J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN 1226-0657
Volume 15, Number 4 (November 2008), Pages 393–400

ON AN ADDITIVE FUNCTIONAL INEQUALITY IN NORMED MODULES OVER A C^*-ALGEBRA

JONG SU AN

ABSTRACT. In this paper, we investigate the following additive functional inequality

$$\|f(x)+f(y)+f(z)+f(w)\| \le \|f(x+y)+f(z+w)\| \tag{0.1}$$

in normed modules over a C^*-algebra. This is applied to understand homomorphisms in C^*-algebras.

Moreover, we prove the generalized Hyers-Ulam stability of the functional inequality

$$\begin{aligned} &\|f(x)+f(y)+f(z)+f(w)\| \\ &\le \|f(x+y+z+w)\| + \theta\|x\|^p\|y\|^p\|z\|^p\|w\|^p \end{aligned} \tag{0.2}$$

in real Banach spaces, where θ, p are positive real numbers with $4p \neq 1$.

1. INTRODUCTION

Ulam [15] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

We are given a group G and a metric group G' with metric $\rho(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies $\rho(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then a homomorphism $h : G \to G'$ exists with $\rho(f(x), h(x)) < \epsilon$ for all $x \in G$?

By now an affirmative answer has been given in several cases, and some interesting variations of the problem have also been investigated.

Received by the editors February 19, 2008. Revised September 8, 2008. Accepted October 28, 2008.

2000 *Mathematics Subject Classification.* Primary 39B72, Secondary 46L05.

Key words and phrases. Jordan-von Neumann functional equation, functional inequality, linear mapping in normed modules over a C^*-algebra.

Hyers [4] considered the case of approximately additive mappings $f: E \to E'$, where E and E' are Banach spaces and f satisfies *Hyers inequality*

$$\|f(x+y)-f(x)-f(y)\| \leq \epsilon$$

for all $x, y \in E$. It was shown that the limit

$$L(x)=\lim_{n\to\infty} 2^{-n} f(2^n x)$$

exists for all $x \in E$ and that $L: E \to E'$ is the unique additive mapping satisfying

$$\|f(x)-L(x)\| \leq \epsilon.$$

No continuity conditions are required for this result, but if $f(tx)$ is continuous in the real variable t for each fixed x, then L is linear, and if f is continuous at a single point of E then $L: E \to E'$ is also continuous.

In 1982-1994, a generalization of this result was proved by the author J.M. Rassias. He introduced the following weaker condition (or weaker inequality or Cauchy inequality)

$$\|f(x+y)-f(x)-f(y)\| \leq \theta ||x||^p ||y||^q$$

for all x, y in E, controlled by (or involving) a product of different powers of norms, where $\theta \geq 0$ and real $p,\ q: r = p+q \neq 1$, and retained the condition of continuity of $f(tx)$ in t for fixed x. Besides he investigated that it is possible to replace ϵ in the above Hyers inequality by a non-negative real-valued function such that the pertinent series converges and other conditions hold and still obtain stability results. In all the cases investigated in these results, the approach to the existence question was to prove asymptotic type formulas of the form

$$L(x)=\lim_{n\to\infty} 2^{-n} f(2^n x),$$

or

$$L(x)=\lim_{n\to\infty} 2^{n} f(2^{-n} x).$$

Theorem 1.1 ([7]-[13]). *Let X be a real normed linear space and Y a real Banach space. Assume in addition that $f: X \to Y$ is an approximately additive mapping for which there exist constant $\theta \geq 0$ and $p, q \in \mathbb{R}$ such that $r = p+q \neq 1$ and f satisfies the Cauchy-Rassias inequality*

$$\|f(x+y)-f(x)-f(y)\| \leq \theta ||x||^p ||y||^q$$

for all $x, y \in X$. Then there exists a unique additive mapping $L : X \to Y$ satisfying

$$\|f(x) - L(x)\| \leq \frac{\theta}{|2^r - 2|}\|x\|^r$$

for all $x \in X$. If in addition $f : X \to Y$ is a mapping such that the transformation $t \to f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then L is an $\mathbb{R}$-linear mapping.

Gilányi [2] showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(x-y)\| \leq \|f(x+y)\| \tag{1.1}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(x+y) + f(x-y).$$

See also [14]. Fechner [1] and Gilányi [3] proved the generalized Hyers-Ulam stability of the functional inequality (1.1). Park, Cho and Han [6] investigated the functional inequality

$$\|f(x) + f(y) + f(z)\| \quad \leq \quad \|f(x+y+z)\| \tag{1.2}$$

in Banach spaces, and proved the generalized Hyers-Ulam stability of the functional inequality (1.2) in Banach spaces.

Throughout this paper, let A be a unital C^*-algebra with unitary group $U(A)$ and unit e, and let B be a C^*-algebra. Assume that X is a normed A-module with norm $\|\cdot\|_X$ and that Y is a normed A-module with norm $\|\cdot\|_Y$.

In this paper, we investigate an A-linear mapping associated with the functional inequality (0.1). This is applied to understand homomorphisms in C^*-algebras. Moreover, we prove the generalized Hyers-Ulam stability of the functional inequality (0.2) in real Banach spaces.

2. Functional Inequalities in Normed Modules over a C^*-Algebra

Theorem 2.1. *Let $f : X \to Y$ be a mapping such that*

$$\|uf(x) + f(y) + f(z) + f(w)\|_Y \leq \|f(ux+y) + f(z+w)\|_Y \tag{2.1}$$

for all $x, y, z, w \in X$ and all $u \in U(A)$. Then the mapping $f : X \to Y$ is A-linear.

Proof. Letting $x = y = z = w = 0$ and $u = e \in U(A)$ in (2.1), we get

$$\|4f(0)\|_Y \leq \|2f(0)\|_Y.$$

So $f(0) = 0$.

Letting $z = w = 0$ in (2.1), we get

$$\|f(x) + f(y)\|_Y \le \|f(x+y)\|_Y \tag{2.2}$$

for all $x, y \in X$.

Replacing x and y by $x+y$ and $z+w$ in (2.2), respectively, we get

$$\|f(x+y) + f(z+w)\|_Y \le \|f(x+y+z+w)\|_Y$$

for all $x, y, z, w \in X$. So

$$\|f(x) + f(y) + f(z) + f(w)\|_Y \le \|f(x+y+z+w)\|_Y \tag{2.3}$$

for all $x, y, z, w \in X$.

Letting $z = w = 0$ and $y = -x$ in (2.3), we get

$$\|f(x) + f(-x)\|_Y \le \|f(0)\|_Y = 0$$

for all $x \in X$. Hence $f(-x) = -f(x)$ for all $x \in X$.

Letting $z = -x - y$ and $w = 0$ in (2.3), we get

$$\|f(x) + f(y) - f(x+y)\|_Y = \|f(x) + f(y) + f(-x-y)\|_Y \le \|f(0)\|_Y = 0$$

for all $x, y \in X$. Thus

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$.

Letting $y = -ux$ and $y = w = 0$ in (2.1), we get

$$\|f(ux) - f(ux)\|_Y = \|f(ux) + f(-ux)\|_Y \le \|2f(0)\|_Y = 0$$

for all $x \in X$ and all $u \in U(A)$. Thus

$$f(ux) = uf(x) \tag{2.4}$$

for all $u \in U(A)$ and all $x \in X$.

Now let $a \in A$ $(a \neq 0)$ and M an integer greater than $4|a|$. Then $|\frac{a}{M}| < \frac{1}{4} < 1 - \frac{2}{3} = \frac{1}{3}$. By Theorem 1 of [5], there exist three elements $u_1, u_2, u_3 \in U(A)$ such that $3\frac{a}{M} = u_1 + u_2 + u_3$. So by (2.4)

$$\begin{aligned} f(ax) &= f\left(\frac{M}{3} \cdot 3\frac{a}{M}x\right) = M \cdot f\left(\frac{1}{3} \cdot 3\frac{a}{M}x\right) \\ &= \frac{M}{3} f\left(3\frac{a}{M}x\right) = \frac{M}{3} f(u_1x + u_2x + u_3x) \\ &= \frac{M}{3}(f(u_1x) + f(u_2x) + f(u_3x)) = \frac{M}{3}(u_1 + u_2 + u_3)f(x) \\ &= \frac{M}{3} \cdot 3\frac{a}{M} f(x) = af(x) \end{aligned}$$

for all $x \in X$. So $f : X \to Y$ is A-linear, as desired. □

Corollary 2.2. *Let $f : A \to B$ be a multiplicative mapping such that*

$$\|\mu f(x) + f(y) + f(z) + f(w)\| \leq \|f(\mu x + y) + f(z + w)\| \tag{2.5}$$

for all $x, y, z, w \in A$ and all $\mu \in \mathbb{T} := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$. Then the mapping $f : A \to B$ is a C^*-algebra homomorphism.

Proof. By Theorem 2.1, the multiplicative mapping $f : A \to B$ is $\mathbb{C}$-linear, since C^*-algebras are normed modules over $\mathbb{C}$. So the multiplicative mapping $f : A \to B$ is a C^*-algebra homomorphism, as desired. □

3. Generalization of Cauchy-Rassias Inequalities

In this section, we prove the generalized Hyers-Ulam stability of the functional inequality (2.3) in real Banach spaces.

Theorem 3.1. *Let X be a real normed linear space and Y a real Banach space. Assume in addition that $f : X \to Y$ is an approximately additive odd mapping for which there exist a constant $\theta \geq 0$ and $p \in \mathbb{R}$ such that $4p \neq 1$ and f satisfies the general Cauchy-Rassias inequality*

$$\begin{aligned} &\|f(x) + f(y) + f(z) + f(w)\| \\ &\leq \|f(x + y + z + w)\| + \theta||x||^p||y||^p||z||^p||w||^p \end{aligned} \tag{3.1}$$

for all $x, y, z, w \in X$. Then there exists a unique additive mapping $L : X \to Y$ satisfying

$$\|f(x) - L(x)\| \leq \frac{3^p\theta}{|81^p - 3|}||x||^{4p} \tag{3.2}$$

for all $x \in X$. If in addition $f : X \to Y$ is a mapping such that the transformation $t \to f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then L is an $\mathbb{R}$-linear mapping.

Proof. Since f is odd, $f(0) = 0$ and $f(-x) = -f(x)$ for all $x \in X$.

Letting $y = z = x$ and $w = -3x$ in (3.1), we get

$$\|3f(x) - f(3x)\| \leq 3^p\theta||x||^{4p} \tag{3.3}$$

for all $x \in X$. So

$$\left\|f(x) - 3f\left(\frac{x}{3}\right)\right\| \leq \frac{\theta}{27^p}||x||^{4p}$$

for all $x \in X$. Hence

$$\left\|3^l f\left(\frac{x}{3^l}\right) - 3^m f\left(\frac{x}{3^m}\right)\right\| \leq \frac{\theta}{27^p}\sum_{j=l}^{m-1}\frac{3^j}{81^{pj}}||x||^{4p} \tag{3.4}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$.

Assume that $p > \frac{1}{4}$. It follows from (3.4) that the sequence $\{3^k f(\frac{x}{3^k})\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{3^k f(\frac{x}{3^k})\}$ converges. So one can define the mapping $L : X \to Y$ by

$$L(x) := \lim_{k\to\infty} 3^k f\left(\frac{x}{3^k}\right)$$

for all $x \in X$.

Letting $l = 0$ and $m \to \infty$ in (3.4), we get

$$\|f(x) - L(x)\| \leq \frac{3^p\theta}{81^p - 3}||x||^{4p}$$

for all $x \in X$.

It follows from (3.1) that

$$\left\|3^k f\left(\frac{x}{3^k}\right) + 3^k f\left(\frac{y}{3^k}\right) + 3^k f\left(\frac{z}{3^k}\right) + 3^k f\left(\frac{w}{3^k}\right)\right\|$$

$$\leq \left\|3^k f\left(\frac{x+y+z+w}{3^k}\right)\right\| + \frac{3^k\theta}{81^{pk}}||x||^p||y||^p||z||^p||w||^p \tag{3.5}$$

for all $x, y, z, w \in X$. Letting $k \to \infty$ in (3.5), we get

$$\|L(x) + L(y) + L(z) + L(w)\| \leq \|L(x + y + z + w)\| \tag{3.6}$$

for all $x, y, z, w \in X$. It is easy to show that $L : X \to Y$ is odd. Letting $z = -x - y$ and $w = 0$ in (3.6), we get $L(x + y) = L(x) + L(y)$ for all $x, y \in X$. So there exists an additive mapping $L : X \to Y$ satisfying (3.2) for the case $p > \frac{1}{4}$.

Now, let $T : X \to Y$ be another additive mapping satisfying (3.2). Then we have

$$\begin{aligned} \|L(x) - T(x)\| &= 3^q \left\| L\left(\frac{x}{3^q}\right) - T\left(\frac{x}{3^q}\right) \right\| \\ &\le 3^q \left(\left\| L\left(\frac{x}{3^q}\right) - f\left(\frac{x}{3^q}\right) \right\| + \left\| T\left(\frac{x}{3^q}\right) - f\left(\frac{x}{3^q}\right) \right\| \right) \\ &\le \frac{3^p\theta}{81^p - 3} \cdot \frac{2 \cdot 3^q}{81^q} ||x||^{4p}, \end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $L(x) = T(x)$ for all $x \in X$. This proves the uniqueness of L.

Assume that $p < \frac{1}{4}$. It follows from (3.3) that

$$\left\| f(x) - \frac{1}{3} f(3x) \right\| \le 3^{p-1}\theta ||x||^{4p}$$

for all $x \in X$. Hence

$$\left\| \frac{1}{3^l} f(3^l x) - \frac{1}{3^m} f(3^m x) \right\| \le 3^{p-1}\theta \sum_{j=l}^{m-1} \frac{81^{pj}}{3^j} ||x||^{4p} \tag{3.7}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$.

It follows from (3.7) that the sequence $\{\frac{1}{3^k} f(3^k x)\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{3^k} f(3^k x)\}$ converges. So one can define the mapping $L : X \to Y$ by

$$L(x) := \lim_{k\to\infty} \frac{1}{3^k} f\left(3^k x\right)$$

for all $x \in X$.

Letting $l = 0$ and $m \to \infty$ in (3.7), we get

$$\|f(x) - L(x)\| \le \frac{3^p\theta}{3 - 81^p} ||x||^{4p}$$

for all $x \in X$.

The rest of the proof is similar to the above proof. So there exists a unique additive mapping $L : X \to Y$ satisfying

$$\|f(x) - L(x)\| \le \frac{3^p\theta}{|81^p - 3|} ||x||^{4p}$$

for all $x \in X$.

Assume that $f : X \to Y$ is a mapping such that the transformation $t \to f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$. By the same reasoning as in the proof of Theorem 1.1, one can prove that L is an $\mathbb{R}$-linear mapping. □

REFERENCES

1. W. Fechner: Stability of a functional inequalities associated with the Jordan-von Neumann functional equation. *Aequationes Math.* **71** (2006), 149-161.
2. A. Gilányi: Eine zur Parallelogrammgleichung äquivalente Ungleichung. *Aequationes Math.* **62** (2001), 303-309.
3. ______ : On a problem by K. Nikodem. *Math. Inequal. Appl.* **5** (2002), 707-710.
4. D.H. Hyers: On the stability of the linear functional equation. *Proc. Nat. Acad. Sci. U.S.A.* **27** (1941), 222-224.
5. R.V. Kadison & G. Pedersen: Means and convex combinations of unitary operators. *Math. Scand.* **57** (1985), 249-266.
6. C. Park, Y. Cho & M. Han: Stability of functional inequalities associated with Jordan-von Neumann type additive functional equations. *J. Inequal. Appl.* **2007**, Art. ID 41820 (2007).
7. J.M. Rassias: On approximation of approximately linear mappings by linear mappings. *J. Funct. Anal.* **46** (1982), 126-130.
8. ______ : On approximation of approximately linear mappings by linear mappings. *Bull. Sci. Math.* **108** (1984), 445-446.
9. ______ : Solution of a problem of Ulam. *J. Approx. Th.* **57** (1989), 268-273.
10. ______ : Solution of a stability problem of Ulam. *Discuss. Math.* **12** (1992), 95-103.
11. ______ : On the stability of the Euler-Lagrange functional equation. *Chinese J. Math.* **20** (1992), 185-190.
12. ______ : Complete solution of the multi-dimensional problem of Ulam. *Discuss. Math.* **14** (1994), 101-107.
13. ______ : *On the stability of a multi-dimensional Cauchy type functional equation, in: Geometry, Analysis and Mechanics (dedicated to Archimedes)*, 365-376, World Sci. Publ., River Edge, NJ, 1994.
14. J. Rätz: On inequalities associated with the Jordan-von Neumann functional equation. *Aequationes Math.* **66** (2003), 191-200.
15. S.M. Ulam: *A Collection of the Mathematical Problems.* Interscience Publ. New York, 1960.

DEPARTMENT OF MATHEMATICS EDUCATION, PUSAN NATIONAL UNIVERSITY, PUSAN 609–735, REPUBLIC OF KOREA
Email address: jsan63@pusan.ac.kr