Optimal Control Policy for Linear-Quadratic Control Problems with Delay and Intermediate State Constraints

Kil To Chong, Olga Kostyukova, and Mariya Kurdina

Abstract: In this paper, we consider a terminal, linear control system with delay, subject to unknown but bounded disturbances. For this system, we consider the problem of constructing a worst-case optimal feedback control policy, which can be corrected at fixed, intermediate time instants. The policy should guarantee that for all admissible uncertainties the system states are in prescribed neighborhoods of predefined system states, at all fixed, intermediate time instants, and in the neighborhood of a given state at a terminal time instant, and the value of the cost function is the best guaranteed value. Simple explicit rules (which can be easily implemented on-line) for constructing the optimal control policy in the original control problem are derived.

Keywords: Linear-quadratic control problems, worst-case feedback policies.

1. INTRODUCTION

To control dynamic systems with uncertainties, there are several approaches for constructing corresponding optimization problems; refer to [9]. We briefly describe some of these approaches.

Suppose that we need to control the dynamic system:

$$\dot{z}(t) = f(z(t),u(t),t,w(t)),\ z(0)=z_0,\ t\in T=[0,t_*], \quad (1)$$

$$\text{s.t. } z(t_i)\in X_i, i=1,\dots,m, \quad (2)$$

where $z(t)\in R^n$, and $u(t)\in R$ are the state vector and scalar input, respectively, $t_i \in T\left(0=t_0 < t_1 < \dots < t_m < t_{m+1}=t_*\right)$ and $X_i \subset R^n$ are given instants and sets, respectively and $i=1,\dots,m.$ Here, $w(t)\in R$ is an unknown disturbance. It is supposed that at each time interval $T_i=[t_i,t_{i+1})$ the disturbance function $w(t), t\in T_i,$ belongs to a given bounded set of functions defined at this interval. The quality of the control function is evaluated by a given cost function:

$$J(z(\cdot),u(\cdot)), \quad (3)$$

where $z(\cdot)=(z(t),t\in T)$, $u(\cdot)=(u(t),t\in T)$, is the system trajectory and control function, respectively.

I. The first approach involves constructing and solving an optimization problem, based on the assumption that the uncertainty is completely deterministic. In our case, this means that we are solving the following open-loop optimal control (OLOC) problem:

$$\min J(z(\cdot),u(\cdot))$$
$$\text{s.t. } \dot{z}(t)=f(z(t),u(t),t,w^*(t)),\ t\in T=[0,t_*], \quad (4)$$
$$z(0)=z_0, z(t_i)\in X_i, i=1,\dots,m+1,$$

where there is a fixed, admissible disturbance $w^*(t), t\in T.$ An optimal control function $u^0(t), t\in T,$ for the deterministic problem (4) is used as the input function, to control a real-world non-deterministic system (1), where $w(t), t\in T,$ which may be any arbitrary, admissible disturbance.

There are clear disadvantages to this approach, because constraints (2) are not satisfied for the trajectory of a real-world system (1) generated by an open-loop optimal control policy and a realized disturbance $w^{real}(t), t\in T.$

II. The second approach relies on min-max optimization and involves the following. We search for a control function $u(t), t\in T,$ that guarantees that constraints (2) are satisfied for all admissible disturbances, and has the best value of the cost function (3), based on the worst-case, realized

Manuscript received March 3, 2006; revised July 31, 2007 and February 1, 2008; accepted July 17, 2008. Recommended by Editorial Board member Poo Gyeon Park under the direction of Editor Jae Weon Choi.

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disturbance. The control policy that solves this problem is called open-loop worst-case optimal control (OLWOC). The use of this control policy in a real-world system (1) guarantees that the constraints (2) are satisfied, for all admissible disturbances.

Note that this approach (similar to the first) optimizes a single control policy over all possible disturbances, and therefore does not consider the fact that more information about states becomes available as time advances. It is often unrealistic to presume that a unique, open-loop control function results in the expected behavior for all possible disturbances [4]. As a result, there may be problems with the feasibility of this approach; refer to [10,13,16].

III. In the third approach we formulate an optimization problem for more realistic assumptions, including the possibility of correcting a control function depending on the measurements of states. In such an approach, it is supposed that at every sampling instant $t_i, i=1,...,m,$ (1) We know the corresponding real-world system state $z_i = z(t_i)$ generated by the control input $u(t), t \in [0,t_i)$, and realized disturbance $w(t), t \in [0,t_i)$, and (2) We can correct the future control function that will be used at the next control interval $T_i = [t_i, t_{i+1})$.

The solution to the optimization problem is not a single, fixed, optimal control function $u^0(t), t \in T$, but the so-called worst-case optimal control policy, viz. Closed-loop worst-case optimal control (CLWOC)

$$\pi = \{u_0(\cdot \mid z), u_1(\cdot \mid z), ..., u_m(\cdot \mid z)\}.$$

This consists of control functions $u_i(\cdot \mid z) = (u_i(t \mid z), t \in T_i)$, $z \in R^n$. In a real-world control function (1), the i-th control function $u_i(\cdot \mid z)$ is used at the corresponding time interval T_i, with $z = z(t_i)$, where $z(t_i)$ is the real-world system state at the instant $t = t_i$. Hence, the realized values of the control functions depend on the realized states $z_i = z(t_i)$ of the system at instants $t = t_i$, $i = 0,1,...,m,$ and could differ according to the realizations of $w(\cdot)$.

This often implies improved performance compared to OLWOC schemes. The approach also avoids the unfeasibility problems that may result from the use of OLWOC approaches. The cost of these benefits is that the computational burden of the feedback min-max algorithm for constructing the policy may be very high. Since determination of a control policy is usually impractical, research has focused on simplifying the closed-loop, worst-case problem, such as a means of approximation or parametrization of the policy [8,9].

Some approaches to overcoming the aforementioned difficulties are described in the literature.

Methods combining dynamic and parametric programming for discrete time min-max problems, based on the assumption that disturbances take values in a polytope, are proposed in [3,6,10]. Methods for solving the problem using single, finite-dimensional optimization are proposed in [5,16]. For very general parametrization of uncertainties, in [9], the authors propose to solve a problem in the form of state feedback via dynamic programming, by discretization of the state space. Several other types of control policies are considered in [8] (also, refer to the references in [8]) on the basis of parametrization. In [8], the main assumption is that the control laws that define the policy are expressed via given functions with unknown parameters. The parameters of these functions are chosen in an optimal manner, according to a cost function. An approach which relied upon a slight modification of the objective function is presented in [1], for implementing a constrained quadratic min-max optimization problem.

In all cases, the resulting min-max feedback has a high computational burden. This is either because the problem (solved on-line) usually has rather high dimensionality, or the method suffers from the burden of dimensionality; from storing a huge volume of information, if the majority of processing is done off-line.

Thus, most reported feedback min-max methods are considered theoretical, rather than practical [4,13].

In this paper, we study a continuous, linear-quadratic, optimal control problem, subject to bounded additive uncertainties. In the problem being considered for the third approach, to construct a guaranteed control strategy we have the capability to correct the strategy at fixed, intermediate time instants $t_i \in [0,t_*]$, $i = 1,...,m.$ But, we consider more a realistic case, assuming that there is a communication delay in the control system: it is supposed that at the current instant t_i we do not know the current system state, $z(t_i)$ but we do know the real-world system state $z(t_i - h)$ at the previous instant $t = t_i - h,$ where $h > 0$.

We propose a method for constructing the worst-case, optimal control policy π^0. For this method, we present not only the theoretical scheme, but the justification and the detailed description of constructive rules for its implementation; it can easily be realized on-line.

We show that processing the policy is equivalent to solving a corresponding convex mathematical programming (MP) problem with $(m-1)$ decision variables. The MP problem may be solved off-line.

Based on the solution to the MP problem, we propose simple explicit rules (which can easily be implemented on-line) for constructing the corresponding control policy in the original control problem.

The paper is organized as follows. In Section 2, we consider a terminal, linear-quadratic optimal control problem with delay, in the presence of an additive, unknown, bounded uncertainty. For this system, we define the optimization problem in an open-loop, worst-case approach, involving construction of a corresponding single control function. Subsequently, we define the optimization problem in a closed-loop worst-case approach that involves constructing a guaranteed optimal strategy, which can be corrected at m fixed, intermediate, time instants. We show that the second approach is better than the first.

In the beginning of Section 3, we present theoretical relations defining an optimal control policy $\pi^0(Y^0)$, which solves the optimization problem. It is shown that construction of the policy $\pi^0(Y^0)$ is equivalent to solving a bilevel, min-max problem.

In Section 3.1, we prove that a bilevel min-max problem is equivalent to a convex mathematical programming problem with $m-1$ decision variables. Based on the solution to the mathematical programming problem, we derive simple, explicit rules (which can be easily implemented on-line) for constructing the optimal control policy in the original control problem.

In Section 3.2 we describe the characteristics of the proposed optimal control policy $\pi^0(Y^0)$.

In Section 4, other types of control policies are proposed. The results of numerical experiments are presented in Section 5.

Hereafter, we use the following notation: $\lambda_{\max}(S)$ denotes the maximum eigenvalue of a matrix S; $\|y\|_S$ denotes the weighted norm with the positive definite matrix $S: \|y\|_S^2 := y^T S y$, $\|y\|_2^2 := y^T y$; $L_2(T_i)$ defines the set of square-integrable functions determined at the interval T_i; $u_i(\cdot\,|\,a,b)$ denotes the function $u_i(\cdot\,|\,a,b) = (u_i(t\,|\,a,b),\ t\in T_i)$ from $L_2(T_i)$.

2. PROBLEM STATEMENTS

In this section, we consider a terminal, linear, control system with delay, subject to unknown but bounded disturbances.

Let the dynamics of an object be defined by the following differential equation:

$$\dot z(t) = Az(t) + bu(t) + gw(t),\ \ t\in T=[0,t_*],\qquad z(0)=z_0,\ \ u(t)=u^*(t),\ \ t\in[0,h], \tag{5}$$

$$\operatorname{rank}(b, Ab, \dots, A^{n-1}b) = n,\qquad \operatorname{rank}(g, Ag, \dots, A^{n-1}g) = n. \tag{6}$$

Here, $z(t)\in R^n$ denotes the state of the system, $u(t)\in R$ denotes the control function at the time instant where $t\ge 0$, $h>0$ is a parameter, and the initial system state z_0 and initial control function $u^*(t), t\in[0,h]$, are supposed to be known, $w(\cdot)=(w(t), t\in[0,t_*])$ is an unknown disturbance from a bounded set $\Omega\subset L_2[0,t_*]$, which is defined in a subsequent section, $A\in R^{n\times n}$, and $b,\ g\in R^n$ are given matrix and vectors.

Let:

$$z(t\,|\,u_t(\cdot), w_t(\cdot)),\ t\in[0,t_*],$$

denote the state of the system (5) at the instant t, which is generated by the control function $u_t(\cdot)=(u(s),\ s\in[0,t])$ and the disturbance $w_t(\cdot)=(w(s), s\in[0,t])$.

Suppose that time instants:

$$0=t_0<t_1<\dots<t_m<t_{m+1}=t_*,$$

values $\delta_i, i=1,\dots,m+1$, and the terminal system state $z_*\in R^n$ are given. We are interested in the control function:

$$u(\cdot)=(u(t), t\in[0,t_*]),$$

and the set of system states:

$$Y=\{y_1,\dots,y_m\},\ y_i\in R^n, i=1,\dots,m, \tag{7}$$

such that the following relations are satisfied:

$$\begin{aligned}&\|z(t_i\,|\,u_{t_i}(\cdot), w_{t_i}(\cdot)) - y_i\|_2^2 \le \delta_i^2,\ \text{ for all } w_{t_i}(\cdot)\in\Omega,\\ &i=1,\dots,m,\\ &\|z(t_*\,|\,u_{t_*}(\cdot), w_{t_*}(\cdot)) - z_*\|_2^2 \le \delta_{m+1}^2 \text{ for all } w_{t_*}(\cdot)\in\Omega.\end{aligned} \tag{8}$$

The control function $u(\cdot)=u_{t_*}(\cdot)$ is said to be feasible if the relations (8) are satisfied.

The quality of the control function is defined by the cost function:

$$\int_0^{t_*} u^2(t)dt. \tag{9}$$

Then, the problem may be approached in the following manner:

Open-loop, worst-case approach: Find a control function $u(\cdot)$ and system states (7) for which trajectories of the system (5) satisfy relations (8), and for which the cost function (9) is a minimum.

This approach belongs to the second type of approach mentioned in the introduction. In such an approach, it is often the case that no feasible control function $u(\cdot)$ exists. Stated crudely, in order to ensure feasibility, the set of admissible disturbances Ω should be a "small" neighborhood of the zero disturbance, and parameters δ_i, $i=1,\dots,m+1$, should be "large".

We can change the approach by including feedback. To simplify calculations, we consider that:

$$t_i = ih,\ i=0,1,\dots,m,\ t_{m+1}=t_*. \tag{10}$$

Note that as a rule: $t_{m+1} \neq (m+1)h$. We suppose that

a) At each (current) time instant t_i, $1 \le i \le m$, we know the state:

$$z_{i-1} := z(t_{i-1} \mid u_{t_{i-1}}(\cdot), w_{t_{i-1}}(\cdot)) \tag{11}$$

of the real-world system (5) at the previous instant $t_{i-1} = t_i - h$. Note that the state (11) is generated by the control function $u_{t_{i-1}}(\cdot)$ chosen at the instant t_{i-1} and a realized admissible disturbance $w_{t_{i-1}}(\cdot)$. It is natural to suppose that we also know the system input $u(t), t \in [t_{i-1}, t_i)$ that was used at the previous interval.

b) At the instant t_i, we may correct the future control function using new information available at this instant, i.e. the control function $u(t), t \in [t_i, t_{i+1})$, that will be applied to the real-world system at time interval $t \in [t_i, t_{i+1})$, is a function of a known state $z(t_{i-1})$ of the real-world system and a known control function $u(t)$, $t \in [t_{i-1}, t_i)$, constructed at the previous instant t_{i-1}.

Such a case may occur in the following case. Suppose that at the instant t_{i-1} the current state $z_{i-1} = z(t_{i-1})$ of the real-world system (5) is measured by a sensor, which passes the information to a controller that constructs a control input for the system (5). It is supposed that the controller can obtain and use this information with delay h.

Here, we give a new mathematical approach to the problem being investigated.

Closed-loop worst-case approach: Find a set of system states (7) and construct a corresponding control policy:

$$\pi(Y) = (u_i(\cdot \mid z_{i-1}, u_{i-1}(\cdot)), i=1,\dots,m), \tag{12}$$

consisting of control functions:

$$u_i(\cdot \mid z_{i-1}, u_{i-1}(\cdot)) = (u_i(t \mid z_{i-1}, u_{i-1}(\cdot)),\ t \in T_i), \tag{13}$$

where

$$i=1,\dots,m,$$
$$z_i - 1 \in R^n,\ u_{i-1}(\cdot) = (u_{i-1}(t), t \in T_{i-1}) \in L_2(T_{i-1}),$$

at each control interval $T_i = [t_i, t_{i+1})$, where $i=1,\dots,m$, such that:

- The trajectory $z(t) = z(t \mid \pi(Y), w(\cdot)), t \in [0, t_*]$, of the system:

$$\begin{aligned} &\dot z(t) = Az(t) + bu^*(t) + gw(t), t \in [0, t_1),\ z(0) = z_0, \\ &\dot z(t) = Az(t) + bu_i(t \mid z(t_i - h), u^*_{i-1}(\cdot)) + gw(t), \\ &t \in [t_i, t_{i+1}),\ i=1,\dots,m, \end{aligned} \tag{14}$$

where

$$u^*_{i-1}(\cdot) = (u_{i-1}(t \mid z(t_{i-2}), u^*_{i-2}(\cdot)), t \in T_{i-1}),\ i=2,\dots,m;$$
$$u^*_0(\cdot) = (u^*(t), t \in [0,h]),$$

satisfies conditions:

$$\begin{aligned} &\|z(t_i \mid \pi_i(Y), w_i(\cdot)) - y_i\|_2^2 \le \delta_i^2, \\ &\text{for } \forall w_i(\cdot) \in \Omega,\ i=1,\dots,m, \\ &\|z(t_* \mid \pi(Y), w(\cdot)) - z_*\|_2^2 \le \delta_{m+1}^2 \text{ for } \forall w(\cdot) \in \Omega; \end{aligned} \tag{15}$$

- the guaranteed value of the cost function:

$$J(\pi(Y)) = \max_{w(\cdot)\in\Omega} \sum_{i=1}^{m} \int_{T_i} u_i^2(t \mid z(t_{i-1}), u^*_{i-1}(\cdot))dt \tag{16}$$

is the minimum value:

$$\min_Y \min_\pi J(\pi(Y)).$$

where

$$z(t_{i-1}) = z(t_{i-1} \mid \pi_{i-1}(Y), w_{i-1}(\cdot)),$$
$$\pi_{i-1}(Y) = (u_s(\cdot \mid z_{s-1}, u_{s-1}(\cdot)), s=1,\dots,i-2),$$
$$w_{i-1}(\cdot) = (w(t), t \in [0, t_{i-1})).$$

Thus, instead of one control function in the open-loop approach, we use a control policy (strategy) that considers possible corrections in the future, based on available information about real-world system behavior.

Note that the dynamics of an object is defined by system (14), which is a closed-loop system, where the time delay $h > 0$, in a state and control function.

The set (7) and control policy (12) that solve the problem are called optimal, and are denoted by:

$$Y^0 = \{y_1^0, \dots, y_m^0\},\ y_i^0 \in R^n, i=1,\dots,m,$$
$$u_i^0(\cdot \mid z_{i-1}, u_{i-1}(\cdot)) = (u_i^0(t \mid z_{i-1}, u_{i-1}(\cdot)), t \in T_i),$$
$$i=1,\dots,m,$$
$$z_{i-1} \in R^n,\ u_{i-1}(\cdot) = (u_{i-1}(t), t \in T_{i-1}) \in L_2(T_{i-1}).$$

This approach belongs to the third type of approach mentioned in the introduction. The solution to such problems is not a fixed set Y^0 and a single, fixed optimal control function $u^0(t), t\in[0,t_*]$, but a fixed set Y^0 and a control policy $\pi^0(Y^0)$ consisting of control functions. For each $i=1,...,m,$ the realized value of control function $u_i^0(\cdot\,|\,z_{i-1},u_{i-1}(\cdot))$ depends on the set Y^0, and on the realized state $z(t_{i-1})=z_{i-1}$ of the system at the instants $t=t_{i-1}$ and a known control $u_{i-1}(\cdot)=(u_{i-1}(t), t\in T_{i-1})$.

We show that the class of feasible (guaranteed) control functions can be essentially extended, if we use a control policy (12) instead of one control function $u(t), t\in[0,t_*]$, for a given set of system states Y.

Firstly, we introduce a class of admissible disturbances Ω. We define the set Ω as follows:

$$\Omega:=\{w(\cdot):\int_{t_i}^{t_{i+1}} w^2(t)dt\le v_{i+1},\ i=0,...,m\}, \qquad (17)$$

where the given values are: $v_i>0, i=1,...,m, v_{m+1}=v_*>0.$ This choice of ellipsoidal uncertainty is motivated by the fact that the expression (17) has been shown to provide a good representation of uncertainties arising in many real-world control problems [12,15]. Moreover, the class of bounded disturbances:

$$|w(t)|\le\alpha,\ t\in[0,t_*],$$

belongs to the class of admissible disturbances (17) with the following choice of values for v_i:

$$v_i=\alpha^2 h,\ \ i=1,...,m-1,\ \ v_{m+1}=\alpha^2(t_*-t_m+h).$$

It is clear that for an open-loop problem approach, there exits an admissible control function if and only if:

$$\min_{u(\cdot)}\min_{Y}\max_{w(\cdot)\in\Omega}\max_{i=1,...,m+1}\{\left\|z(t_i\,|\,u_{t_i}(\cdot),w_{t_i}(\cdot))-y_i\right\|_2^2-\delta_i^2,\ i=1,...,m+1\}\le 0, \qquad (18)$$

with $y_{m+1}:=z_*$.

For $i=2,...,m+1$, denote:

$$\gamma_i=\max_{w}\left\|\int_{t_{i-2}}^{t_i}F(t_i,t)gw(t)dt\right\|_2^2, \qquad (19)$$

$$\text{s.t. }\int_{t_{i-2}}^{t_{i-1}}w^2(t)dt\le v_{i-1},\ \ \int_{t_{i-1}}^{t_i}w^2(t)dt\le v_i,$$

$$Q:=\int_0^h F(h,t)g(F(h,t)g)^T dt,\ \ \gamma_1=v_1\lambda_{\max}(Q),$$

$F(t,\tau):=F(t)F^{-1}(\tau)$, where $F(t)\in R^{n\times n}$ is the solution to the equation $\dot F(t)=AF(t)$, $F(0)=I$. From (6) it follows that: $\det Q\ne 0$, $\det Q_*\ne 0$.

From [7], we can show that the relations:

$$\delta_i^2\ge\gamma_i,\ i=1,...,m+1, \qquad (20)$$

are necessary and sufficient for the existence of a feasible policy $\pi(Y)$ (refer to (12)) satisfying (15).

Note that the relations (20) are significantly less stringent than the relation (18) that guarantees the existence of a feasible control in the open-loop, worst-case approach.

For the sake of notational simplicity, we assume that parameters $\delta_i, i=1,...,m+1$, take the minimum possible values, i.e. the equality in (20) is always satisfied:

$$\delta_i^2=\gamma_i,\ i=1,...,m+1. \qquad (21)$$

3. OPTIMAL CONTROL POLICY $\pi^0(Y^0)$

Here, where $i=1,...,m+1$, at the interval $t\in[t_{i-1},t_i)$ we distinguish two systems:
The real-world (realized) system, subject to the disturbance:

$$\dot z(t)=Az(t)+bu_{i-1}^*(t)+gw(t),$$
$$z(t_{i-1})=z_{i-1},\ t\in[t_{i-1},t_i), \qquad (22)$$

and the nominal system (without the disturbance):

$$\dot x(t)=Ax(t)+bu_{i-1}^*(t),$$
$$x(t_{i-1})=z_{i-1},\ t\in[t_{i-1},t_i). \qquad (23)$$

Suppose that the control function $u_{i-1}^*(t)$, $t\in[t_{i-1},t_i)$, is fixed; $Z_i\subset R^n$ denotes the set of all system (22) states at the instant t_i that are generated by all admissible disturbances $w(t), t\in[t_{i-1},t_i)$, and $x(t_i)$ denotes states of the system (23) at the instant t_i

Lemma 1: Define

$$Z_i(x):=\{z\in R^n:\left\|z-x\right\|_{Q^{-1}}^2\le v_i\}. \qquad (24)$$

Then, the relation $Z_i=Z_i(x(t_i))$ is satisfied.

The Lemma is proved in [7].

For a fixed set of system states $y_1, y_2,...,y_m$, $y_{m+1}=z_*$, we consider the control policy $\pi(Y)$ (12) with some control functions (13). From (21), the policy is admissible if and only if the following conditions are satisfied:

• The first state y_1 has the given value:

$$y_1 = x_1 = F(t_1,0)z_0 + \int_0^{t_1} F(t_1,t)bu^*(t)dt.$$

• For all $i = 1,\dots,m,$ the control function:

$$u_i(t \mid z_{i-1},u_{i-1}(\cdot)), t \in T_i = [t_i,t_{i+1}), \quad (25)$$

changes the trajectory of the nominal system:

$$\dot{x}(t) = Ax(t) + bu(t)$$

from the state:

$$x_i = x_i(z_{i-1},u_{i-1}(\cdot)) := F(t_i,t_{i-1})z_{i-1} + \int_{t_{i-1}}^{t_i} F(t_i,t)bu_{i-1}(t)dt \quad (26)$$

at the instant $t = t_i$, to the fixed state y_{i+1}, at the instant $t = t_{i+1}$.

The latest condition can be expressed as follows:

$$y_{i+1}=F(t_{i+1},t_i)x_i + \int_{t_i}^{t_{i+1}} F(t_{i+1},t)bu_i(t \mid z_{i-1},u_{i-1}(\cdot))dt. \quad (27)$$

Recall that the assumption is that at every time instant $t_i, i = 1,\dots,m,$ we know the system state $z_{i-1} = z(t_i - h)$ at the previous instant $t_{i-1} = t_i - h$ and the control function $u_{i-1}(t), t \in [t_{i-1},t_i),$ that was used at the previous interval, and we want to determine the control function $u_i(t), t \in [t_i,t_{i+1}),$ that will be used at the next interval.

To derive the worst-case, optimal policy $\pi^0(Y^0)$ we apply Bellman's principle of optimality [2] and dynamic programming.

Consider the real-world time instant $t = t_m$. Recall that the assumption is that we know the real-world system state z_{m-1}, and the control function $u_{m-1}(\cdot)$ and consequently we know the vector:

$$x_m = F(t_m,t_{m-1})z_{m-1} + \int_{t_{m-1}}^{t_m} F(t_m,t)bu_{m-1}(t)dt. \quad (28)$$

Recall the cost function (9); we conclude that where $i = m$, we must choose the control function (25) such that it minimizes the function:

$$\int_{t_m}^{t_{m+1}} u_m^2(t \mid z_{m-1},u_{m-1}(\cdot))dt \quad (29)$$

subject to (27). It follows from classical results of optimal control theory [14] that such a control function has the form:

$$u_m(t \mid z_{m-1},u_{m-1}(\cdot)) = \psi_m^T F(t_{m+1},t)b, t \in [t_m,t_{m+1}),$$

where

$$\psi_m^T = (y_{m+1} - F_{m+1}x_m)^T G_m^{-1},$$

and $i = m,\dots,1$

$$F_{i+1} := F(t_{i+1},t_i),\ G_i := \int_{t_i}^{t_{i+1}} F(t_{i+1},t)b(F(t_{i+1},t)b)^T dt,$$

and x_m is defined in (28). The corresponding value of function (29) can be expressed in the form:

$$J_m(z_{m-1},u_{m-1}(\cdot),y_{m+1}) = \bar{J}_m(x_m(z_{m-1},u_{m-1}(\cdot)),y_{m+1}),$$

$$\bar{J}_m(x_m,y_{m+1}) := \|y_{m+1} - F_{m+1}x_m\|^2_{G_m^{-1}}.$$

Suppose that the real-time instant is $t = t_i$, $1 \le i \le m-1$. Recall that the assumption is that we know real-world system state z_{i-1}, and control function $u_{i-1}(\cdot)$ and consequently we know the vector x_i defined in (26). Suppose also that at this instant we know functions:

$$J_{i+1}(z_i,u_i(\cdot),y_{i+2},\dots,y_{m+1}) := \bar{J}_{i+1}(x_{i+1}(z_i,u_i(\cdot)),y_{i+2},\dots,y_{m+1}).$$

We consider a control function $u_i(t), t \in T_i,$ such that equality (27) is satisfied. The corresponding guaranteed value of the cost function at the interval $[t_i,t_*]$ can be expressed as follows:

$$\max_{z_i \in Z_i(x_i)} \Big[\int_{t_i}^{t_{i+1}} u_i^2(t)dt + \bar{J}_{i+1}(F(t_{i+1},t_i)z_i + \int_{t_i}^{t_{i+1}} F(t_{i+1},t)bu_i(t)dt, y_{i+2},\dots,y_{m+1})\Big]. \quad (30)$$

Here, we consider that for any admissible distur-bance $w(t), t \in T_i,$ the state of the realized system z_i and the state of the nominal system x_i are related via $z_i \in Z_i(x_i),$ where $Z_i(x)$ is defined by (24).

Consequently, we must choose the control function $u_i(\cdot)$ such that it minimizes function (30), subject to (27). Considering the last equality, we can express the problem as follows:

$$\bar{J}_i(x_i,y_{i+1},\dots,y_{m+1}) := \min_{u_i(\cdot)\text{s.t.}(27)} \max_{z_i \in Z_i(x_i)} \Big[\int_{t_i}^{t_{i+1}} u_i^2(t)dt + \bar{J}_{i+1}(F(t_{i+1},t_i)(z_i - x_i) + y_{i+1},y_{i+2},\dots,y_{m+1})\Big]. \quad (31)$$

Based on classical results of optimal control theory [14], we conclude that the control function that solves the problem (31) can be expressed as follows:

$$u_i(t \mid z_{i-1},u_{i-1}(\cdot)) = \psi_i^T F(t_{i+1},t)b, t \in [t_i,t_{i+1}),$$
$$\psi_i = (y_{i+1} - F_{i+1}x_i)^T G_i^{-1}, \quad (32)$$

and function $\bar{J}_i(x_i,y_{i+1},\dots,y_{m+1})$ has the form:

$$\bar{J}_i(x_i, y_{i+1},\dots,y_{m+1}) = \| y_{i+1} - F_{i+1}x_i \|^2_{G_i^{-1}} + \max_{\substack{\|d_{s-1}\|^2_{Q^{-1}} \le v_{s-1}, \\ s=i+1,\dots,m}} \sum_{s=i+1}^{m} \| y_{s+1} - F_{s+1}F_s d_{s-1} - F_{s+1}y_s \|^2_{G_s^{-1}}, \quad (33)$$

where $d_{s-1} := z_{s-1} - x_{s-1},\ s = i+1,\dots,m$.

Function $\bar{J}_i(x_i, y_{i+1},\dots,y_{m+1})$ and relations (31) may be interpreted as Bellman's function and Bellman's equation, respectively. Then, control function (32) is the solution to Bellman's equation.

Calculating the optimal control $u_i(\cdot)$ recursively where $i = m,\dots,1$, we find that for a fixed set of system states $Y = \{y_1,\dots,y_m\}$ the best admissible control policy $\pi^0(Y)$ consists of the control functions defined by the rules:

$$u_i(t \mid z_{i-1}, u_{i-1}(\cdot)) = \psi_i^T F(t_{i+1},t)b, t \in [t_i, t_{i+1}), \quad \psi_i^T = (y_{i+1} - F_{i+1}x_i)^T G_i^{-1}, i = 1,\dots,m; \quad (34)$$

$$y_1 = x_1 = F(t_1,0)z_0 + \int_0^{t_1} F(t_1,t)bu^*(t)dt, \quad x_i = F_i z_{i-1} + G_{i-1}\psi_{i-1}, i = 2,..,m; y_{m+1} = z_*, \quad (35)$$

and the best guaranteed value of the cost function (16) is equal to:

$$J(\pi^0(Y)) = \bar{J}_1(x_1, y_2,\dots,y_m,y_{m+1}) = \max_{\substack{\|d_i\|_{Q^{-1}} \le v_i, \\ i=1,\dots,m-1}} \sum_{i=1}^{m-1} \| y_{i+1} - F_{i+1}y_i - F_{i-1}F_i d_{i-1} \|^2_{G_i^{-1}}, \quad (36)$$

where $d_0 = 0,\ d_i = z_i - x_i,\ i = 1,\dots,m-1,\ y_1 = x_1,$ $y_{m+1} = z_*$.

Here, among all policies $\pi^0(Y)$ with control laws (34), we choose a policy that minimizes the function (36) over all sets Y. This implies the following problem:

$$J(\pi^0(Y^0)) = V^0 := \min_{y_i \in R^n, i=2,\dots,m} \bar{J}_1(x_1, y_2,\dots,y_m,y_{m+1}). \quad (37)$$

Let $y_1^0 = y_1, y_2^0,\dots,y_m^0$ be the solution to the problem (37). $\pi^0(Y^0)$ denotes the policy (34) with $y_2 = y_2^0$, ..., $y_m = y_m^0$, $y_{m+1} = z_*$.

Hence, to construct the optimal policy $\pi^0(Y^0)$ we must solve problem (37). We examine the problem (37) more closely:

$$V^0 = \min_{\substack{y_i, \\ i=2,\dots,m}} \max_{\substack{\|d_i\|^2_{Q^{-1}} \le v_i, \\ i=1,\dots,m-1}} \sum_{i=1}^{m} \|y_{i+1} - F_{i+1}y_i - F_{i+1}F_i d_{i-1}\|^2_{G_i^{-1}}. \quad (38)$$

Note that based on (10) it is true that: $F_i =: F$, $G_i =: G, i = 1,\dots,m,\ F_{m+1} =: F_*, G_{m+1} =: G_*$.

The problem (38) is a bilevel optimization problem in the variables $y_i \in R^n,\ i = 2,\dots,m,$ (upper-level) and $d_i \in R^n,\ i = 1,\dots,m-1,$ (lower-level). In general, such problems are non-convex and non-smooth, and are assumed to be very difficult to process. However, the special characteristics of problem (38) enable us to derive an effective method for its solution. These characteristics are investigated and justified in the next sections.

3.1. Algorithm for solving a bilevel problem

3.1.1 Auxiliary results

First, we formulate some auxiliary results that are needed to understand the characteristics of the problem (38).

The given parameters are: vectors $a, b \in R^n$, non-singular matrices A and $B \in R^{n\times n}$, and positive definite matrices G, S, $Q \in R^{n\times n}$, and a positive number $v \in R$. We consider two optimization problems:

$$I^0 := \min_{y \in R^n} \max_{d \in R^n} (\|y + a\|^2_{G^{-1}} + \|b - Ay - ABd\|^2_{S^{-1}}), \quad \text{s.t.} \ \|d\|^2_{Q^{-1}} \le v, \quad (39)$$

and

$$I^* := \min_{\lambda \ge \lambda_*} \left(\lambda v + (b + Aa)^T D(\lambda)(b + Aa)\right), \quad (40)$$

where $D(\lambda) = \left(S + AGA^T - KK^T/\lambda\right)^{-1}$, $K = ABM^{-1}$, $Q^{-1} = M^T M$, $\lambda_* := \lambda_{\max}(K^T S^{-1} K)$.

First, note that by design, $D(\lambda)$ is a positive, definite matrix, for all $\lambda \ge \lambda_*$ (refer to Lemma 3 in Appendix). Second, it seems that the two problems are equivalent, in the following sense:

Lemma 2: The equality $I^0 = I^*$ is satisfied. Given an optimal solution $\lambda^0 \in R$ to the problem (40), a solution $y^0 \in R^n$ to the problem (39) is defined via:

$$y^0 = GA^T D(\lambda^0)(b + Aa) - a. \quad (41)$$

The lemma is proved in [7].

3.1.2 Reducing a min-max problem (38) to a convex mathematical programming problem

Consider again problem (38), and express it in the following form:

$$V^0 = \min_{y_i, i=3,\dots,m} (S_2(y_3) + \max_{\substack{\|d_i\|^2_{Q^{-1}} \le v_i,\\ i=2,\dots,m-1}} \sum_{i=3}^{m} \|y_{i+1} - F_{i+1}y_i - F_{i+1}F_i d_{i-1}\|^2_{G_i^{-1}}), \tag{42}$$

where

$$S_2(y_3) := \min_{y_2} \max_{\|d_1\|^2_{Q^{-1}} \le v_1} (\|y_2 - F_2 y_1\|^2_{G_1^{-1}} + \|y_3 - F_3 y_2 - F_3 F_2 d_1\|^2_{G_2^{-1}}).$$

The application of Lemma 2 to $S_2(y_3)$ implies that: $S_2(y_3) = S_2^*(y_3)$,

$$S_2^*(y_3) := \min_{\lambda_1 \ge \mu_1} (\lambda_1 v_1 + (y_3 + a_2)^T D_1(\lambda_1)(y_3 + a_2)), \tag{43}$$

where $a_2 = F_3 a_1, a_1 = -F_2 y_1, \quad D_0 = G_1^{-1}$,

$$D_1(\lambda_1) = (G_2 + F_3 D_0^{-1} F_3^T - \frac{K_2 K_2^T}{\lambda_1})^{-1},$$

$$\mu_1 = \lambda_{\max}(K_2^T G_2^{-1} K_2^{-1}), K_2 = F_3 F_2 M^{-1}.$$

Substituting, using (43) in (42) yields:

$$\begin{aligned} V^0 &= \min_{\substack{y_i,\\ i=3,\dots,m}} (\min_{\lambda_1 \ge \mu_1} (\lambda_1 v_1 + (y_3 + a_2)^T D_1(\lambda_1)(y_3 + a_2)) \\ &\quad + \max_{\substack{\|d_i\|^2_{Q^{-1}} \le v_i,\\ i=2,\dots,m-1}} \sum_{i=3}^{m} \|y_{i+1} - F_{i+1}y_i - F_{i+1}F_i d_{i-1}\|^2_{G_i^{-1}}) \\ &= \min_{\lambda_1 \ge \mu_1} \min_{y_i, i=4,\dots,m} (S_3(y_4, \lambda_1) + \lambda_1 v_1 \\ &\quad + \max_{\substack{\|d_i\|^2_{Q^{-1}} \le v_i,\\ i=3,\dots,m-1}} \sum_{i=4}^{m} \|y_{i+1} - F_{i+1}y_i - F_{i+1}F_i d_{i-1}\|^2_{G_i^{-1}}), \end{aligned} \tag{44}$$

where $i = 3,\dots,m$,

$$\begin{aligned} &S_i(y_{i+1}, \lambda_1, \dots, \lambda_{i-2}) \\ &:= \min_{y_i} \max_{\|d_{i-1}\|^2_{Q^{-1}} \le v_{i-1}} ((y_i + a_{i-1})^T D_{i-2}(\lambda_1, \dots, \lambda_{i-2}) \\ &\quad \times (y_i + a_{i-1}) + \|y_{i+1} - F_{i+1}y_i - F_{i+1}F_i d_{i-1}\|^2_{G_i^{-1}}). \end{aligned}$$

Here, where $i = 3,\dots,m, \quad a_i = F_{i+1}a_{i-1}, \quad K_i = F_{i+1}F_i M^{-1}$,

$$\begin{aligned} &D_{i-1}(\lambda_1, \dots, \lambda_{i-1}) \\ &= \left(G_i + F_{i+1} D_{i-2}^{-1}(\lambda_1, \dots, \lambda_{i-2}) F_{i+1}^T - \frac{K_i K_i^T}{\lambda_{i-1}} \right)^{-1}. \end{aligned}$$

The application of Lemma 2 to $S_3(y_4, \lambda_1)$ implies that: $S_3(y_4, \lambda_1) = S_3^*(y_4, \lambda_1)$, where $i = 3,\dots,m$,

$$\begin{aligned} S_i^*(y_{i+1}, \lambda_1, \dots, \lambda_{i-2}) &:= \min_{\lambda_{i-1} \ge \mu_{i-1}} (\lambda_{i-1} v_{i-1} \\ &\quad + (y_{i+1} + a_i)^T D_{i-1}(\lambda_1, \dots, \lambda_{i-1})(y_{i+1} + a_i)), \end{aligned}$$

$$\mu_i = \lambda_{\max}(K_i^T G_i^{-1} K_i).$$

Substituting, using $S_3^*(y_4, \lambda_1)$ in (44), and recursively applying Lemma 2 $m-2$ times, yields an expression for V^0:

$$V^0 = \min_{\lambda_1 \ge \mu_1} \dots \min_{\lambda_{m-2} \ge \mu_{m-2}} S_m(y_{m+1}, \lambda_1, \dots, \lambda_{m-2}). \tag{45}$$

The application of Lemma 2 to $S_m(y_{m+1}, \lambda_1, \dots, \lambda_{m-2})$ implies that:

$$S_m(y_{m+1}, \lambda_1, \dots, \lambda_{m-2}) = S_m^*(y_{m+1}, \lambda_1, \dots, \lambda_{m-2}), \tag{46}$$

and

$$\begin{aligned} V^0 &= \min_{\lambda_1 \ge \mu_1} \dots \min_{\lambda_{m-1} \ge \mu_{m-1}} \Big(\sum_{i=1}^{m-1} \lambda_i v_i \\ &\quad + (y_{m+1} + a_m)^T D_{m-1}(\lambda_1, \dots, \lambda_{m-1})(y_{m+1} + a_m) \Big). \end{aligned}$$

Consequently, the problem (45) can be expressed in the form:

$$V^0 = \min_{\lambda \in R^{m-1}} f(\lambda), \text{ s.t. } \lambda \ge \mu, \tag{47}$$

where $f(\lambda) := v^T \lambda + c^T D_{m-1}(\lambda_1, \dots, \lambda_{m-1}) c$, the given vectors are: $\mu = (\mu_1, \dots, \mu_{m-1})^T$, $v = (v_1, \dots, v_{m-1})^T$, $c = c(z_*, z_0, u^*(\cdot)) = z_* - F_{m+1}F_m \dots F_2 x_1(z_0, u^*(\cdot))$ and $x_1 = x_1(z_0, u^*(\cdot))$ is defined as in (35).

Remark 1: Note that from (10):

$$F = F(h, 0) = F_i, i = 1,\dots,m, G_i = G, i = 0,\dots,m-1,$$

$$\begin{aligned} &D_{i-1}(\lambda_1, \dots, \lambda_{i-1}) \\ &= \left(\sum_{s=1}^{i+1} F^{s-1} G F^{s-1} - \sum_{s=1}^{i-1} \frac{F^{s+1} Q F^{(s+1)T}}{\lambda_{i-s}} \right), i = 2,\dots,m, \end{aligned}$$

and consequently, $\mu_i = \text{const}, \quad i = 1,\dots,m-1$.

Proposition 1: The function $f(\lambda)$ is continuous and convex at $\Lambda = \{\lambda \in R^{m-1} : \lambda \ge \mu\}$.

Proof: The proposition is implied by Lemma 3 and Lemma 4 (refer to Appendix), where $S(\lambda) = D_{m-1}(\lambda)$, $C = c$ and $r = 1$.

Obviously, the cost function of the problem (47) is bounded. Hence, the problem (47) has a solution that can be found by standard methods of convex programming.

3.1.3 Constructing the optimal solution Y^0

Here, we show the means of obtaining a solution $Y^0 = \{y_1^0 = y_1, y_2^0, ..., y_m^0\}$ to problem (38) using a solution, e.g., $\lambda^0 = (\lambda_1^0, ..., \lambda_{m-1}^0)$, to problem (47).

The application of Lemma 2 to two equivalent problems implies that:

$S_m(y_{m+1}, \lambda_1^0, ..., \lambda_{m-2}^0)$, $\quad S_m^*(y_{m+1}, \lambda_1^0, ..., \lambda_{m-2}^0)$
(refer to (46)):

$$y_m^0 = D_{m-2}^{-1}(\lambda_1^0, ..., \lambda_{m-2}^0) F_{m+1} D_{m-1}(\lambda_1^0, ..., \lambda_{m-1}^0) \times (y_{m+1}^0 + F_{m+1} a_{m-1}) - a_{m-1}, \; y_{m+1}^0 = z_*.$$

Then, where $i = m-1, ..., 2$, we recursively apply Lemma 2 to equivalent problems $S_i(y_{i+1}^0, \lambda_1^0, ..., \lambda_{i-2}^0)$, $S_i^*(y_{i+1}^0, \lambda_1^0, ..., \lambda_{i-2}^0)$, and compute the vector

$$y_i^0 = D_{i-2}^{-1}(\lambda_1^0, ..., \lambda_{i-2}^0) F_{i+1} D_{i-1}(\lambda_1^0, ..., \lambda_{i-1}^0) \times (y_{i+1}^0 + F_{i+1} a_{i-1}) - a_{i-1}. \quad (48)$$

Based on vectors (48), we construct the optimal control policy $\pi^0(Y^0)$ by the rules (34), where $y_i, i = 1, ..., m$, are replaced with $y_i^0, i = 1, ..., m$.

3.2. Properties of proposed optimal control policy $\pi^0(Y^0)$ and generalization

Summarizing the aforementioned results, we state the following proposition.

Proposition 2: For any admissible disturbance $w(\cdot) \in \Omega$, the policy $\pi^0(Y^0)$ guarantees that:

- The initial state z_0 of the realized system (22) is directed towards the δ_{m+1}-neighborhood of the given terminal state z_* in $m-1$ steps;
- For all $i = 1, ..., m$, the state $z(t_i)$ of the realized system (5) at the instant t_i is in the δ_i-neighborhood of the fixed (found) state y_i^0,
- The value of the cost function for the realized control function does not exceed V^0,
- The estimate V^0, is the optimal, guaranteed value of the cost function.

We constructed the optimal set Y^0 and optimal control policy based on the assumption that in system (5) the initial system state z_0 and control function $u(t)$, $t \in [t_0, t_1]$, which is used until the time delay, are given. Here, we show that this assumption is not onerous, and the proposed approach can be easily applied to more general practical cases.

Suppose that in dynamic system (5):

- Control function $u(t)$, $t \in [t_0, t_1]$ is not fixed and can be arbitrary,
- In the starting instant $t_0 = 0$, the initial system state z_0 is not known exactly, but it is known that:

$$z_0 \in Z_0(x_0) = \{z \in R^n : \|z - x_0\|_{Q_0^{-1}}^2 \le v_0\}, \quad (49)$$

where the given parameters are: vector $x_0 \in R^n$, positive defined matrix $Q_0 \in R^{n \times n}$, and value $v_0 \ge 0$. In such a case, for a fixed set Y of system states (refer to (7)), the control policy $\pi^0(Y)$ consists of control functions:

$$u_0(\cdot \mid x_0), u_i(\cdot \mid z_{i-1}, u_{i-1}(\cdot)), i = 1, ..., m. \quad (50)$$

As before, the last control functions have the form (32), where x_i is defined in (26). Control function $u_0(\cdot \mid x_0)$ in the first interval has a similar form:

$$u_0(t \mid x_0) = \psi_0^T F(t_1, t) b, t \in [t_0, t_1), \quad \psi_0 = (y_1 - F_1 x_0)^T G_0^{-1}, \quad (51)$$

where x_0 is a given, based on an assumption. In relations (36), function $\bar{J}_1(x_1, y_2, ..., y_m, y_{m+1})$ is replaced by function $\bar{J}_0(x_0, y_1, ..., y_m, y_{m+1})$ that is defined as before by (33), where $i = 0$ and $Q = Q_0$, where $s = 1$. All other arguments and constructions are completely the same as before, with the only difference being that $\min_{y_i, i=2,...,m}$ is replaced by $\min_{y_i, i=1,...,m}$. As a result, the optimal set $Y^0 = \{y_1^0, ..., y_m^0\}$ can be found. In the first control interval we use control function (51) where $y_1 = y_1^0$. Where $i = 1, ..., m$, in the interval $[t_i, t_{i+1})$ we use control function (32) where $y_{i+1} = y_{i+1}^0$ and the vector $x_i = F_i z_{i-1} + G_{i-1} \psi_{i-1}$ that is known in the instant t_i.

Note that the equality $v_0 = 0$ in (49) implies the case where the initial system state z_0 is known exactly, in the starting instant t_0.

4. OTHER TYPES OF CONTROL POLICIES

To illustrate the good performance of the proposed approximative control policy $\pi^0(Y^0)$ we compare it with other reasonable control policies. In this section we briefly describe two other control policies.

In the proposed policies we can correct future control functions at the current instant t_i, on the basis of information that is available at this instant. The information is the same as that used for construction of the policy $\pi^0(Y^0)$. However, in the policy $\pi^0(Y^0)$ at the instant t_i we perform the correction based on the fact that we can correct the control function at future instants $t_{i+1},\ldots,\ t_m$, on the basis of new information that will be available at these instants. In the policies described in this section, which are based on the principles of classical feedback, we do not consider the fact that more information about states becomes available as time advances.

In the next section, we compare the characteristics of these policies with numerical examples.

4.1. Control policy based on classical feedback

Consider a realized time instant t_i. Recall that the assumption is that we know the system state z_{i-1} at the previous time instant $t_{i-1}=t_i-h$ and the control function $u^*(t), t\in[t_{i-1},t_i)$. Hence we know the state:

$$x_i = x(t_i) = Fz_{i-1} + \int_{t_{i-1}}^{t_i} F(t_i,t)bu^*(t)dt \tag{52}$$

of the nominal system (23) at $t=t_i$ We compute a control function $u(t), t\in[t_i,t_*]$, which changes the state of the nominal system (23) from the state x_i at the time instant t_i, to the given state z_* at the instant t_*, and minimizes the cost function:

$$\int_{t_i}^{t_*} u^2(t)dt \to \min.$$

This optimal control function is given by:

$$\bar{u}(t\,|\,x_i) = \psi_i^T(x_i)F(t_*,t)b, t\in[t_i,t_*], \tag{53}$$

$$\psi_i^T(x_i) = (z_* - F(t_*,t_i)x_i)^T \bar{G}_i^{-1},$$
$$\bar{G}_i = \int_{t_i}^{t_*} F(t_*,t)b(F(t_*,t)b)^T dt. \tag{54}$$

This control function is applied to the realized system at the interval $[t_i,t_{i+1})$, which changes the realized (perturbed) system to the state z_{i+1} at the instant t_{i+1}:

$$z_{i+1} = x_{i+1} + \int_{t_i}^{t_{i+1}} F(t_{i+1},t)gw(t)dt.$$

Here, $x_{i+1} = x(t_{i+1}) = Fz_i + GF_{i+1}^T\psi_i(x_i),\ F_i = F(t_*,t_i)$.

At the instant t_{i+1} we correct the control function by the rules (52)-(54), using the new, known state z_i of the real-world system at $t=t_i$ and the new, known position (t_{i+1},x_{i+1}) of the nominal system. This process can be continued. As a result, we obtain a policy of type (12):

$$\bar{\pi} = (\bar{u}_i(\cdot\,|\,z_{i-1},\bar{u}_{i-1}(\cdot)), i=1,\ldots,m) \tag{55}$$

with the control functions (13):

$$\bar{u}_i(\cdot\,|\,z_{i-1},\bar{u}_{i-1}(\cdot))$$
$$= (\bar{u}_i(t\,|\,x_i) = (z_* - F_i x_i)^T \bar{G}_i^{-1}F(t_*,t)b, t\in[t_i,t_{i+1})),$$
$$x_i = Fz_{i-1} + GF_i^T\psi_{i-1}(x_{i-1}), i=1,\ldots,m,$$

$z_{i-1}=z(t_{i-1})$ is a known state of the real-world system (5) at the instant t_{i-1}.

The (guaranteed) value of the cost function (16) of the policy $\bar{\pi}$ is equal to:

$$J(\bar{\pi}) = \max_{w(\cdot)\in\Omega} J(\bar{\pi},w(\cdot)),$$

$$J(\bar{\pi},w(\cdot)) = \sum_{i=1}^{m}\int_{T_i}\bar{u}_i^2(t\,|\,z_{i-1},\bar{u}_{i-1}(\cdot)),$$

where $z_{i-1}=z(t_{i-1}\,|\,\bar{\pi},w_{t_{i-1}}(\cdot))$ is the state of the real-world system at the instant t_{i-1} generated by control policy $\bar{\pi}$ and a realized admissible disturbance $w_{t_{i-1}}(\cdot)$.

The described policy $\bar{\pi}$ is reasonable, but it is not feasible for our problem: we cannot find a set $Y=\{y_1,\ldots,y_m\}$ such that relations (8) are satisfied for the policy $\bar{\pi}$.

4.2. Control policy $\pi^0(Y^*)$

Here, we construct a control policy $\pi^0(Y^*)$ consisting of control functions (34), where $Y=Y^*$. The set of system states $Y^*=\{y_1^*,\ldots,y_m^*\}$ is defined by the following "reasonable" rules $y_i^*=x^*(t_i)$, $i=1,\ldots,m$. Here, $u^*(\cdot)$ and $x^*(\cdot)$ are the optimal program control function and the corresponding trajectory of the problem, respectively:

$$\int_0^{t_*} u^2(t)dt \to \min, \dot{x}(t) = Ax(t) + bu(t),$$

$$x(0)=z_0, u(t)=u^*(t), t\in[0,t_1), x(t_*)=z_*.$$

We obtain the latest problem if we use $w(t)=0$, $t\in[0,t_*]$, in the original problem (5). Policy $\pi^0(Y^*)$ consists of control functions that are constructed by the rules (34), where $y_i, i=1,...,m,$ are replaced with $y_i^*, i=1,...,m.$

It is clear that the policy $\pi^0(Y^*)$ is (guaranteed) feasible. However it is not optimal: $J(\pi^0(Y^0)) \le J(\pi^0(Y^*))$.

5. NUMERICAL EXPERIMENTS

In this section, we present the results of a numerical comparison of the policies suggested in this paper. In our numerical experiments, we consider a dynamic system (5) where $h = 6.6$,

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \qquad (56)$$

$b^T = (1,2,0,1)$, $g^T = (-2, 0.1, 0.5, -2)$, $z_0^T = (1,-1,3,3)$, $u^*(t) = 0, t \in [0,h)$, $m = 5$, $n = 4$, $z_*^T = (-8,-6,1,0)$, $z_*^T = (-8,-6,1,0)$, $t_0 = 0$, $t_1 = 6.6$, $t_2 = 13.2$, $t_3 = 19.8$, $t_4 = 26.4$, $t_5 = 33$, $t_6 = t_* = 40$.

The corresponding set of admissible disturbances Ω is defined via (17), where:

$$v_i = \alpha^2(t_i - t_{i-1}),\ i = 1,...,m+1, \text{ with } \alpha = 1/4. \qquad (57)$$

Recall that the class of bounded disturbances:

$$|w(t)| \le \alpha, t \in T,$$

belongs to the set of admissible disturbances.

It is rather difficult to calculate values $\gamma_i, i = 2,...,m,$ by rules (19), but we can obtain estimates, where $\bar{\gamma}_i \ge \gamma_i, i = 2,...,m$. In the example it is true that:

$$\gamma_1 = 2.82,\ \bar{\gamma}_2 = 5.28,\ \bar{\gamma}_3 = 5.28,$$
$$\bar{\gamma}_4 = 5.28,\ \bar{\gamma}_5 = 5.28,\ \bar{\gamma}_6 = 5.23.$$

We construct a convex problem (47) with $m-1=4$ variables, and find its optimal solution $\lambda^0 = (\lambda_1^0, \lambda_2^0, \lambda_3^0, \lambda_4^0)$ and the corresponding optimal value $V^0 = 17.19$.

We obtain the set $Y^0 = \{y_1^0, y_2^0, y_3^0, y_4^0, y_5^0\}$ using the solution λ^0. To this end, starting from $y_6^0 = z_*$ we recursively compute y_i^0 according to (48), where $i = 5,4,3,2,$ and find $y_1^0 = y_1$ by (35).

Based on this set we construct the optimal control policy $\pi^0(Y^0)$ by the rules (34), where $y_i, i=1, ...,5,$ are replaced with $y_i^0, i=1,...,5$.

The realized values of the control functions $u_i^0(t \mid z(t_{i-1}), u_{i-1}^0(\cdot))$, $t \in T_i$, $i = 1,..,m,$ will depend on the realized disturbances, because real-world system states $z(t_i - h) = z(t_i - h \mid \pi^0(Y^0), w_{i-1}(\cdot))$, $i = 2,.., m,$ do depend on $w_{i-1}(\cdot) = w_{t_{i-1}}(\cdot)$. The corresponding value of the cost function is denoted by:

$$J(w) := J(\pi^0(Y^0), w(\cdot)) = \sum_{i=1}^{m} \int_{T_i} (u_i^0(t \mid z(t_{i-1} \mid \pi^0(Y^0), w_{i-1}(\cdot)), u_{i-1}^0(\cdot)))^2 dt.$$

Then, for all admissible disturbances $w(\cdot)$ it is true that: $J(w) \le V^0 = 17.19$. It easy to construct admissible disturbances $w_\pi(\cdot)$ such that $J(w_\pi) = V^0$.

Table 1 presents, for different admissible disturbances $w(\cdot)$, values of cost functions $J(w)$ and deviations of real-world system states $z_i = z(t_i \mid \pi^0(Y^0), w_{t_i}(\cdot))$ from designed states $y_i^0, i = 1, ...5,$ and $y_6^0 = z_*$.

The results of the numerical experiments show that the constructed policy $\pi^0(Y^0)$ for all $i = 1,...,m+1$ guarantees that the state $z(t_i)$ of the realized system (5) at the instant t_i is in the δ_i-neighborhood of the fixed (found) state y_i^0, and the value of the cost function for the realized control function does not exceed $V^0 = 17.19$.

For comparison, we constructed the admissible policy $\pi^0(Y^*)$ by the rules described in Section 4.2. This control policy $\pi^0(Y^*)$ is admissible, but not optimal: the guaranteed value of the cost function is equal to $J(\pi^0(Y^*)) = 25.01$.

Here, we show that, in this example, the control policy constructed on the basis of classical feedback is not admissible, because this strategy does not guarantee that the intermediate constraints are satisfied (8).

We continued the numerical experiment with data (56), (57) and constructed two admissible disturbances $w_1(\cdot)$ and $w_2(\cdot)$ by the rules: $w_s(\cdot) = \varphi_i^{sT} F(t_i, t) g$, $t \in T_i$, $i = 1,...,6,$ where $\varphi_i^s \in R^n$:

$$(\varphi_i^1, i = 1,...,6)$$

$$=\begin{pmatrix} 0.0847 & 0.3635 & 0.1132 & 0.0945 & 0 & 0 \\ 0.1367 & 0.2866 & 0.0808 & 0.0378 & 0 & 0 \\ 0.1508 & 0.1347 & 0.0356 & 0.0057 & 0 & 0 \\ 0.1511 & 0.074 & 0.0189 & -0.0019 & 0 & 0 \end{pmatrix},$$

$(\varphi_i^2, i=1,\ldots,6)$

$$=\begin{pmatrix} -0.1021 & -0.1065 & -0.1194 & -0.1256 & 0 & 0 \\ 0.0056 & 0.007 & 0.0137 & 0.024 & 0 & 0 \\ -0.0812 & -0.0965 & -0.0897 & -0.0813 & 0 & 0 \\ -0.0789 & -0.0513 & -0.0186 & 0.0057 & 0 & 0 \end{pmatrix}.$$

For $s=1,2,$ denotes the trajectory generated by control policy $\bar{\pi}$ and admissible disturbance $w_s(\cdot)$, and $z_i^s = z^s(t_i),\quad i=1,\ldots,6$.

Table 2 contains information about deviations of trajectories at the instants $t_i, i=1,\ldots,6$.

It is clear that the deviations $z^1(t)$ of $z^2(t), t\in T$, at the instants t_3 and t_4 are greater than $2\bar{\gamma}_3$ and $2\bar{\gamma}_4$, respectively. Consequently, in this example, it is impossible to find y_3 and y_4 such that relations (8) are satisfied. Hence, a control policy based on classical feedback is not admissible.

6. CONCLUSION

The proposed policy has several applications for Model Predictive Control of time-varying systems with delay and uncertainties (e.g., refer to [8]): in moving horizon style or shrinking horizon style. In the first case, one should solve a problem of type (47) for any initial system state $z_0 = z(\tau-h)$, and control function $u^*(\cdot)=(u^*(t), t\in[\tau-h,\tau])$ and, depending on its solution, construct a control function only at the first control interval $T_1=[\tau,\tau+h]$. Here τ is the current time instant; $z(\tau-h)$ is the real-world system state at the instant $\tau-h$; $u^*(t), t\in[\tau-h,\tau]$, is the control function that was applied to the real-world system during time interval $[\tau-h,\tau]$.

In shrinking horizon style, at every current instant t_i with a known system state $z(t_{i-1})$ and control function $u_{i-1}(t), t\in T_{i-1}$, we should solve a problem of type (47)

$$V_i^0 = \min_{\lambda_i\ge\mu_i} \ldots \min_{\lambda_{m-1}\ge\mu_{m-1}} \Big(\sum_{s=i}^{m-1}\lambda_s v_s \tag{58}$$
$$+(y_{m+1}+a_{m-i+1})^T D_{m-i}(\lambda_i,\ldots,\lambda_{m-1})(y_{m+1}+a_{m-i+1})\Big)$$

Table 1. Values of cost functionals $J(w)$ and deviations of real system states z_i from constructed states $y_i^0, i=1,\ldots 6,$ for different admissible disturbances $w(\cdot)$.

$w(\cdot)$	$J(w)$	$\|z_1-y_1^0\|_2$	$\|z_2-y_2^0\|_2$	$\|z_3-y_3^0\|_2$	$\|z_4-y_4^0\|_2$	$\|z_5-y_5^0\|_2$	$\|z_6-z_*\|_2$
$w_{\bar{\pi}}(\cdot)$	17.19	2.40	4.75	4.74	4.69	4.70	4.55
0	2.68	0	0	0	0	0	0
$\alpha\cos(t^2)$	2.78	0.43	0.39	0.03	0.02	0.01	0.01
$\alpha\sin(t^2)$	2.76	0.37	0.37	0.05	0.01	0.01	0.01
α	3.54	2.77	4.45	4.45	4.45	4.45	4.53
$-\alpha$	2.25	2.77	4.45	4.45	4.45	4.45	4.53
$\alpha\cos(t)$	9.14	1.78	3.54	3.47	3.38	3.30	3.31
$-\alpha\cos(t)$	7.98	1.78	3.54	3.47	3.38	3.30	3.31
$\alpha\sin(t)$	8.88	1.63	3.29	3.36	3.45	3.53	3.73
$-\alpha\sin(t)$	5.43	1.63	3.29	3.36	3.45	3.53	3.73
$\alpha\sin(\sqrt{t})$	2.94	2.27	1.48	2.36	4.03	3.62	1.83
$-\alpha\sin(\sqrt{t})$	3.03	2.27	1.48	2.36	4.03	3.62	1.83
$\alpha\cos(\sqrt{t})$	2.95	1.04	3.15	3.20	0.89	2.14	3.91
$-\alpha\cos(\sqrt{t})$	3.57	2.27	1.48	2.36	4.03	3.62	1.83

Table 2. Deviations of the trajectories $z^s(t), t\in T,\ s=1,2,$ at the moments $t_i, i=1,\ldots,6$.

$\|z_1^1-z_1^2\|_2$	$\|z_2^1-z_2^2\|_2$	$\|z_3^1-z_3^2\|_2$	$\|z_4^1-z_4^2\|_2$	$\|z_5^1-z_5^2\|_2$	$\|z_6^1-z_6^2\|_2$
5.27	9.75	13.37	14.58	7.34	0

with respect to decision variables $\lambda_i,...,\lambda_{m-1}$. Here, for $s=i+1,...,m$, $a_s=F_{i+1}a_{s-1}$, $a_i=-F_{i+1}y_i$,

$$y_i = F(t_i,t_{i-1})z(t_{i-1}) + \int_{t_{i-1}}^{t_i} F(t_i,t)bu_{i-1}(t)dt.$$

Depending on the solution to the problem (58), we construct a control policy at the interval $[t_i,t_*]$ and only use the first control function from the policy at the next control interval T_i.

Several other policies are considered in [8] (also, refer to the references in [8]). There, the main assumption is that the control laws that define the policy are expressed via given functions with unknown parameters. Parameters of these functions are chosen in an "optimal" manner according to a cost function. The policy $\pi^0(Y^0)$ can be interpreted similarly: we may make the assumption that the control functions that define the policy are expressed via given functions (34) with unknown parameters $y_i \in R^n, i=1,..,m,$ and we try to choose the parameters in an optimal manner according to (38).

However, in this paper, we introduced this policy based on another principle: we did not make any assumptions that control functions are predetermined functions with unknown parameters, but we showed that the policy $\pi^0(Y^0)$ is optimal for the optimization problem being considered.

APPENDIX A

Lemma 3: Let $S,A \in R^{n\times n}$ be positive defined matrices and let $K \in R^{n\times n}$ be a nonsingular matrix. Then, the matrix $D(\lambda) := S + A - \frac{KK^T}{\lambda}$ is positive, defined for all $\lambda \ge \lambda_* := \lambda_{\max}(K^T S^{-1} K)$.

Proof: We prove that the matrix $\bar{D}(\lambda) := S - \frac{KK^T}{\lambda}$ is semi-positive, defined for all $\lambda \ge \lambda_*$.

It is clear that the matrix $\bar{D}(\lambda)$ is semi-positive, defined if and only if the matrix $\tilde{D}(\lambda) := K^{-1}SK^{-T} - I\frac{1}{\lambda}$ is semi-positive. It is well known (refer to [11]) that the matrix $\tilde{D}(\lambda)$ is semi-positive defined if and only if $\lambda_{\min}(K^{-1}SK^{-T}) \ge \frac{1}{\lambda}$ or equivalently $\lambda \ge \lambda_{\max}(K^T S^{-1} K)$. Hence, we prove that, for $\lambda \ge \lambda_*$., the matrix $\bar{D}(\lambda)$ is semi-positive defined.

Based on the fact that $D(\lambda) = A + \bar{D}(\lambda)$, where the matrix A is positive defined, we conclude that the matrix $D(\lambda)$ is positive defined for all $\lambda \ge \lambda_*$.

Lemma 4: Consider the matrix function $S(\lambda)$ $=(A - \sum_{i=1}^{k} \frac{B_i}{\lambda_i})$, $\lambda = (\lambda_1,...,\lambda_k) \in \Lambda$, where the matrices A, B_i, $i=1,...,k$, are positive defined, and the set $\Lambda \subset R_+^k$ is convex. Assume that $S(\lambda)$ is positive definite for all $\lambda \in \Lambda$. Then, the function $\lambda_{\max}(C^T S^{-1}(\lambda) C)$ is convex at Λ. Here $C \in R^{n\times r}$, rank $C=r$.

Proof: We use known results (refer to [11]) in the following:

For any positive defined matrices $A,B \in R^{n\times n}$ and a matrix $C \in R^{n\times r}$, rank $C=r$, the following relations are satisfied:

$$\begin{aligned}&\lambda_{\max}(A) = 1/\lambda_{\min}(A^{-1}),\\ &\lambda_{\min}(A+B) \ge \lambda_{\min}(A), \qquad\qquad (A1)\\ &[C^T(A+B)^{-1}C]^{-1} \ge [C^T A^{-1} C]^{-1} + [C^T B^{-1} C]^{-1}.\end{aligned}$$

Since $\lambda_{\max}(C^T S^{-1}(\lambda)C) = 1/\lambda_{\min}([C^T S^{-1}(\lambda)C]^{-1})$ we prove the convexity of the function $\lambda_{\max}(C^T S^{-1}(\lambda)C)$ at Λ by showing the concavity of the function $q(\lambda) := \lambda_{\min}([C^T S^{-1}(\lambda)C]^{-1})$. Consider the matrix:

$$S(\alpha x + (1-\alpha)y) = A - \sum_{i=1}^{k} \frac{B_i}{\alpha x_i + (1-\alpha)y_i}, \qquad (A2)$$

$x,y \in \Lambda, \alpha \in [0,1]$.

Using the inequality $\frac{1}{\alpha a + (1-\alpha)b} \le \frac{\alpha}{a} + \frac{1-\alpha}{b}$, for all $a>0, b>0, \alpha \in [0,1]$, we may express the matrix (A2) in the form:

$$\begin{aligned}&S(\alpha x + (1-\alpha)y)\\ &= \alpha(A - \sum_{i=1}^{k}\frac{B_i}{x_i}) + (1-\alpha)(A - \sum_{i=1}^{k}\frac{B_i}{y_i}) + \sum_{i=1}^{k} B_i\beta_i \quad (A3)\\ &= \alpha S(x) + (1-\alpha)S(y) + \bar{B},\end{aligned}$$

where $\beta_i \ge 0$, $i=1,..,k,$ are values and the matrix $\bar{B} = \sum_{i=1}^{k} B_i\beta_i$ is semi-positive defined. The concavity of the function $q(\lambda), \lambda \in \Lambda$, follows from the inequalities (A1) and the expression (A3)

$$\begin{aligned}&q(\alpha x + (1-\alpha)y)\\ &= \lambda_{\min}([C^T S^{-1}(\alpha x + (1-\alpha)y)C]^{-1})\\ &= \lambda_{\min}([C^T(\alpha S(x) + (1-\alpha)S(y) + \bar{B})^{-1}C]^{-1})\\ &\ge \alpha\lambda_{\min}[C^T S^{-1}(x)C]^{-1} + (1-\alpha)\lambda_{\min}[C^T S^{-1}(y)C]^{-1}\\ &= \alpha q(x) + (1-\alpha)q(y).\end{aligned}$$

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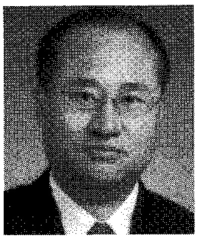

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