J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math.
Volume 15, Number 4 (November 2008), Pages 423–432
ISSN 1226-0657

MINTY'S LEMMA FOR STRONG IMPLICIT VECTOR VARIATIONAL INEQUALITY SYSTEMS

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ABSTRACT. In this paper, we consider a new Minty's Lemma for strong implicit vector variational inequality systems and obtain some existence results for systems of strong implicit vector variational inequalities which generalize some results in [1].

1. INTRODUCTION AND PRELIMINARIES

In [2], Huang and Fang introduced system of order complementarity problems and established some existence theorem by using Ky Fan Lemma and then Kassay, Kolumban and Pales [3] introduced and studied Minty and Stampaccia variational inequality system by using Kakutani-Fan-Glicksberg fixed point theorem. Recently, by those works and by using Kakutani-Fan-Glicksberg fixed point theorem, Fang and Huang [1] provided some existence results for systems of strong implicit vector variational inequalities, for a constant cone, in reflexive Banach spaces.

In this paper, we consider a new Minty's Lemma for strong implicit vector variational inequality systems and obtain some existence results for a system of strong implicit vector variational inequalities which generalize some results in [1].

Throughout this paper, unless other specified, X_i and Y_i are Banach spaces, $K_i \subset X_i$ are nonempty, bounded, closed and convex sets and $C_i \subset Y_i$ be pointed, closed and convex cones with $intC_i \neq \emptyset$. Let $T_i : K \to L(X_i, \widehat{Y}_i)$, where $\widehat{Y}_1 = Y_2 \times Y_3$, $\widehat{Y}_2 = Y_3 \times Y_1$, $\widehat{Y}_3 = Y_1 \times Y_2$ and $K = \prod_{i=1}^{3} K_i$, and $h_i : K_i \times K_i \to X_i$ $(i = 1, 2, 3)$ be mappings. A nonempty subset C of a Hausdorff topological vector space X is said to be a pointed convex cone if

$$C + \lambda C \subseteq C \text{ and } C \cap (-C) = \{\overline{0}\}, \text{ for all } \lambda \geq 0,$$

Received by the editors June 27, 2008. Revised October 16, 2008. Accepted October 28, 2008.
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2000 *Mathematics Subject Classification.* 49J40.
Key words and phrases. strong implicit vector variational inequality system, pseudomonotone, properly quasimonotone, Minty's Lemma, hemi-continuous.

where $\bar{0}$ denotes the zero vector. If $C_1 \subset Y_1$ and $C_2 \subset Y_2$ are pointed convex cones, then $C_1 \times C_2 \subset Y_1 \times Y_2$ is also a pointed convex cone.

Now we consider the following system of strong implicit vector variational inequalities of Stampacchia type (**SSIVVI-S**) and Minty type (**SSIVVI-M**);

(**SSIVVI-S**) Find $x = (x_1, x_2, x_3) \in K$ such that

$$\langle T_i(x), h_i(x_i, y_i)\rangle \geq_{\widehat{C}_i} 0, \text{ for } y_i \in K_i \ (i = 1, 2, 3)$$

and

(**SSIVVI-M**) Find $x = (x_1, x_2, x_3) \in K$ such that

$$\langle T_i(\widehat{x}_i), h_i(y_i, x_i)\rangle \leq_{\widehat{C}_i} 0, \text{ for } y_i \in K_i \ (i = 1, 2, 3),$$

where $\widehat{C}_1 = C_2 \times C_3$, $\widehat{C}_2 = C_3 \times C_1$, $\widehat{C}_3 = C_1 \times C_2$, $\widehat{x}_1 = (y_1, x_2, x_3)$, $\widehat{x}_2 = (x_1, y_2, x_3)$, $\widehat{x}_3 = (x_1, x_2, y_3)$.

Definition 1.1. Let $T_i : K \to L(X_i, \widehat{Y}_i)$ and $h_i : K_i \times K_i \to X_i$ be mappings. $\{T_1, T_2, T_3\}$ is said to be co-pseudomonotone with respect to $\{h_1, h_2, h_3\}$ if for any $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in K$,

$$\langle T_i x, h_i(x_i, y_i)\rangle \geq_{\widehat{C}_i} 0 \ \Rightarrow \ \langle T_i y, h_i(y_i, x_i)\rangle \leq_{\widehat{C}_i} 0.$$

Example 1.1. Let $X_i, Y_i = \mathbb{R}$, $K_i = [0, 10]$, $C_i = \mathbb{R}_+$,

$$T_i(x) = \begin{pmatrix} 2ix_1^2 \\ x_2 + x_3, \end{pmatrix}$$

$h_i(x_i, y_i) = iy_i - i(x_i + 1)^2$ for all $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in K$.

Let x, $y \in K$ such that

$$\begin{aligned}\langle T_i(x), h_i(x_i, y_i)\rangle &= \begin{pmatrix} 2ix_1^2 \\ x_2 + x_3 \end{pmatrix} (iy_i - i(x_i + 1)^2) \\ &= \begin{pmatrix} 2ix_1^2(iy_i - i(x_i + 1)^2) \\ (x_2 + x_3)(iy_i - i(x_i + 1)^2) \end{pmatrix} \geq_{\widehat{C}_i} 0.\end{aligned}$$

The inequality above implies

$$\begin{aligned} iy_i - i(x_i + 1)^2 \geq 0 \ &\Rightarrow y_i \geq (x_i + 1)^2 \\ &\Rightarrow (y_i + 1)^2 \geq y_i + 1 \geq y_i \geq (x_i + 1)^2 \geq x_i \end{aligned}$$

It follows that

$$\begin{aligned}\langle T_i(y), h_i(y_i, x_i)\rangle &= \begin{pmatrix} 2iy_1^2 \\ y_2 + y_3 \end{pmatrix} (ix_i - i(y_i + 1)^2) \\ &= \begin{pmatrix} 2iy_1^2(ix_i - i(y_i + 1)^2) \\ (y_2 + y_3)(ix_i - i(y_i + 1)^2) \end{pmatrix} \leq_{\widehat{C}_i} 0.\end{aligned}$$

Hence $\{T_1, T_2, T_3\}$ is co-pseudomonotone with respect to $\{h_1, h_2, h_3\}$.

Definition 1.2. Let $T_i : K \to L(X_i, \widehat{Y_i})$ and $h_i : K_i \times K_i \to X_i$ be mappings.

(1) $\{T_1, T_2, T_3\}$ is said to be properly co-quasimonotone of Stampacchia type with respect to $\{h_1, h_2, h_3\}$ if for all $m \in \mathbb{N}$, for all vectors $v_i^1, \cdots, v_i^m \in K_i$, and scalars $\lambda^1, \cdots, \lambda^m > 0$ with $\sum_{j=1}^{m} \lambda^j = 1$ and $u_i := \sum_{j=1}^{m} \lambda^j v_i^j$, $\langle T_i \tilde{x_i}, h_i(u_i, v_i^j)\rangle \geq_{\widehat{C_i}} 0$ holds for all j, where

$$\tilde{x_1} = (u_1, x_2, x_3), \ \tilde{x_2} = (x_1, u_2, x_3) \text{ and } \tilde{x_3} = (x_1, x_2, u_3).$$

(2) $\{T_1, T_2, T_3\}$ is said to be properly co-quasimonotone of Minty type with respect to $\{h_1, h_2, h_3\}$ if for all $m \in \mathbb{N}$, for all vectors $v_i^1, \cdots, v_i^m \in K_i$ and scalars $\lambda^1, \cdots, \lambda^m > 0$ with $\sum_{j=1}^{m} \lambda^j = 1$ and $u_i := \sum_{j=1}^{m} \lambda^j v_i^j$, $\langle T_i \overline{x_i}, h_i(v_i^j, u_i)\rangle \leq_{\widehat{C_i}} 0$ holds for all i, where

$$\overline{x_1} = (v_1^j, x_2, x_3), \ \overline{x_2} = (x_1, v_2^j, x_3) \text{ and } \overline{x_3} = (x_1, x_2, v_3^j)$$

Definition 1.3 ([1]). Let X and Y be Banach spaces, and K be a nonempty, closed and convex subset of X. A mapping $h : K \to Y$ is said to be hemicontinuous if, for any fixed x, $y \in K$, a mapping $L : [0,1] \to Y$ defined by $L(t) = h((1-t)x + ty)$ is continuous at 0^+, i.e., $\lim_{t \to 0^+} L(t) = L(0)$.

The following lemma is obtained from Theorem 3.3 in [4].

Lemma 1.1. *Let X_i be reflexive Banach spaces, and let $T_i : K \to L(X_i, \widehat{Y_i})$, where $\widehat{Y_1} = Y_2 \times Y_3$, $\widehat{Y_2} = Y_3 \times Y_1$ and $\widehat{Y_3} = Y_1 \times Y_2$, and $h_i : K_i \times K_i \to X_i$ be mappings satisfying the following conditions ($i = 1, 2, 3$):*

(1) $\langle T_i(x), h_i(x_i, x_i)\rangle \in -\widehat{C_i}$ $(i = 1, 2, 3)$;

(2) *for any given $x = (x_1, x_2, x_3) \in K$, $\{T_1, T_2, T_3\}$ are properly co-quasimonotone of Minty type with respect to $\{h_1, h_2, h_3\}$;*

(3) *h_i is continuous.*

Then the following variational inequality (VI) has a solution;

(VI) Find $x_0 = (x_1^0, x_2^0, x_3^0) \in \prod_{i=1}^{3} K_{M_i}$, where $K_{M_i} = K_i \cap M_i \neq \phi$ for M_i are finite-dimensional subspaces of X_i such that

$$\langle T_i(x^i), g(z_i, x_i^0)\rangle \leq_{\widehat{C_i}} 0, \ \ z_i \in K_{M_i} \textit{ for } i = 1, 2, 3$$

where $x^1 = (z_1, x_2, x_3)$, $x^2 = (x_1, z_2, x_3)$ and $x^3 = (x_1, x_2, z_3)$.

Definition 1.4 ([6]). Let X, Y be Hausdorff topological spaces and $T : X \to 2^Y$ be a set-valued mapping. T is said to be *upper semicontinuous* (shortly, u.s.c.) at $x_0 \in X$ if for any neighborhood $N(T(x_0))$ of $T(x_0)$, there exists a neighborhood $N(x_0)$ of x_0 such that

$\forall x \in N(x_0), T(x) \subset N(T(x_0))$. We say that T is u.s.c. if T is u.s.c. at every point $x \in X$.

Lemma 1.2 ([5]). *Let X and Y be Hausdorff topological spaces, and $F : X \to 2^Y$ be a multivalued mapping. If Y is compact and F is closed, then F is u.s.c..*

Theorem 1.1 ([6, Kakutani-Fan-Glicksberg fixed point theorem]).
Let X be a nonempty compact convex subset of a locally convex Hausdorff topological vector space E. Assume that $T : X \to 2^X$ is an u.s.c. mapping with nonempty closed convex values. Then T has a fixed point on X.

Lemma 1.3 ([1]). *Let C be a pointed, closed and convex cone of a real Banach space E. Then for any, $a \in -C$ and $b \notin C$, we have $t_1 a + t_2 b \notin C$ for all t_1, $t_2 > 0$.*

2. MAIN RESULTS

First, we consider a new Minty's Lemma for a system of strong implicit vector variational inequalities.

Theorem 2.1. *Let $T_i : K \to L(X_i, \widehat{Y_i})$, and $h_i : K_i \times K_i \to X_i$ be mappings satisfying the following conditions $(i = 1, 2, 3)$; for any given $x = (x_1, x_2, x_3) \in K$*

1. *$\{T_1, T_2, T_3\}$ is co-pseudomonotone with respect to $\{h_1, h_2, h_3\}$;*
2. *h_i is bilinear such that $h_i(a, b) + h_i(b, a) = 0$ for $a, b \in K_i$;*
3. *for fixed $v = (v_1, v_2, v_3) \in K$, $u \mapsto \langle T_i(u), h_i(u_i, v_i)\rangle$ is hemicontinuous $(i = 1, 2, 3)$.*

Then for a given point $x \in K$, the following conclusions are equivalent

(i) $\langle T_i(x), h_i(x_i, y_i)\rangle \geq_{\widehat{C}_i} 0$, *for* $y_i \in K_i$;

(ii) $\langle T_i(\widehat{x}_i), h_i(y_i, x_i)\rangle \leq_{\widehat{C}_i} 0$, *for* $y_i \in K_i$; $(i = 1, 2, 3)$.

Proof. (ii) is easily shown from (i) by the condition (1).

Conversely, for any given $y = (y_1, y_2, y_3) \in K$ and $t \in (0, 1)$, let $y^t = x + t(y - x)$. It follows from (ii) that

$$\langle T_i(y^t), h_i(y_i^t, x_i)\rangle \leq_{\widehat{C}_i} 0.$$

Now we show that $\langle T_i(y^t), h_i(y_i^t, y_i)\rangle \geq_{\widehat{C}_i} 0$ for all $t \in (0, 1)$. Suppose that there exists some $s \in (0,1)$ such that

$$\langle T_i(y^s), h_i(y_i^s, y_i)\rangle \not\geq_{\widehat{C}_i} 0.$$

By Lemma 1.3 and the bilinearity of h_i, we have

$$\begin{aligned}\langle T_i(y^s), h_i(y_i^s, y_i^s)\rangle &= \langle T_i(y^s), h_i(y_i^s, x_i + s(y_i - x_i))\rangle \\ &= \langle T_i(y^s), h_i((1+s-s)y_i^s, (1-s)x_i + sy_i)\rangle \\ &= s\langle T_i(y^s), h_i(y_i^s, y_i)\rangle + (1-s)\langle T_i(y^s), h_i(y_i^s, x_i)\rangle \\ &\notin \widehat{C}_i,\end{aligned}$$

which contradicts condition (2).

Hence $\langle T_i(y^t), h_i(y_i^t, y_i)\rangle \geq_{\widehat{C}_i} 0$, for $t \in (0,1)$. From condition (3), for fixed $v \in K$, a mapping $L_i : K \to \widehat{Y}_i$ defined by

$$L_i(u) = \langle T_i u, h_i(u_i, v_i)\rangle$$

for $u = (u_1, u_2, u_3) \in K$, is hemicontinuous, i.e., a mapping from $[0,1]$ to $\widehat{Y}_i$

$$t \mapsto \langle T_i(x + t(y-x)), h_i(x_i + t(y_i - x_i), y_i)\rangle$$

is continuous at 0^+ for all $x, y \in K$.

Thus

$$\begin{aligned}\langle T_i x, h_i(x_i, y_i)\rangle &= \lim_{t\to 0^+}\langle T_i(x + t(y-x)), h_i(x_i + t(y_i - x_i), y_i)\rangle \\ &= \lim_{t\to 0^+}\langle T_i(y^t), h_i(y_i^t, y_i)\rangle \geq_{\widehat{C}_i} 0, \ \ \forall y \in K.\end{aligned}$$

□

Now, we consider some existence results for systems of strong implicit vector variational inequalities.

Theorem 2.2. *Let X_i be reflexive Banach spaces, $T_i : K \to L(X_i, \widehat{Y}_i)$, and $h_i : K_i \times K_i \to X_i$ be mappings satisfying the following conditions ($i = 1,2,3$);*

(1) $\langle T_i(x), h_i(x_i, x_i)\rangle \leq_{\widehat{C}_i} 0$ $(i = 1,2,3)$;

(2) *for any given $x = (x_1, x_2, x_3) \in K$, $\{T_1, T_2, T_3\}$ are properly co-quasimonotone of Minty type with respect to $\{h_1, h_2, h_3\}$;*

(3) *for any given $x = (x_1, x_2, x_3) \in K$ and $z = (z_1, z_2, z_3) \in \prod_{i=1}^{3} X_i$, $\langle T_i(\check{x}_i), z_i\rangle$ is continuous from the weak topology of X_k to the norm topology of $\widehat{Y}_l$, where for $k = 1$, $l = 3$, for $k = 2$, $l = 1$ and for $k = 3$, $l = 2$, and $\check{x}_1 = (x_1, \cdot, x_3)$, $\check{x}_2 = (x_1, x_2, \cdot)$, $\check{x}_3 = (\cdot, x_2, x_3) \in K$.*

(4) *h is linear and continuous such that $h_i(a,b) + h_i(b,a) = 0$ for $a, b \in K_i$.*

Then the problem (**SSIVVI − M**) *is solvable.*

Proof. Let $\mathcal{A}_i = \{M_i : M_i$ is a finite dimensional subspace of X_i with $K_{M_i} = K_i \cap M_i \neq \emptyset\}$ for $i = 1, 2, 3$. Define a multivalued mapping $G : \prod_{i=1}^{3} K_{M_i} \to 2^{\prod_{i=1}^{3} K_{M_i}}$ by

$$G(x) = \left\{ x_0 \in \prod_{i=1}^{3} K_{M_i} : x_0 \text{ solves problem } (VI) \right\}, \quad \forall x \in \prod_{i=1}^{3} K_{M_i}.$$

By Lemma 1.1, (VI) is solvable, $G(x)$ is nonempty. Since K_i is bounded, K is bounded. Now we claim that $G(x)$ is closed. Let $\langle (x_1^n, x_2^n, x_3^n) \rangle$ be a sequence in $G(x)$ converging to $(x_1^0, x_2^0, x_3^0) \in \prod_{i=1}^{3} K_{M_i}$. Then

$$\begin{aligned} \langle T_i(x^i), g_i(z_i, x_i^0) \rangle &= \left\langle T_i(x_i), h_i\left(z_i, \lim_{n\to\infty} x_i^n\right) \right\rangle \\ &= \left\langle T_i(x^i), \lim_{n\to\infty} h_i(z_i, x_i^n) \right\rangle \\ &= \lim_{n\to\infty} \langle T_i(x^i), g_i(z_i, x_i^n) \rangle \leq_{\widehat{C}_i} 0. \end{aligned}$$

Hence $G(x)$ is closed. And $G(x)$ is convex, in fact, for $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in X$ and for $t \in (0, 1)$,

$$\begin{aligned} & t(x_1, x_2, x_3) + (1-t)(y_1, y_2, y_3) \\ &= (tx_1 + (1-t)y_1, tx_2 + (1-t)y_2, tx_3 + (1-t)y_3) \in G(x), \\ & \langle T_i(x^i), h_i(z_i, tx_i + (1-t)y_i) \rangle \\ &= \langle T_i(x^i), h_i(z_i, tx_i) \rangle + \langle T_i(x^i), h_i(z_i, (1-t)y_i) \rangle \\ &= \langle T_i(x^i), th_i(z_i, x_i) \rangle + \langle T_i(x^i), (1-t)h_i(z_i, y_i) \rangle \\ &= t\langle T_i(x^i), h_i(z_i, x_i) \rangle + (1-t)\langle T_i(x^i), h_i(z_i, y_i) \rangle \leq_{\widehat{C}_i} 0. \end{aligned}$$

Hence $G(x)$ is convex. Now we show that $G\left(\prod_{i=1}^{3} K_{M_i}\right)$ is closed in $\prod_{i=1}^{3} K_{M_i} \times \prod_{i=1}^{3} K_{M_i}$. Let $\langle ((x_1^n, x_2^n, x_3^n), (y_1^n, y_2^n, y_3^n)) \rangle$ be a sequence in $\prod_{i=1}^{3} K_{M_i} \times \prod_{i=1}^{3} K_{M_i}$ such that

$$\begin{aligned} \lim_{n\to\infty} ((x_1^n, x_2^n, x_3^n), (y_1^n, y_2^n, y_3^n)) = ((x_1, x_2, x_3), (y_1, y_2, y_3)), (y_1^n, y_2^n, y_3^n) \\ \in G(x_1^n, x_2^n, x_3^n), \forall n \in \mathbb{N}. \end{aligned}$$

Now we show that $(y_1, y_2, y_3) \in G(x_1, x_2, x_3)$.

$$\langle T_1(z_1, x_2, x_3), h_1(z_1, y_1) \rangle = \left\langle \lim_{n\to\infty} T_1(z_1, x_2^n, x_3), h_1\left(z_1, \lim_{n\to\infty} y_1^n\right) \right\rangle$$

$$
\begin{aligned}
&= \left\langle \lim_{n\to\infty} T_1(z_1, x_2^n, x_3), \lim_{n\to\infty} h_1(z_1, y_1^n) \right\rangle \\
&= \lim_{n\to\infty} \langle T_1(z_1, x_2^n, x_3), h_1(z_1, y_1^n)\rangle \leq_{\widehat{C}_1} 0.
\end{aligned}
$$

$$
\begin{aligned}
\langle T_2(x_1, z_2, x_3), h_2(z_2, y_2)\rangle &= \left\langle \lim_{n\to\infty} T_2(x_1, z_2, x_3^n), h_2\left(z_2, \lim_{n\to\infty} y_2^n\right)\right\rangle \\
&= \left\langle \lim_{n\to\infty} T_2(x_1, z_2, x_3^n), \lim_{n\to\infty} h_2(z_2, y_2^n)\right\rangle \\
&= \lim_{n\to\infty} \langle T_2(x_1, z_2, x_3^n), h_2(z_2, y_2^n)\rangle \leq_{\widehat{C}_2} 0. \\
\langle T_3(x_1, x_2, z_3), h_3(z_3, y_3)\rangle &= \left\langle \lim_{n\to\infty} T_3(x_1^n, x_2, z_3), h_3\left(z_3, \lim_{n\to\infty} y_3^n\right)\right\rangle \\
&= \left\langle \lim_{n\to\infty} T_3(x_1^n, x_2, z_3), \lim_{n\to\infty} h_3(z_3, y_3^n)\right\rangle \\
&= \lim_{n\to\infty} \langle T_3(x_1^n, x_2, z_3), h_3(z_3, y_3^n)\rangle \leq_{\widehat{C}_3} 0.
\end{aligned}
$$

Hence $G(\prod_{i=1}^{3} K_{M_i})$ is closed in $\prod_{i=1}^{3} K_{M_i} \times \prod_{i=1}^{3} K_{M_i}$. Since G is closed, G has a closed graph. Since G is closed and bounded, G is u.s.c.. The Kakutani-Fan-Glicksberg fixed point theorem implies that there exists $x_0 \in \prod_{i=1}^{3} K_{M_i}$ such that

$$\langle T_i(x_0^i), h_i(z_i, x_i^0)\rangle \leq_{\widehat{C}_i} 0, \ \ \forall z_i \in K_{M_i} \ (i = 1, 2, 3),$$

where $x_0^1 = (z_1, x_2^0, x_3)$, $x_0^2 = (x_1, z_2, x_3^0)$ and $x_0^3 = (x_1^0, x_2, z_3)$.

For any $M := (M_1, M_2, M_3) \in \prod_{i=1}^{3} \mathcal{A}_i$, let S_M be the solution set of the following vector variational inequality;

Find $x \in K$ such that

$$\langle T_i(x^i), h_i(z_i, x_i)\rangle \leq_{\widehat{C}_i} 0, \ \ \forall z_i \in K_{M_i} \ (i = 1, 2, 3).$$

By the similar argument, S_M is nonempty and bounded for all $M \in \prod_{i=1}^{3} K_{M_i}$. Denote by $\overline{S}_M$ the weak closure of S_M in $\prod_{i=1}^{3} X_i$. Since X_i $(i = 1, 2, 3)$ are reflexive, $\overline{S}_M$ is weakly compact.

Let $M_i^k \in \mathcal{A}_i$ for $k = 1, \cdots, n$. For any $M^k = (M_1^k, M_2^k, M_3^k) \in \prod_{i=1}^{3} \mathcal{A}_i$ for $k = 1, \cdots, n$,

$$S_{L_M} \subset \bigcap_k S_{M^k},$$

where L_M denotes the linear subspace spanned by $\bigcup_k M^k$.

Hence $\{\overline{S}_M : M \in \prod_{i=1}^{3} \mathcal{A}_i\}$ has the finite intersection property

$$\bigcap_{M\in\prod_{i=1}^{3}\mathcal{A}_i} \overline{S}_M \neq \emptyset.$$

Let

$$x^* = (x_1^*, x_2^*, x_3^*) \in \bigcap_{M\in\prod_{i=1}^{3}\mathcal{A}_i} \overline{S}_M.$$

We claim that

$$\langle T_i(x_*^i), h_i(z_i, x_i^*)\rangle \leq_{\widehat{C}_i} 0, \ \ \forall z_i \in K_i \ (i = 1, 2, 3),$$

where $x_*^1 = (z_1, x_2^*, x_3)$, $x_*^2 = (x_1, z_2, x_3^*)$ and $x_*^3 = (x_1^*, x_2, x_3)$.

In fact, for any given $x_i \in K_i$, choose $M_i \in \mathcal{A}_i$ such that x_i, $x_i^* \in M_i$. Since $x^* \in \overline{S}_M$, there exists a net $\langle x^\alpha\rangle = \langle(x_1^\alpha, x_2^\alpha, x_3^\alpha)\rangle \in S_M$ converging to x^* weakly in S_M. Hence $\langle T_i(x_\alpha^i), h_i(z_i, x_i^\alpha)\rangle \leq_{\widehat{C}_i} 0$. By the condition (3),

$$\langle T_i(x_*^i), h_i(z_i, x_i^*)\rangle \leq_{\widehat{C}_i} 0, \ \ \forall z_i \in K_i \ (i = 1, 2, 3).$$

□

Theorem 2.3. *Let X_i are reflexive Banach spaces, $T_i : K \to L(X_i, \widehat{Y}_i)$ and $h_i : K_i \times K_i \to X_i$ be mappings $(i = 1, 2, 3)$.*

(1) *for fixed $v = (v_1, v_2, v_3) \in K$, $u \mapsto \langle T_i(u), h_i(u_i, v_i)\rangle$ is hemicontinuous $(i = 1, 2, 3)$;*
(2) *for any given $x = (x_1, x_2, x_3) \in K$ and $\{T_1, T_2, T_3\}$ is co-pseudomonotone and properly co-quasimonotone of Stampacchia type with respect to $\{h_1, h_2, h_3\}$;*
(3) *$\langle T_i(x), h_i(x_i, x_i)\rangle \geq_{\widehat{C}_i} 0$ for all $x \in K$;*
(4) *for any given $x \in K$ and $z = (z_1, z_2, z_3) \in \prod_{i=1}^{3} X_i$, $\langle T_i(\check{x}_i), z_i\rangle$ is continuous from the weak topology of X_k to the norm topology of $\widehat{Y}_l$, where for $k = 1$, $l = 3$, for $k = 2$, $l = 1$ and for $k = 3$, $l = 2$, and $\check{x}_1$, $\check{x}_2$, $\check{x}_3 \in K$. ;*
(5) *h is bilinear and continuous such that $h_i(a, b) + h_i(b, a) = 0$ for $a, b \in K_i$.*

Then $(\mathbf{SSIVVI - S})$ *is solvable.*

Proof. From the existence results for strong implicit vector variational inequality (in [4]), there exists $x^* = (x_1^*, x_2^*, x_3^*) \in K$ such that

$$\langle T_i(x_{i*}), h_i(x_i, x_i^*)\rangle \leq_{\widehat{C}_i} 0, \ \ \forall x_i \in K_i \ (i = 1, 2, 3),$$

where $x_{1*} = (x_1, x_2^*, x_3)$, $x_{2*} = (x_1, x_2, x_3^*)$ and $x_{3*} = (x_1^*, x_2, x_3)$.

From conditions (1), (2) and (3) imply that $T_i(\overline{x}_i^*)$ $(i = 1, 2, 3)$ satisfy all the assumptions of Minty's lemma. Hence,

$$\langle T_i(x^*), h_i(x_i^*, x_i)\rangle \geq_{\widehat{C}_i} 0, \ \ \forall x_i \in K_i \ (i = 1, 2, 3).$$

□

By putting $h_i(x, y) = y - g_i(x)$, where $g_i : K_i \to X_i (i = 1, 2, 3)$, in Theorem 2.2 and Theorem 2.3, we obtain the following Corollary 2.1 and Corollary 2.2, respectively, which extend some results in [1].

Corollary 2.1. *Let $T_i : K \to L(X_i, \widehat{Y}_i)$, $g_i : K_i \to X_i$ $(i = 1, 2, 3)$ be mappings.*

(1) *for any given $x = (x_1, x_2, x_3) \in K$, $\{T_1, T_2, T_3\}$ is properly co-quasimonotone of Minty type with respect to $\{g_1, g_2, g_3\}$;*

(2) *for any given $x \in K$ and $z = (z_1, z_2, z_3) \in \prod_{i=1}^{3} X_i$, $\langle T_i(\check{x}_i), z_i\rangle$ is continuous from the weak topology of X_k to the norm topology of $\widehat{Y}_l$, where for $k = 1$, $l = 3$, for $k = 2$, $l = 1$ and for $k = 3$, $l = 2$, and $\check{x}_1$, $\check{x}_2$, $\check{x}_3 \in K$. .*

Then there exists $x^* = (x_1^*, x_2^*, x_3^*) \in K$ such that

$$\langle T_i(x_{i*}), x_i^* - g_i(x)\rangle \leq_{\widehat{C}_i} 0, \ \ \forall x_i \in K_i \ (i = 1, 2, 3).$$

Corollary 2.2. *Let $T_i : K \to L(X_i, \widehat{Y}_i), g_i : K_i \to X_i$ $(i = 1, 2, 3)$ be mappings.*

(1) *for fixed $v = (v_1, v_2, v_3) \in K$, $u \mapsto \langle T_i(u), h_i(u_i, v_i)\rangle$ is hemicontinuous $(i = 1, 2, 3)$;*

(2) *for any given $x \in K$, $\{T_1, T_2, T_3\}$ is co-pseudomonotone with respect to $\{g_1, g_2, g_3\}$;*

(3) *for any given $x \in K$, $\{T_1, T_2, T_3\}$ is properly co-quasimonotone of Stampacchia type with respect to $\{g_1, g_2, g_3\}$;*

(4) *for any given $x \in K$ and $z \in \prod_{i=1}^{3} X_i$, $\langle T_i(\check{x}_i), z_i\rangle$ is continuous from the weak topology of X_k to the norm topology of $\widehat{Y}_l$, where for $k = 1$, $l = 3$, for $k = 2$, $l = 1$ and for $k = 3$, $l = 2$, and $\check{x}_1$, $\check{x}_2$, $\check{x}_3 \in K$. .*

Then there exists $x^* \in K$ such that

$$\langle T_i(x_{i*}), x_i - g_i(x_i^*)\rangle \geq_{\widehat{C}_i} 0, \ \ \forall x_i \in K_i \ (i = 1, 2, 3).$$

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