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MONOTONICITY AND LOGARITHMIC CONVEXITY OF THREE FUNCTIONS INVOLVING EXPONENTIAL FUNCTION

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ABSTRACT. In this note, an alternative proof and extensions are provided for the following conclusions in [6, Theorem 1 and Theorem 3]: The functions $\frac{1}{x^2}-\frac{e^{-x}}{(1-e^{-x})^2}$ and $\frac{1}{t}-\frac{1}{e^t-1}$ are decreasing in $(0,\infty)$ and the function $\frac{t}{e^{at}-e^{(a-1)t}}$ for $a\in\mathbb{R}$ and $t\in(0,\infty)$ is logarithmically concave.

1. INTRODUCTION

Let

$$f(t)=\begin{cases}\dfrac{1}{t^2}-\dfrac{e^{-t}}{(1-e^{-t})^2}, & t\neq 0;\\ \dfrac{1}{12}, & t=0.\end{cases}\tag{1}$$

It is obvious that

$$f(t)=\frac{1}{t^2}-\frac{e^t}{(e^t-1)^2},\quad t\neq 0.\tag{2}$$

For $a\in\mathbb{R}$, let

$$F_a(t)=\begin{cases}\dfrac{t}{e^{at}-e^{(a-1)t}}, & t\neq 0;\\ 1, & t=0.\end{cases}\tag{3}$$

Some inequalities, connections with the famous Binet's formula for Euler's gamma function, applications, origin, background of $f(t)$ have been investigated in [1, 3, 6, 7, 11], [2, p. 217], [5, p. 295 and p. 704] and related references therein.

In [6, Theorem 1 and Theorem 3], among other things, the following conclusions were established by using two different approaches: The function $f(t)$ defined by (1)

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is strictly decreasing in $(0,\infty)$; The function $F_a(t)$ defined by (3) is logarithmically concave in $(0,\infty)$; When $a \ge \frac{1}{2}$, the function $F_a(t)$ is decreasing in $(0,\infty)$; The function

$$h(t)=\begin{cases}\dfrac{1}{t}-\dfrac{1}{e^t-1}, & t\ne 0\\ \dfrac{1}{2}, & t=0\end{cases} \tag{4}$$

is decreasing in $(0,\infty)$.

It is remarked that the function $h(t)$ is also related to Bernoulli's numbers, see [4, 8] and related references therein.

Recall that a r-time differentiable function $f(x)>0$ is said to be r-log-convex (or r-log-concave) on an interval I with $r\ge 2$ if and only if $[\ln f(x)]^{(r)}$ exists and $[\ln f(x)]^{(r)}\ge 0$ (or $[\ln f(x)]^{(r)}\le 0$) on I.

The aim of this note is to provide an alternative proof of the above conclusions obtained in [6, Theorem 1 and Theorem 3] and to extend them by using the power series expansion of e^t at $t=0$ and the celebrated Hermite-Hadamard's integral inequality [9, 10] for convex functions.

Our main results are stated as follows.

Theorem 1.1. *The function $F_a(t)$ is decreasing on $\mathbb{R}$ if $a\ge 1$, increasing on $\mathbb{R}$ if $a\le 0$, increasing in $(-\infty,0)$ if $a\le\frac{1}{2}$, and decreasing in $(0,\infty)$ if $a\ge\frac{1}{2}$.*

The function $F_a(t)$ for $a\in\mathbb{R}$ is logarithmic concave on $\mathbb{R}$. Equivalently, the function $h(t)$ is decreasing on $\mathbb{R}$.

The function $F_a(t)$ for $a\in\mathbb{R}$ is 3-log-concave in $(-\infty,0)$ and 3-log-convex in $(0,\infty)$. Equivalently, the function $h(t)$ is concave in $(-\infty,0)$ and convex in $(0,\infty)$.

For $t\in\mathbb{R}$, the following inequalities are valid:

$$\frac{e^t}{(e^t-1)^2}\le\frac{1}{t^2}\quad\text{and}\quad\left(\frac{e^t-1}{t}\right)^3\ge\frac{e^t(e^t+1)}{2}. \tag{5}$$

The function $f(t)$ is even in $\mathbb{R}$ and decreasing in $(0,\infty)$.

2. Proof of Theorem 1.1

For $t\ne 0$, taking the logarithm of $F_a(t)$ and differentiating yield

$$\ln F_a(t)=\ln|t|-\ln\left|1-e^{-t}\right|-at,$$
$$[\ln F_a(t)]'=\frac{1}{t}-\frac{1}{e^t-1}-a,$$

$$
\begin{aligned}
[\ln F_a(t)]'' &= \frac{e^t}{(e^t-1)^2} - \frac{1}{t^2} \\
&= \frac{1}{(e^{-t}-1)^2}\left[e^{-t} - \left(\frac{e^{-t}-1}{-t}\right)^2\right], \\
[\ln F_a(t)]^{(3)} &= \frac{2}{t^3} - \frac{e^t(1+e^t)}{(e^t-1)^3} \\
&= \frac{2}{t^3}\left(\frac{-t}{e^{-t}-1}\right)^3\left[\left(\frac{e^{-t}-1}{-t}\right)^3 - \frac{e^{-t}(e^{-t}+1)}{2}\right].
\end{aligned}
$$

The well-known Hermite-Hadamard's integral inequality [9, 10] reads that if $f(x)$ is a convex function on the closed interval $[a,b]$ then

$$
0 \le \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - f\left(\frac{a+b}{2}\right) \le \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t. \tag{6}
$$

Applying the left-hand side inequality in (6) to e^{-t} for $t \in \mathbb{R}$ gives

$$
e^{-t/2} \le \frac{e^{-t}-1}{-t}
$$

for $t \in \mathbb{R}$. Hence, the second derivative $[\ln F_a(t)]''$ is non-positive, and so the first inequality in (5) holds, the function $F_a(t)$ for $a \in \mathbb{R}$ is logarithmic concave and the function $h(t)$ is decreasing on $\mathbb{R}$.

By L'Hôspital's rule, it follows that

$$
\lim_{t\to-\infty} h(t) = 1, \quad \lim_{t\to\infty} h(t) = 0 \quad \text{and} \quad \lim_{t\to 0} h(t) = \frac{1}{2}. \tag{7}
$$

Consequently, the function $F_a(t)$ is decreasing on $\mathbb{R}$ if $a \ge 1$, increasing on $\mathbb{R}$ if $a \le 0$, increasing in $(-\infty, 0)$ if $a \le \frac{1}{2}$, and decreasing in $(0,\infty)$ if $a \ge \frac{1}{2}$.

By virtue of the power series expansion of e^t at $t = 0$, it is deduced that

$$
\begin{aligned}
&\left(\frac{e^t-1}{t}\right)^3 - \frac{e^t(e^t+1)}{2} \\
&= \frac{e^{3t} - 3e^{2t} + 3e^t - 1}{t^3} - \frac{e^{2t}+e^t}{2} \\
&= 3\sum_{k=0}^{\infty}\left(3^{k-1} - 2^k + 1\right)\frac{t^{k-3}}{k!} - \frac{1}{t^3} - \frac{1}{2}\sum_{k=0}^{\infty}\left(2^k+1\right)\frac{t^k}{k!} \\
&= \sum_{k=4}^{\infty}\left[\frac{3\left(3^{k+2} - 2^{k+3} + 1\right)}{(k+3)(k+2)(k+1)} - \frac{2^k+1}{2}\right]\frac{t^k}{k!}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k=4}^{\infty} \frac{6+6\left(3^{k+2}-2^{k+3}\right)-(k+3)(k+2)(k+1)\left(2^{k}+1\right)}{2(k+3)!} t^{k} \\
&\triangleq \sum_{k=4}^{\infty} \frac{Q(k)}{2(k+3)!} t^{k}.
\end{aligned}
$$

Furthermore, the sequence $Q(k)$ for $k \geq 4$ can be calculated as follows:

$$
\begin{aligned}
Q(k) &= 54 \times 3^{k}-\left(k^{3}+6 k^{2}+11 k+54\right) 2^{k}-\left(k^{3}+6 k^{2}+11 k\right) \\
&= 4 \times 3^{k}-\left(k^{3}+6 k^{2}+11 k\right) \\
&\quad + 2^{k}\left[50\left(\frac{3}{2}\right)^{k}-\left(k^{3}+6 k^{2}+11 k+54\right)\right] \\
&\triangleq Q_{1}(k)+2^{k} Q_{2}(k), \\
Q_{1}^{\prime}(t) &= (4 \ln 3) 3^{t}-3 t^{2}-12 t-11, \\
Q_{1}^{\prime\prime}(t) &= (2 \ln 3)^{2} 3^{t}-6 t-12, \\
Q_{1}^{(3)}(t) &= 4(\ln 3)^{3} 3^{t}-6, \\
Q_{2}^{\prime}(t) &= 50\left(\ln \frac{3}{2}\right)\left(\frac{3}{2}\right)^{t}-3 t^{2}-12 t-11, \\
Q_{2}^{\prime\prime}(t) &= 50\left(\ln \frac{3}{2}\right)^{2}\left(\frac{3}{2}\right)^{t}-6 t-12, \\
Q_{2}^{(3)}(t) &= 50\left(\ln \frac{3}{2}\right)^{3}\left(\frac{3}{2}\right)^{t}-6.
\end{aligned}
$$

Since $\ln 3 > 1$, it is clear that $Q_1^{(3)}(4) > 0$ and that $Q_1^{(3)}(t)$ is increasing, so $Q_1^{(3)}(t) > 0$ for $t \geq 4$, and then $Q_1''(t)$ is increasing for $t \geq 4$. Since $Q_1''(4) > 0$, then $Q_1''(t) > 0$ for $t \geq 4$, and so $Q_1'(t)$ is increasing for $t \geq 4$. Since $Q_1'(4) > 0$, then $Q_1'(t) > 0$, and so $Q_1(t)$ is increasing for $t \geq 4$. From $Q_1(4) = 120$, it follows that $Q_1(t) > 0$ for $t \geq 4$.

Since $\ln \frac{3}{2} > 0.4$, it is easy to see that $Q_2^{(3)}(t)$ is increasing and $Q_2^{(3)}(4) > 0$, so $Q_2^{(3)}(t) > 0$, and then $Q_2''(t)$ is increasing, for $t \geq 4$. Since $Q_2''(4) > 0$, then $Q_2''(t) > 0$, and so $Q_2'(t)$ is increasing, for $t \geq 4$. Since $Q_2'(5) > 0$, then $Q_2'(t) > 0$, and so $Q_2(t)$ is increasing, for $t \geq 5$. From $Q_2(6) > 0$, it is concluded that $Q_2(t) > 0$ for $t \geq 6$.

It is standard to obtain that $Q(4) = 42$ and $Q(5) = 504$. Combining this with properties of $Q_1(t)$ and $Q_2(t)$ discussed above reveals that $Q(k) > 0$ for $k \geq 4$. This implies that the second inequality in (5) holds for $t > 0$, and so $[\ln F_a(t)]^{(3)} < 0$, $F_a(t)$ is 3-log-concave and $[\ln F_a(t)]'' = -f(t)$ is decreasing for $t \in (-\infty, 0)$. It is obvious that the function $f(t)$ is even on $\mathbb{R}$, as a result, the function $f(t) = -[\ln F_a(t)]''$ is decreasing in $(0,\infty)$, and then $f'(t) = -[\ln F_a(t)]^{(3)} < 0$ and $F_a(t)$ is 3-log-convex in $(0,\infty)$. This implies that the second inequality in (5) for $t < 0$. Therefore, the second inequality in (5) is valid for $t < 0$. The proof is complete. □

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