r(\mathbb{R}^n\times\mathbb{R}^m)$, then the components f_1 and f_2 are also C^r, by Taylor's expansion we obtain:

$$f_1(x,y)=f_1(x,\phi(x))+G_1(x,y,\phi(x))(y-\phi(x)),$$
$$f_2(x,y)=f_2(x,\phi(x))+G_2(x,y,\phi(x))(y-\phi(x)),$$

where G_1 is $p\times m$ and G_2 is $(n-p)\times m$ are two matrix functions of C^{r-1}.

With the preliminary feedback

$$\begin{aligned}\bar{u}(x,y)&=\frac{\partial\phi}{\partial x_1}f_1(x,y)+\frac{\partial\phi}{\partial x_2}f_2(x,y)\\ &-G_1^T(x,y,\phi(x))\frac{\partial V}{\partial x_1}\\ &-G_2^T(x,y,\phi(x))\frac{\partial V}{\partial x_2}+\phi(x)-y,\end{aligned} \tag{13}$$

we obtain $\forall(x,y)\in\mathbb{R}^n\times\mathbb{R}^m$

$$\dot{W}(x,y)=\dot{V}(x)-|y-\phi(x)|^2 \tag{14}$$

we use (10) and (14) we obtain:

$$\dot{W}(x,y)=0\Leftrightarrow(x_1,y)=(0,\phi(0,x_2))=(0,0). \tag{15}$$

Then W is a candidate Lyapunov function, we conclude that by Risito-Rumyantsev theorem (see for instance Vorotnikov [33]) that $(x_1,y)=(0,0)$ is asymptotically stable.

Proposition 2: Considering the system (7). If there exists continuous function $\psi:\mathbb{R}^n\to(0,+\infty)$, strictly positive such that for each solution $(x(t), y(t))$, $|(x(0), y(0))| < \eta$, we get:

$$\dot{V}(x) - |y - \phi(x)|^2 + |f_2(x,y)| \leq |y - \phi(x)|^2 \psi(x,y). \tag{16}$$

Then with the state feedback law

$$u(x,y) = \bar{u}(x,y) - \psi(x,y)(y - \phi(x)), \tag{17}$$

the system (7) is partially asymptotically stable in the sense that $(x_1, y) = (0,0)$ is asymptotically stable and the state x_2 converges.

Proof: Let the function T defined by

$$T(x,y,t) = W(x,y) + \int_0^t |f_2(x,y)(s)| ds, \tag{18}$$

where W is the Lyapunov function introduced in (11). It is clear that T is a positive function.
With the new feedback (17), the system (7) is asymptotically stable with respect to (x_1, y).
The derivation of T along the system (7) and with the new feedback (17), we obtain:

$$\dot{T} = \dot{V}(x) - |y - \phi(x)|^2 - |y - \phi(x)|^2 \psi(x,y) + |f_2(x,y)|. \tag{19}$$

From the assumption (16), we have

$$\dot{T} \leq 0.$$

Thus, T admits finite limits as t tends to $+\infty$, let be

$$\lim_{t \to +\infty} T(x,y,t) = T_\infty.$$

Since T is continuous with respect to all variables and admits a finite limits, then it is bounded. Consequently, the function g defined by

$$g(t) = \int_0^t |f_2(x_1, x_2, y)(s)| ds \tag{20}$$

is also bounded. Since $g'(t) = |f_2(x_1, x_2, y)(t)| \geq 0$ then we obtain the convergence of the integral $\int_0^{+\infty} |f_2(x_1, x_2, y)(s)| ds < +\infty$. Therefore, by the Cauchy condition, we deduce the convergence of the solution $x_2(t)$ starting from the η neighborhood of the origin of the system (7).

Remark 1: Let $(x_2^i)_{1 \leq i \leq n-p}$ the components of x_2, then:

$$\| x_2 \|_\infty \leq |x_2(0)| + W(x(0), y(0)), \tag{21}$$

where $\|x_2\|_\infty := sup\{|x_2^i|, 1 \leq i \leq n-p\}$.
Indeed, with the assumption (16) and with the help of the state feedback $u(x,y)$ introduced in (17), the function T is decreasing, then

$$T(t) := T(x(t), y(t), t) \leq T(0). \tag{22}$$

Also, we have

$$\int_0^t |f_2(x(s), y(s)| ds \leq T(t). \tag{23}$$

The solution x_2 can be written as $x_2(t) = x_2(0) + \int_0^t f_2(x(s), y(s)) ds$, then

$$|x_2(t)| \leq |x_2(0)| + \int_0^t |f_2(x(s), y(s))| ds. \tag{24}$$

From (22), (23), and (24), we obtain

$$\| x_2 \|_\infty \leq |x_2(0)| + W(x(0), y(0)).$$

Let $a = \lim_{t \to +\infty} x_2(t)$, then by using the latter inequality, we have

$$\| a \|_\infty \leq |x_2(0)| + W(x(0), y(0)). \tag{25}$$

Definition 3: The system (1) is called partially exponentially stable if

1. The system (1) is stable,
2. there exists $r > 0$ such that, if $|x_1(0)| + |x_2(0)| < r$, then there exists two reals $c > 0$ and $k > 0$ such that $|x_1(t)| \leq c|(x_1(0), x_2(0))| e^{-kt}$,
3. there exists $r > 0$ such that, if $|x_1(0)| + |x_2(0)| < r$, there exists a depending on initial conditions such that $\lim_{t \to +\infty} x_2(t) = a(x_1(0), x_2(0))$.

Proposition 3: Considering the control system (6). Assuming that there exists $u \in C^r$ satisfying $u(0, x_2) = 0$ such that:

1. there exists $r > 0$ such that, if $|x_1(0)| + |x_2(0)| < r$, then there exist two reals $c > 0$ and $k > 0$ such that $|x_1(t)| \leq c|(x_1(0), x_2(0))| e^{-kt}$,
2. in closed-loop the matrix $\frac{\partial f_1}{\partial x_1}$ is bounded.
3. the function f_2 verifies $|f_2(x_1, x_2, u)| \leq a|x_1| + b|u|$ where $a, b > 0$.

Then the extended system

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2, y) \\ \dot{y} = u \\ \dot{x}_2 = f_2(x_1, x_2, y) \end{cases} \tag{26}$$

is partially exponentially stabilizable by smooth feedback laws of class C^{k-1} in the following sense: (x_1, y) is asymptotically stabilizable and x_2 converges.

Proof: We assume that the Assumptions 1 and 2 hold, then by Lyapunov inversion Theorem (Khalil [18], Theorem 4.14, p. 162), there exists a C^∞ Lyapunov function V such that:

1. there exist a positive constants c_1 and c_2 such that

$$c_1 |x_1|^2 \le V(x_1, x_2) \le c_2 |x_1|^2, \tag{27}$$

2. there exists a positive constant $c_3 > 0$ such that

$$\dot{V} \le -c_3 |x_1|^2, \tag{28}$$

By using the assertions 1 and 2, it is easy to obtain

$$\dot{V} \le -\frac{c_3}{c_1} V. \tag{29}$$

For the extended system (26), we consider the Lyapunov function defined in (11):

$$W = V(x_1, x_2) + \frac{1}{2} | y - \phi(x)|^2,$$

with the action of the feedback control $\bar{u}(x, y)$ defined by (13) we obtain the following equation

$$\dot{W} = \dot{V} - | y - \phi(x)|^2, \tag{30}$$

we deduce from (27) and (28)

$$\dot{W} \le -min(\frac{1}{2}, \frac{c_3}{c_1}) W, \tag{31}$$

which implies the exponential stabilizability of the system (26) with respect to (x_1, y).
Therefore, there exists $r > 0$, $\mu > 0$, and $c > 0$ such that from $|(x(0), y(0))| < r$. It follows that

$$|x_1(t)| + |y(t)| \le c |(x(0), y(0))| e^{-\mu t},$$

hence

$$|x_1(t)|, |y(t)| \le c |(x(0), y(0))| e^{-\mu t}. \tag{32}$$

From the assumption 3, we deduce

$$\begin{aligned} |f_2(x_1, x_2, y)| &\le a |x_1| + b |y| \\ &\le c(a+b) |(x(0), y(0))| e^{-\mu t}. \end{aligned} \tag{33}$$

The inequality (33) shows that the vector field f_2 of the augmented system (26) is Lebesgue integrable, therefore the state x_2 converges to constant vector depending on initial conditions.

Remark 2: 1. Let be $x(t)$ a function defined on $[0, +\infty)$, we assume that $x \in C^1$. If $\lim_{t \to +\infty} x(t) = l$ and $\lim_{t \to +\infty} \dot{x}(t) = l'$ then $l' = 0$.

2. If the system (7) is partially asymptotically stabilizable in the sense of Definition 2, then this does not implies that the reduced system (6) is partially asymptotically stabilizable in the sense of Definition 2. Indeed, we consider the system [10]:

Let, for a constant c and $n \ge 2$ a function $f : \mathbb{R}^n \times \mathbb{R} \simeq \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^n$ defined by

$$\begin{aligned} f(x, y) = -[(|x_1|^2 + x_2^2)^3 \\ - c^2 (y^3 - |x_1|^2 y + x_2^3)^2] x, \end{aligned} \tag{34}$$

where $(x, y) = (x_1, x_2, y) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}$. Coron *et al.* [10] showed that for c large enough, the system $\dot{x} = f(x, u)$ is not locally asymptotically stabilizable but the system $\dot{x} = f(x, y)$, $\dot{y} = u$ is globally asymptotically stabilizable. Consequently, the latter system is partially asymptotically stabilizable.

We assume that $x = (x_1, x_2) \in \mathbb{R}^{n-1} \times \mathbb{R}$, and the system:

$$\begin{aligned} \dot{x}_1 &= -[(|x_1|^2 + x_2^2)^3 - c^2 (u^3 - |x_1|^2 u + x_2^3)^2] x_1 \\ \dot{x}_2 &= -[(|x_1|^2 + x_2^2)^3 - c^2 (u^3 - |x_1|^2 u) + x_2^3)^2] x_2 \end{aligned} \tag{35}$$

is partially asymptotically stabilizable in the sense of definition 2. Then there exists a continuous feedback $u(x)$, such that $u(0, x_2) = 0$, and the system (35) in closed loop is completely stable, asymptotically stable with respect to x_1, and x_2 converges.

Let $a = \lim_{t \to +\infty} x_2(t)$.

We have

$$\begin{aligned} \dot{x}_2 = -[(|x_1|^2 + x_2^2)^3 - c^2 (u(x)^3 \\ - |x_1|^2 u(x) + x_2^3)^2] x_2. \end{aligned}$$

Then $\lim_{t \to +\infty} \dot{x}_2(t) = a^7 - c^2 a^7 = a^7 (1 - c^2)$, therefore for c large enough ($c^2 > 1$) we have necessarily $a = 0$, i.e., the system (34) is asymptotically stabilizable, which is contradiction (remark 1).

In the following sections, we give two applications of our theoretical results, the first one addressed to the partial asymptotic stabilization of the rigid spacecraft and the second is interested in the ship problem.

4. APPLICATIONS: PARTIAL ATTITUDE CONTROL

4.1. Partial stabilization of rigid spacecraft with two controls

The attitude control of rigid spacecraft with only two controllers has been the subject of numerous research articles in the literature. This type of system is used to illustrate several aspects of nonlinear controllability. Let us mention the results of Bonnard [2] and Crouch [12] proving that the system is globally controllable in large time, Keraï [17] proving that the system satisfies Sussmann's condition, and so is small time locally controllable. The attitude stabilization is studied by the work of Byrnes and Isidori [4] have shown that all rigid satellite with (one or) two independent actuators cannot be locally asymptotically stabilized using continuously differentiable static or dynamic state feedback. However stabilization about an attractor is possible, inducing a closed loop system with trajectories tending to a revolute motion about a principal axis.

Also, the same problem is studied by Morin *et al.* [25] where the concept of time-varying feedback laws stabilizing locally the system is explicitly derived by using center manifold theory combined with averaging techniques. Coron and Keraï established the time-varying homogenous and periodic feedback laws stabilizing the satellite with two controllers. The method described by Morin *et al.* [24] that studied the attitude of underactuated rigid spacecraft was locally, exponential stabilized with respect to a given dilation. The controllers was periodic, time-varying and non-differentiable at the origin and the construction relied on the proprieties of homogenous systems.

To get around the problem of impossibility to stabilize many controllable systems by continuous feedback laws, the strategies of asymptotic stabilization by means of continuous time-varying feedback laws has been proposed. Nevertheless, the fact to introduce the time in these feedback laws can produce "undesirable" oscillations of the system (see for instance, Morin *et al.* [22,25], [23,26,29,30]). In order to overcome these difficulties, we present the partial attitude control method.

Many alternative coordinate choices exist for the description of the rotational motion of a rigid body. Not all choices are equivalent with respect to their domain of validity for accurate attitude representation or the ease they offer in the control design process and the final properties of the proposed control law. Furthermore, the Eulerian angles, the Cayley-Rodrigues parameters are examples of parameterizations of $SO(3)$. Two dimensional parameterizations introduce necessarily a singularity, as it is not possible to find a globally diffeomorphic transformation between $SO(3)$ (which is compact) and the euclidian space $\mathbb{R}^3$ (which is not). In this context we have chosen the Euler-Poisson parameterizations for describing the rigid spacecraft [32,33,37]. In this case we have shown that we can solve this problem by C^∞ continuous feedback controller functions only on the state. This work improves Zuyev's [37] result, and proves that the velocity ω_3 converges.

Equation of motion: We consider the Euler-Poisson parameterizations ([32,33,37]) which describe the motion of the rigid-body, it is written in the following form:

$$\begin{cases} \dot\omega_1 &= u_1 \\ \dot\omega_2 &= u_2 \\ \dot\omega_3 &= \omega_1\omega_2 \\ \dot\nu_1 &= \omega_3\nu_2 - \omega_2\nu_3 \\ \dot\nu_2 &= \omega_1\nu_3 - \omega_3\nu_1 \\ \dot\nu_3 &= \omega_2\nu_1 - \omega_1\nu_2. \end{cases} \tag{36}$$

We will be interested in to stabilizing partially the equilibrium $\omega_1 = \omega_2 = \omega_3 = 0, \nu_1 = \nu_2 = 0, \nu_3 = 1$. We notice that $\nu_1^2 + \nu_2^2 + \nu_3^2$ =constant. Then we can suppose that:

$$\nu_1^2 + \nu_2^2 + \nu_3^2 = 1.$$

We choose the hemisphere $\nu_3 > 0$, the equality $\nu_1^2 + \nu_2^2 + \nu_3^2 = 1$ implies:

$$\nu_3 = \sqrt{1 - (\nu_1^2 + \nu_2^2)}.$$

We consider the unit open ball $B(0,1)$ and the function $\sigma(\nu_1,\nu_2) = \sqrt{1-(\nu_1^2+\nu_2^2)}$.

σ is smooth on the open ball $B(0,1)$, we develop σ in the first order by Taylor's formula in the neighborhood of $(0,0)$ we obtain:

$$\sigma(\nu_1,\nu_2) = 1 + g(\nu_1,\nu_2),$$

where the function g is smooth and satisfies

$$g(0,0) = g'(\nu_1,\nu_2)(0,0) = 0.$$

To simplify the work we use the backstepping theorem, then it is sufficient to study the reduced system given by:

$$\begin{cases} \dot\omega_3 &= u_1u_2 \\ \dot\nu_1 &= \omega_3\nu_2 - u_2\nu_3 \\ \dot\nu_2 &= u_1\nu_3 - \omega_3\nu_1 \\ \dot\nu_3 &= u_2\nu_1 - u_1\nu_2. \end{cases} \tag{37}$$

We replace ν_3 by $1 + g(\nu_1,\nu_2)$ in the system (37), we obtain:

$$\begin{cases} \dot\omega_3 &= u_1u_2 \\ \dot\nu_1 &= -u_2 - u_2\, g(\nu_1,\nu_2) + \omega_3\nu_2 \\ \dot\nu_2 &= u_1 + u_1\, g(\nu_1,\nu_2) - \omega_3\nu_1 \\ \dot\nu_3 &= u_2\nu_1 - u_1\nu_2. \end{cases} \tag{38}$$

In order to study the partial asymptotic stabilizability of the reduced system, we need the following theorem

Theorem 2 (Lyapunov-Malkin [36])**:** Consider the system of differential equations:

$$\begin{cases} \dot{x}_1 &= A x_1 + R(x_1,x_2) \\ \dot{x}_2 &= S(x_1,x_2), \end{cases} \tag{39}$$

where $x_1 \in \mathbb{R}^p$, $x_2 \in \mathbb{R}^{n-p}$, A is a $p \times p$ -matrix, and $R(x_1,x_2)$, $S(x_1,x_2)$ represent higher order nonlinear terms. If all eigenvalues of the matrix A have negative real parts, and, $R(x_1,x_2)$, $S(x_1,x_2)$ vanish when $x_1 = 0$, then the solution $x_1 = 0$, $x_2 = 0$ of this system is stable with respect to $(x_1,x_2)'$, and asymptotically stable with respect to x_1. If $(x_1(0),x_2(0))'$ is small enough, then there is a constant n-vector c (depending on the initial conditions) such that

$$\lim_{t\to+\infty} x_1(t) = 0, \ \lim_{t\to+\infty} x_2(t) = c.$$

Proposition 4: Let $\alpha > 0$, we choose the feedbacks u_1 and u_2 with this manner:

$$u_1 = -\alpha \nu_2 + \nu_2 \omega_3, \ \ u_2 = \alpha \nu_1 - \nu_1 \omega_3.$$

Then:

1. the system (38) is completely stable,
2. the system (38) is exponentially stable with respect to (ν_1,ν_2), the angular velocity ω_3 converges to ω_3^∞ and the point ν_3 converges to 1.

Proof: In closed loop the system (38) can be written in Lyapunov-Malkin form [36], to this end we put the system (38) in the following form:

$$\begin{cases} \dot{x}_1 &= A x_1 + R(x_1,x_2) \\ \dot{x}_2 &= S(x_1,x_2), \end{cases} \tag{40}$$

where $x_1 = (\nu_1,\nu_2)$, $x_2 = (\omega_3,\nu_3)$,

$$S(x_1,x_2) = \begin{pmatrix} (-\alpha \nu_2 + \nu_2 \omega_3)(\alpha \nu_1 - \nu_1 \omega_3) \\ \nu_1(\alpha \nu_2 - \nu_2 \omega_3) - \nu_2(-\alpha \nu_1 + \nu_1 \omega_3) \end{pmatrix},$$

$$\dot{x}_1 = \begin{pmatrix} \dot{\nu}_1 \\ \dot{\nu}_2 \end{pmatrix} = \begin{pmatrix} -\alpha & 0 \\ 0 & -\alpha \end{pmatrix}\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} + R(x_1,x_2)$$

and

$$R(x_1,x_2) = \begin{pmatrix} \nu_1 \omega_3 - (\alpha \nu_1 - \nu_1 \omega_3) g(\nu_1,\nu_2) + \nu_2 \omega_3 \\ \nu_2 \omega_3 + (-\alpha \nu_2 + \nu_2 \omega_3) g(\nu_1,\nu_2) - \nu_1 \omega_3 \end{pmatrix}.$$

It is clair that the matrix

$$A = \begin{pmatrix} -\alpha & 0 \\ 0 & -\alpha \end{pmatrix}$$

has $-\alpha < 0$ as eigenvalues. Besides the functions $R(x_1,x_2)$ and $S(x_1,x_2)$ having nonlinear terms and vanishing together at $(0,0,\nu_3,\omega_3)'$ and at $(0,0,0,0)'$.

The Lyapunov-Malkin Theorem allows us to conclude.

Remark 3: By using the Lyapunov-Malkin theorem, we can obtain the same results of Proposition 4 by linear feedback laws as follows $u_1 = -\alpha \nu_2$, $u_2 = \alpha \nu_1$, $\alpha > 0$.

In the following proposition we give explicitly the feedback controller that achieves the partial asymptotic stabilization of the system (36).

Proposition 5: With feedback controllers

$$\begin{aligned} \phi_1(x) &= -k(\omega_1 - u_1(x)), \\ \phi_2(x) &= -k(\omega_2 - u_2(x)), \end{aligned} \tag{41}$$

$u_1(x)$ and $u_2(x)$ are given in the Proposition 4, $k, \alpha > 0$ and $x = (\omega_i, \nu_i)$, $i = 1,2,3$.

The system (36) is partially asymptotically stable, more precisely: $(\omega_1,\omega_2,\nu_1,\nu_2,\nu_3)' = (0,0,0,0,1)'$ is asymptotically stable and ω_3 converges.

Proof: With the feedback laws given in (41), the system (36) in closed loop is given by:

$$\begin{cases} \dot{\omega}_1 &= -k\omega_1 - k\alpha\nu_2 + k\nu_2\omega_3 \\ \dot{\omega}_2 &= -k\omega_2 + k\alpha\nu_1 - k\nu_1\omega_3 \\ \dot{\omega}_3 &= \omega_1\omega_2 \\ \dot{\nu}_1 &= -\omega_2 + \omega_3\nu_2 - \omega_2 g(\nu_1,\nu_2) \\ \dot{\nu}_2 &= \omega_1 + \omega_1 g(\nu_1,\nu_2) - \omega_3\nu_1 \\ \dot{\nu}_3 &= \omega_2\nu_1 - \omega_1\nu_2. \end{cases} \tag{42}$$

We can put the system (42) in the following form

$$\begin{cases} \dot{x}_1 &= A x_1 + R(x_1,x_2) \\ \dot{x}_2 &= S(x_1,x_2), \end{cases} \tag{43}$$

where $x_1 = (\omega_1,\omega_2,\nu_1,\nu_2)'$, $x_2 = (\omega_3,\nu_3)$,

$$A = \begin{pmatrix} -k & 0 & 0 & -k\alpha \\ 0 & -k & k\alpha & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$R(x_1,x_2) = \begin{pmatrix} k\nu_2\omega_3 \\ -k\nu_1\omega_3 \\ \omega_3\nu_2 - \omega_2 g(\nu_1,\nu_2) \\ -\omega_3\nu_1 + \omega_1 g(\nu_1,\nu_2) \end{pmatrix},$$

and

$$S(x_1,x_2) = \begin{pmatrix} \omega_1\omega_2 \\ \omega_2\nu_1 - \omega_1\nu_2 \end{pmatrix}.$$

The matrix A has the following characteristic polynomial $P(r) = (r(k+r) + k\alpha)^2$ which has two distinct roots r_1 and r_2.

For all signs of the quantity $\Delta = k^2 - 4k\alpha$, the roots r_1 and r_2 have negative real parts. Indeed,

- If $k^2 - 4k\alpha < 0$, then the solutions of P given by the complex form are:

$$r_j = \frac{-k \pm i\sqrt{-k^2 + 4k\alpha}}{2}, \quad j = 1,2, \tag{44}$$

it is clear that the roots r_1 and r_2 are a negative real parts.

- If $k^2 - 4k\alpha > 0$, in this case the solutions of P are given by the following expression:

$$r_j = \frac{-k \pm \sqrt{k^2 - 4k\alpha}}{2}, \quad j = 1,2. \tag{45}$$

The constants k and α are positive then, because we have the following inequality $k^2 - 4k\alpha < k^2$, the roots r_1 and r_2 are negative.

Therefore, the matrix A is Hurwitz. In addition, the functions R and S do not contain a constant, or linear terms and vanish together in the equilibrium point $(0, x_2)'$. Thus, by Lyapunov-Malkin Theorem, the system (42) is asymptotically stable with respect to x_1, stable with respect to $(x_1, x_2)'$ and the vector x_2 converges to constant vector depending on the initial conditions; then the angular velocity ω_3 converges.

The state ν_3 satisfies the equality $\nu_1^2 + \nu_2^2 + \nu_3^2 = 1$, and because $\nu_3 > 0$, then $\lim_{t \to +\infty} \nu_3(t) = 1$.

Remark 4: 1. The feedback laws given in (41) are $C^\infty(\mathbb{R}^6, \mathbb{R})$.

2. In [37], Zuyev proposes the following feedback laws:

$$u_1 = \omega_2\omega_3 - \frac{1}{A_1}\nu_2\nu_3 - \left(\frac{|A_1 - A_2|}{2A_2}|\omega_3| + \varepsilon A_1\right)\omega_1 \tag{46}$$

$$u_2 = \omega_1\omega_3 - \frac{1}{A_2}\nu_1\nu_3 - \left(\frac{|A_1 - A_2|}{2A_1}|\omega_3| + \varepsilon A_2\right)\omega_2 \tag{47}$$

with $\varepsilon > 0$ an arbitrary real. With these feedbacks, the system (36) is asymptotically stable with respect to $(\omega_1, \omega_2, \nu_1, \nu_2)'$, and bounded with respect to (ω_3, ν_3).

A particular attention is paid to the following points:

- Zuyev's feedbacks are only continuous.
- No arguments permit us to say that the feedback laws (46) and (47) ensure the convergence of the state ω_3 which can be oscillatory. Consequently, these feedback laws do not answer the approach of our partial asymptotic stabilization proposed (Definition 2). On the other hand, our feedback suggested in (41) achieves our objective of partial asymptotic stabilization, and improves the Zuyev's result.

Numerical Simulations: The performances of our feedback laws are tested by numerical simulations on the nonlinear model of satellite. These simulations are presented in Figs. 1, 2, and 3.

The feedback laws applied to the system are:

$$\phi_1 = -10(\omega_1 - u_1(x)),$$
$$\phi_2 = -10(\omega_2 - u_2(x)),$$
$$u_1 = 10\nu_2 + \nu_2\omega_3,$$
$$u_2 = 10\nu_1 - \nu_1\omega_3.$$

It is clear that these feedback laws make partially asymptotically stable the system (36). The advantage of this method resides in obtaining a static stabilization. Moreover, the state variable, which is not "controllable" converges, which makes it possible to

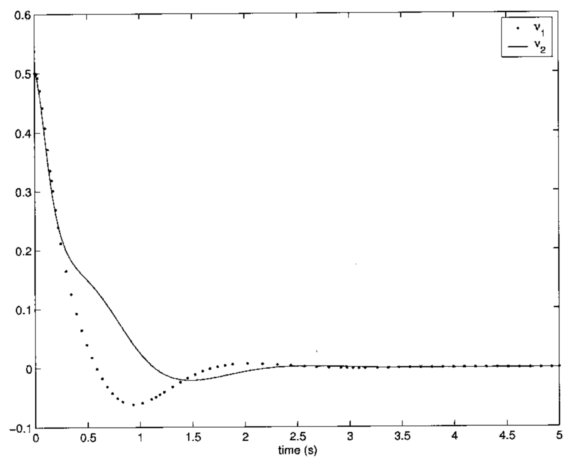

Fig. 1. Comportment of the velocities ν_1, ν_2.

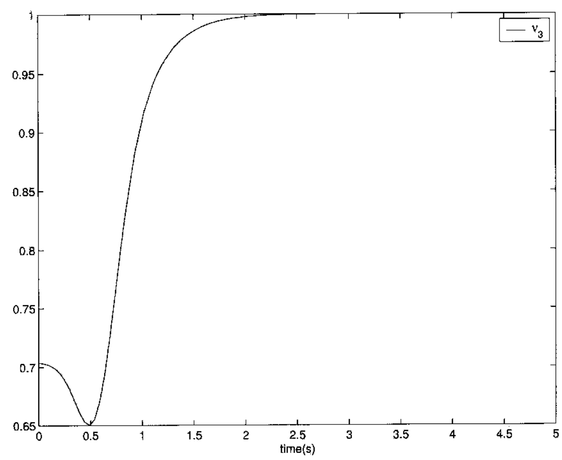

Fig. 2. Comportment of the velocity ν_3.

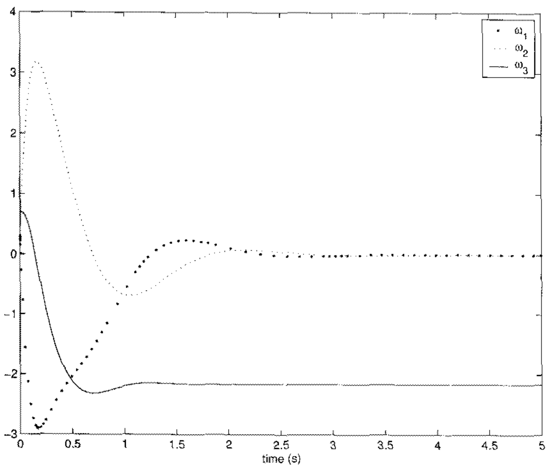

Fig. 3. Trajectories of the angular velocities ω_1, ω_2 and ω_3.

avoid the oscillation of the system in the neighborhood of the equilibrium point.

4.2. Partial stabilization of the ship

This subsection is devoted to study the underactuated ship, it was shown by Pettersen and Egeland [26] that no continuous or discontinuous static-state feedback law exists which makes the origin of the ship system asymptotically stable, because the latter system does not satisfy Brockett's condition [3] and Coron and Rosier conditions [11].
Our treatment enables us to overcome the difficulties imposed by the Brockett's condition. The stabilization problem for the underactuated ship is treated in the sense of asymptotic partial stabilization.

One of the most difficult manoeuvres of the captain of the ship is to put it on the quay. This operation can be done by exerting forces on the engines in order to put the ship in a side way on the quay.
In this work we will prove this mechanical operation and we develop a smooth feedback control that ensures the local "convergence" of the ship on the quay.

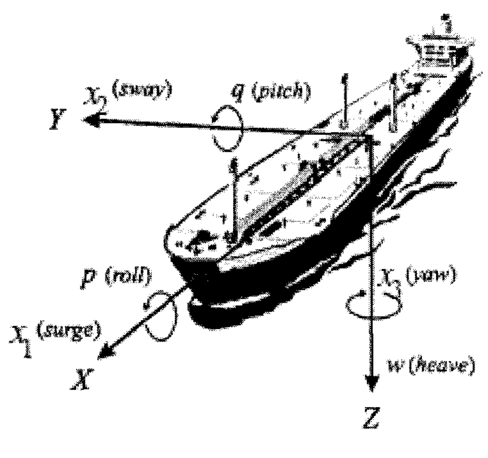

Fig. 4. Inertial frame attached to the Sohip.

Equation of Motion: The ship (see Pettersen and Nijmeijer) [27] can be modeled by the simplified one

$$\begin{cases} \dot{x}_1 &= \dfrac{m_{22}}{m_{11}}x_2x_3 - \dfrac{d_{11}}{m_{11}}x_1 + \dfrac{1}{m_{11}}u_1 \\ \dot{x}_2 &= -\dfrac{m_{11}}{m_{22}}x_1x_3 - \dfrac{d_{22}}{m_{22}}x_2 \\ \dot{x}_3 &= \dfrac{m_{11}-m_{22}}{m_{33}}x_1x_2 - \dfrac{d_{33}}{m_{33}}x_3 + \dfrac{1}{m_{33}}u_2 \\ \dot{\theta} &= x_1 cos\psi - x_2 sin\psi \\ \dot{\phi} &= x_1 sin\psi + x_2 cos\psi \\ \dot{\psi} &= x_3, \end{cases} \tag{48}$$

x_1, x_2, x_3 are the velocities in surge, sway and yaw respectively and θ, ϕ, ψ denote the position and orientation of the ship in the earth frame. u_1 and u_2 are the controls. The parameters m_{ii}, d_{ii} are supposed to be strictly positive.
Since the system (48) does not check the Brockett's condition for stabilization by continuous, state and stationary feedback laws. Stabilization is then treated within the partial asymptotic stabilization sense.
Our objective consists in constructing two feedbacks u_1 and u_2 which maintain the system (48) in the following configuration: The partial state $(x_1, x_2, x_3, \theta, \psi)'$ is asymptotically stable, and ϕ converges.
To achieve this goal, the following section introduces the change of coordinates necessary to the use of the backstepping and partial stabilization techniques.

4.3. Feedback transformation and stability analysis

4.3.1 Feedback transformation

To obtain a simpler form of the system (48), we adopt the following transformation:

$$\begin{aligned} u_1 &:= \frac{m_{22}}{m_{11}}x_2x_3 - \frac{d_{11}}{m_{11}}x_1 + \frac{1}{m_{11}}u_1 \\ u_2 &:= \frac{m_{11}-m_{22}}{m_{33}}x_1x_2 - \frac{d_{33}}{m_{33}}x_3 + \frac{1}{m_{33}}u_2. \end{aligned} \tag{49}$$

We suppose also that $d_{22} = m_{22}$ and $\dfrac{m_{11}}{m_{22}} = c > 0$, the system (48) takes the following form:

$$\begin{cases} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= -c x_1 x_3 - x_2 \\ \dot{x}_3 &= u_2 \\ \dot{\theta} &= x_1 cos\psi - x_2 sin\psi \\ \dot{\phi} &= x_1 sin\psi + x_2 cos\psi \\ \dot{\psi} &= x_3. \end{cases} \tag{50}$$

The system (50) is presented in cascaded form. To study the partial stabilizability of (50), we applied Theorem 1 and it is sufficient to study the reduced system of (50) where it's given by the following form:

$$\begin{cases} \dot{x}_2 &= -c u_1 u_3 - x_2 \\ \dot{\theta} &= u_1 cos\psi - x_2 sin\psi \\ \dot{\phi} &= u_1 sin\psi + x_2 cos\psi \\ \dot{\psi} &= u_2. \end{cases} \tag{51}$$

4.3.2 Stability analysis

As the satellite case, the partial asymptotic stabilization of the system (50) is done in two stages. The first one consists in stabilizing partially the reduced system. When the second is based on the linearization results [18,35] to deduce the suitable feedback laws for the stabilization of the augmented system (50).

Step 1: Stabilization of the reduced system

Our principal objective is to determine C^∞ feedback laws such that the system (51) in closed loop is partially asymptotically stable in the following sense: the partial equilibrium $(x_2,\theta,\psi)' = (0,0,0)'$ is asymptotically stable, and the angle ϕ converges.

The following proposition gives these feedback laws [15].

Proposition 6: Considering the following feedback laws u_1 and u_2

$$u_1 = -k\theta + x_2\psi, \quad u_2 = -k\psi, \tag{52}$$

where $k>0$, then, with the action of u_1 and u_2, the equilibrium $(x_2,\theta,\psi)' = (0,0,0)'$ of the system (51) is asymptotically stable and ϕ converges.

Proof 1) Asymptotic stabilization of $(x_2,\theta,\psi)'$:

It is clear that, with the feedback u_2, the state ψ is exponentially stable, and

$$\psi(t) = \psi(0)e^{-kt}. \tag{53}$$

The equality (53) gives

$$cos\psi(t) \sim 1, \quad if\ t \to +\infty, \tag{54}$$

$$sin\psi(t) \sim \psi(t), \quad if\ t \to +\infty. \tag{55}$$

Then, for $|\psi(0)|$ small and with (54), (55) the dynamic of θ becomes:

$$\dot{\theta} = u_1 - x_2\psi. \tag{56}$$

From our feedback u_1 given in (52) and from (54), the dynamic of θ becomes

$$\dot{\theta} = -k\theta. \tag{57}$$

Then θ is exponentially stabilizable.

Considering the sub-system of (51) in closed loop

$$\begin{cases} \dot{x}_2 &= c k \psi(-k\theta + x_2 \psi) - x_2 \\ \dot{\theta} &= (-k\theta + x_2\psi) cos\psi - x_2 sin\psi \\ \dot{\psi} &= -k\psi. \end{cases} \tag{58}$$

The linearization of the system (58) in the equilibrium point is given by:

$$\dot{x}_2 = -x_2, \quad \dot{\theta} = -k\theta, \quad \dot{\psi} = -k\psi. \tag{59}$$

The system (59) is exponentially stable, then the system (58) is also locally exponentially stable (see [18,35]).

Then, there exist two reals $r>0$ and $\alpha>0$ such that

$$|y(t)| \le r|y(0)|e^{-\alpha t}, \tag{60}$$

where $y = (x_2(t),\theta(t),\psi(t))'$. Therefore, we have the following inequalities

$$|x_2(t)|,|\theta(t)|,|\psi(t)| \le r|y(0)|e^{-\alpha t}.$$

2) Convergence of ϕ:

Since $|sin\psi| \le |\psi|$, then by triangular inequality and simple calculation yields

$$\begin{aligned} |\dot{\phi}(t)| &\le |x_2| + |u_1\psi| \\ &\le r|y(0)|e^{-\alpha t} \\ &\quad + kr^2|y(0)|^2 e^{-2\alpha t} + r^3|y(0)|^3 e^{-3\alpha t}. \end{aligned} \tag{61}$$

The inequality (61) shows that $\dot{\phi}$ is Lebesgue-integrable and therefore ϕ converges.

Step 2: Partial asymptotic stabilization of (50)

The following proposition establishes the feedback laws that ensure partial asymptotic stabilization of the system (50).

Proposition 7: With the action of the following feedback laws v_1 and v_2

$$v_1 = -\mu(x_1 - u_1(x)), \quad v_2 = -\mu(x_3 - u_2(x)), \tag{62}$$

where $0<\mu<4k$ and the feedback $u_1(x)$ and $u_2(x)$ are given in (52).

Then the system (50) is asymptotically stable with respect to $(x_1,x_2,x_3,\theta,\psi)'$, and the state ϕ converges.

Proof: The proof is based on linearization theorems. We take the system (49) in closed loop with our feedback laws (62) it is written in the following form

$$\begin{cases} \dot{x}_1 = -\mu x_1 - k\mu\theta + \mu x_2 \psi \\ \dot{x}_2 = -x_2 - c x_1 x_3 \\ \dot{x}_3 = -\mu x_3 - k\mu\psi \\ \dot{\theta} = x_1 cos\psi - x_2 sin\psi \\ \dot{\phi} = x_1 sin\psi + x_2 cos\psi \\ \dot{\psi} = x_3. \end{cases} \quad (63)$$

The linearization of the system with respect to $(x_1, x_2, x_3, \theta, \psi)'$ gives

$$\begin{cases} \dot{x}_1 = -\mu x_1 - k\mu\theta \\ \dot{x}_2 = -x_2 \\ \dot{x}_3 = -\mu x_3 - k\mu\psi \\ \dot{\theta} = x_1 \\ \dot{\psi} = x_3. \end{cases} \quad (64)$$

The states θ and ψ verify the second order differential equations given by:

$$\ddot{\theta} + \mu\dot{\theta} + k\mu\theta = 0, \quad (65)$$

$$\ddot{\psi} + \mu\dot{\psi} + k\mu\psi = 0. \quad (66)$$

It is clear that the solutions of (65) and (66) are exponentially stable because the polynom $x^2 + \mu x + k\mu = 0$ has two complex roots with negative real parts ($0 < \mu < 4k$).

Thus, the linearized system (64) is exponentially stable, then by the same argument as in step 1, the system (63) is locally exponentially stable with respect to $(x_1, x_2, x_3, \theta, \psi)'$. Then, there exists $c > 0$ and $\beta > 0$ such that $|x(t)| \le c|x(0)|e^{-\beta t}$, where $x(t) = (x_1, x_2, x_3, \theta, \psi)'$. Consequently, $|x_1(t)| \le c|x(0)|e^{-\beta t}$ and $|x_2(t)| \le c|x(0)|e^{-\beta t}$. Also, we have $\dot{\phi} = x_1 sin\psi + x_2 cos\psi$, then $|\dot{\phi}(t)| \le |x_1(t)| + |x_2(t)| \le 2c|x(0)|e^{-\beta t}$, which proves the convergence of ϕ.

4.4. Numerical simulations

For the simulation, we take the following feedback

$$v_1 = -10(x_1 - u_1(x)),\ v_2 = -10(x_3 - u_2(x)),$$
$$u_1 = -5\theta + x_2\psi,\ u_2 = -5\psi.$$

Figs. 5 and 6 show our results. According to numerical simulations, it is clear that the state (x_1, x_2, x_3) is asymptotically stable. When the axes θ and ψ of the orientation of the ship in the terrestrial reference converge asymptotically to equilibrium point 0, so on the other hand, the axe ϕ converges to a constant which depends on initial data.

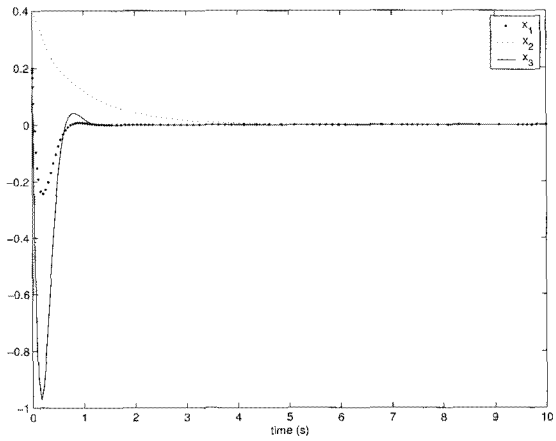

Fig. 5. Comportment of the velocities x_1, x_2, x_3.

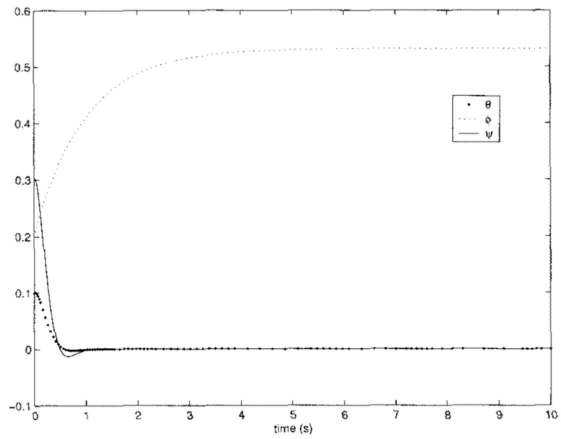

Fig. 6. Positions and orientations of the axes θ, φ, ψ.

4.5. Partial stabilization by linear feedback laws

In this subsection, we will prove that it is possible to stabilize partially the system (48) by linear feedback laws. To this end, by using the variables changing proposed in [26], the authors showed that the system (48) can be also written in the form:

$$\begin{cases} \dot{z}_1 = u + z_2 r \\ \dot{z}_2 = v - z_1 r \\ \dot{z}_3 = r \\ \dot{u} = u_1 \\ \dot{v} = -c_1 v - c_2 u r \\ \dot{r} = u_2. \end{cases} \quad (67)$$

The state is given by $x = (z_1, z_2, z_3, u, v, r)'$, u_1 and u_2 are the input.

Our objective is to construct two commands of class C^∞ such that the equilibrium $(z_1, z_3, u, v, r)' = (0,0,0,0,0)'$ is asymptotically stable, and the state z_2 converges.

Step 1: In a first stage, and in order to apply theorem 1, we study the reduced system obtained from (67). This system is given by:

$$\begin{cases} \dot z_1 &= u_1 + z_2 u_2 \\ \dot z_2 &= v - z_1 u_2 \\ \dot z_3 &= u_2 \\ \dot v &= -c_1 v - c_2 u_1 u_2. \end{cases} \quad (68)$$

By making a change of feedback and by taking the equivalence feedback

$$\bar u = u_1 + z_2 u_2, \ u_2 = u_2. \quad (69)$$

The system (68) will be equivalent to the following system

$$\begin{cases} \dot z_1 &= \bar u \\ \dot z_2 &= v - z_1 u_2 \\ \dot z_3 &= u_2 \\ \dot v &= -c_1 v - c_2 \bar u u_2 + c_2 z_2 u_2^2. \end{cases} \quad (70)$$

We choose the feedback $\bar u$ and u_2 of the form

$$\bar u(z) = -\alpha z_1, \ u_2(z) = -\alpha z_3, \ \alpha > 0. \quad (71)$$

According to (71), in closed loop, the sub-system described by $(z_1, z_3, v)'$ is given by

$$\begin{cases} \dot z_1 &= -\alpha z_1 \\ \dot z_3 &= -\alpha z_3 \\ \dot v &= -c_1 v - c_2 \alpha^2 z_1 z_3 + c_2 \alpha^2 z_2 z_3^2. \end{cases} \quad (72)$$

It is clear that the linearized system of (72) near the equilibrium which is given by

$$\dot z_1 = -\alpha z_1, \ \dot z_3 = -\alpha z_3, \ \dot v = -c_1 v \quad (73)$$

is exponentially stable, then the system (70) is locally exponentially stable with respect to $(z_1, z_3, v)'$, and therefore, there exists $\gamma_1 > 0$ and $\gamma_2 > 0$ such that

$$|(z_1, z_3, v)(t)| \le \gamma_1 |z(0)| e^{-\gamma_2 t}, \quad (74)$$

where $z(t) = (z_1, z_2, z_3, v)'$.
From (70), we have

$$\begin{aligned} |\dot z_2(t)| &\le |v(t)| + \alpha |z_1(t) z_3(t)| \\ &\le \gamma_1 |z(0)| e^{-\gamma_2 t} + \alpha \gamma_1^2 |z(0)|^2 e^{-2\gamma_2 t}. \end{aligned} \quad (75)$$

The inequality (75) implies that $\dot z_2$ is Lebesgue-integrable, then the state z_2 converge.

Step 2: In this step, we propose the following feedback laws

$$\phi_1(x) = -\mu_1(u - \bar u(x)), \ \phi_2(x) = -\mu_2(r - u_2(x)). \quad (76)$$

The gains μ_i are chosen larger enough, $\bar u(x)$ and $u_2(x)$ are given in (71).
Feedback laws (76) yield $(z_1, z_3, u, v, r)' = (0,0,0,0,0)'$ asymptotically stabilizable and the state z_2 converge.
Indeed, for the sub-system (72), we consider the dilation

$$\delta_z^\lambda(z_1, z_3, v) = (\lambda z_1, \lambda z_3, \lambda^2 v)',$$

we remark that (72) is homogeneous of degree zero with respect to dilation δ_z^λ. Then thanks to theorem of Morin *et al.* [23], the extended system (67) is asymptotically stable with respect to $(z_1, z_3, u, v, r)'$. This is obtained by feedbacks given in (76). We show like in the satellite case, that the state z_2 converges to a constant which is not necessarily null.

5. CONCLUSION

In this paper, we have developed the backstepping and partial asymptotic stabilization techniques for solving two partial attitude problems.

The treated examples relate to two dynamic systems made up respectively of a satellite and a ship which present the following properties:

- the two examples are underactuated systems with two controls and six variables of state.
- they do not satisfy the Brockett's necessary condition for stabilization.
- the two systems are in cascaded form with integrators.

For the two systems, the results show that we have stabilized asymptotically five variables of the state, whereas only one converges to a constant value depending on initial conditions. This partial asymptotic stabilization is obtained by indefinitely differentiable feedback functions on the state only.
As we have seen during the synthesis of the paper, the idea of construction of these feedback, is based on the transformation of the initial system in a simple system. This enabled us to establish an algorithm of partial asymptotic stabilization by the application of Lyapunov techniques.

The results obtained for the satellite case improved the work of Zuyev [37], and ensured the convergence of angular velocity ω_3 of the satellite. Moreover, for the example of the ship, our results confirm the physical reality. Indeed it is known that during the loading of the ship on the quay, the captain tries to put it laterally. This explains the convergence of the state ϕ (position in Y direction), also our feedbacks improved those obtained by Wichlund *et al.* [34] which states that the yaw angle ψ is bounded (in our work ψ is asymptotically stabilizable).

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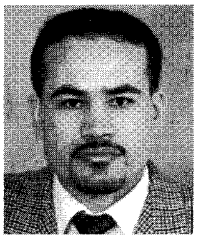

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