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화학반응식 균형과 안정성을 위한 새로운 유사 역행렬법

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A New Pseudoinverse Matrix Method For Balancing Chemical Equations And Their Stability

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"Balancing the equation of the reaction is a matter of mathematics only."
M. J. ten Hoor, J. Chem. Educ. **74** (1997), 1367

요 약. 이 논문에 화학반응식 균형을 맞추기 위한 새로운 유사 역행렬법이 기술되었다. 여기에 제공된 방법은 Moore-Penrose 유사 역행렬을 사용한 Diophantin 행렬식의 해에 기초를 둔다. 방법은 전형적인 여러 화학반응식에 시험적용되었고 폭넓은 균형연구에서 모든 반응식에 매우 성공적이었다. 이 방법은 아무 제한없이 성공적으로 적용되고, 또한 새로운 화학반응식의 타당성에 대한 검증력도 있고, 만일 새식이 타당하다면 화학식 균형을 이룰 것이다. 여기서 다루어진 화학반응식들은 소수산화수를 지닌 원자를 포함하고 있다. 또한, 화학반응식의 확장된 행렬의 안정성에 대한 화학반응식의 안정성의 필요충분조건을 이 연구에 소개하였다.

주제어: 유사 역행렬, 화학반응식 균형 맞추기, 안정성

ABSTRACT. In this work is given a new pseudoniverse matrix method for balancing chemical equations. Here offered method is founded on virtue of the solution of a Diophantine matrix equation by using of a Moore-Penrose pseudoinverse matrix. The method has been tested on several typical chemical equations and found to be very successful for the all equations in our extensive balancing research. This method, which works successfully without any limitations, also has the capability to determine the feasibility of a new chemical reaction, and if it is feasible, then it will balance the equation. Chemical equations treated here possess atoms with fractional oxidation numbers. Also, in the present work are introduced necessary and sufficient criteria for stability of chemical equations over stability of their extended matrices.

Keywords: Pseudoinverse Matrix, Balancing Chemical Equations, Stability

1. INTRODUCTION

Chemical equations are the most exploited part of theoretical as well as industrial chemistry. Every chemistry instructor doubtless has his favorite technique of balancing equations. Chemical equations are commonly written for two purposes: to show the probable reaction products, and to show the quantities of materials that enter into reaction. It is fortunate indeed that an extensive and comprehensive knowledge of sub-atomic phenomena is not essential in the balancing of equations. Every student, which has general chemistry as an obligatory subject, is bound to come across balancing chemical equations. It is highly important for chemistry student to be able to write and balance chemical

equations readily and accurately. Actually, balancing chemical equations provides an excellent demonstrative and pedagogical example of interconnection between stoichiometrical principles and linear algebra.

A *chemical equation* is a symbolic representation of a *chemical reaction*. The substances initially involved in a chemical reaction are called *reactants*, but the newly formed substances are called the *products*. The products are new substances with properties that are different from those of reactants. Classically, chemical reactions encompass changes that strictly involve the motion of *electrons* in the forming and breaking of *chemical bonds*, although the general concept of a chemical reaction, in particular the notion of a *chemical equation*, is applicable to *transformations of elementary particles*.

A chemical equation is not only the shorthand writing of the chemist, but it should be a mental picture of an actual reaction. To the observant investigator, the equation should immediately remind him as to the physical nature and properties of the reactants, *viz.*, color, state, etc., as well as the chemical result and its physical nature. Thus, a great deal of significance should be attached to the writing of chemical equations. The part of chemical mathematics called *Stoichiometry* deals with the weight relations determined by chemical equations and formulas. According to it, the balancing of chemical equations is important in this area. Since a chemical reaction, when it is feasible, is a natural process, the consequent equation is always consistent. Therefore, we must have a nontrivial solution and we should be able to obtain it assuming its existence. Such an assumption is absolutely valid and does not introduce any error. If the reaction is infeasible, then the only solution is a trivial one, *i. e.*, the all coefficients are equal to zero.

2. HISTORICAL SURVEY

During the past years, numerous articles have appeared in scientific literature relative to the balancing of chemical equations. In this section is made a short survey of selected articles for balancing chemical equations. The selection criteria for references were intentionally wide, in order to include a large variety of topics. Balancing chemical equations in the scientific literature is considered from four points of view: mathematical, computational, chemical and pedagogical.

Now, shortly we will describe these glances.

· Jones for the first time in mathematics proposed the general problem for balancing chemical equations.[1] Krishnamurthy in his article[2] gave an algebraic approach for balancing chemical equations founded on virtue of a generalized matrix inverse. Little bit late Das[3] offered a simple scholarly technique, which was discussed in.[4,5] One other technique for balancing chemical equation over an integer programming approach is given in,[6] while in[7] by using of a reflexive generalized inverse matrix is solved the general problem of balancing chemical equations given in.[1] The newest mathematical results for balancing chemical equations and their stability over a nonsingular matrix method are obtained in.[8] Actually, to date in mathematics and chemistry there are only three strictly formalized consistent mathematical methods for balancing chemical equations, particularly they are the methods given in[7,8] and right now presented method in this work, while other so called *methods* in chemical sense have a limited usage, and they are useful only for particular cases, especially for balancing chemical equations which possess atoms with integer oxidation numbers.

· In chemistry there are many published articles,[9-29] which consider the use of computers for balancing chemical equations. All of these computational methods use some commercial softer packet.

· University textbooks of general chemistry generally support the ion-electron technique as basic procedure for balancing chemical equation, because it makes the best use of fundamental chemical principles. Also, some authors advocated other techniques, which involve less algebraic manipulation that may deserve attention - particularly in classes of chemistry and chemical engineering majors.[30-56]

Several simple chemical equations are solved by elementary algebraic techniques in.[45,57-62] Bottom-

ley published the earliest article that makes use of the linear algebra method.[63] A set of various modifications, which implement this approach, is documented in.[31,44,45,64-66] The case when the chemical equation has no unique solution received considerable attention in the education articles.[18,67-79] The equation represents two or more independently occurring reactions can be combined in varying stoichiometric ratios.[80,81] Fixed ratios of reagents, observed experimentally in particular cases, are equivalent to a restriction on the coefficients that make a unique solution.[18]

It is necessary to stress out that balancing chemical equations by inspection is equivalent to using the algebraic method or a computerized matrix algebra approach.[82,83] The valence change method[50,84-106] and the ion-electron method[85-88,92,100,107-113] are also simple algebraic inspection techniques, subjected to exactly the same controls and limitations as the algebraic and matrix methods. Here it is good to emphasis that first Karslake in[114] considered balancing of ionic chemical equations. Actually, the technique suggested by García[115] can reduces the number of algebraic steps for ion-electron method. Above both mentioned methods - the valence change method and ion-electron method begin by establishing the relative proportions of reagents taking part in separate oxidation and reduction components of a redox reaction. Then, each technique uses a lowest common multiplier to enforce a principle of conservancy - for instance, conservation of oxidation number change in the case of the oxidation number method. Johnson in his article[116] defined the equivalent term *oxidation stage change* on this subject.

Stout in[117] presented three redox reactions as puzzles. Each one can be shown as simple redox system, which may easily be balanced using here offered method. After this article was published, the followed other debatable articles with critical accent.[118-125]

· Balancing chemical equations through the pedagogical point of view is given in the articles.[112,124-133] This approach is very interesting for the education of chemical research. A check of the hypothesis that formal reasoning and a sufficiently large mental capacity are required to balance more complex many-step equations is made over a test to determine level of intellectual development, mental capacity, and degree of field dependence/field independence of the students.[134]

3. PRELIMINARIES

Now we will introduce some well known results from the matrix algebra. Throughout, the set of $m \times n$ matrices over a field will be denoted by $\mathbb{R}^{m\ n}$.

Definition 3. 1. *The Moore-Penrose pseudoinverse A^- of a matrix $A \in \mathbb{R}^{m\times n}$ is the unique matrix satisfying the following criteria: $AA^-A = A$, $A^+AA^- = A^-$, $(AA^-)^T = AA^+$, $(A^+A)^T = A^-A$.*

Suppose the matrix A has full rank, then

$A = [a_{ij}]_{n\times n}$ and rank $A = n \Rightarrow A^- = A^{-1}$,

$A = [a_{ij}]_{m\times n}$ $(m > n)$ and rank$A = n \Rightarrow A^- = (A^TA)^{-1}A^T$,

$A = [a_{ij}]_{m\times n}$ $(m < n)$ and rank$A = m \Rightarrow A^+ = A^T(AA^T)^{-1}$.

If the matrix A does not have full rank, *i.e.*, A is an $m\times n$ matrix and rank $A = r < \min(m, n)$, then pseudoinverse A^+ can be constructed from the singular value decomposition $A = UDV^T$ by $A^- = VD^-U^T$.

Also the Moore-Penrose pseudoinverse A^+ of a matrix A, we can derive on this way. There do always exist two matrices $C = [c_{ij}]_{m\times r}$ and $D = [d_{ij}]_{r\times n}$ of rank r, such that $A = CD$. Using these matrices it holds that $A^- = D^T(DD^T)^{-1}(C^TC)^{-1}C^T$.

Pseudoinverse matrix A^- of a matrix A was independently defined by Moore[135] and Penrose[136]. If the matrix A^- satisfies first two equalities of definition 3. 1, then it is called a *reflexive generalized inverse* of A, denoted by $\tilde{A}$. This matrix is not unique.

Definition 3. 2. *The characteristic equation of an $n\times n$ matrix A is the equation in one variable λ*

$$\det(A - \lambda I) = 0 \qquad (3.1)$$

where det(·) *denotes a determinant and I is an $n\times n$ identity matrix.*

Definition 3. 3. *The polynomial*

$$a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0 \qquad (3.2)$$

which results from evaluating the determinant (3.1)

is the characteristic polynomial of the matrix A.

The above polynomial of degree $n \geq 1$ with real coefficients a_i $(0 \leq i \leq n)$, where $a_n \neq 0$, by the fundamental theorem of algebra has n (not necessarily distinct) roots $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Definition 3. 4. *The roots of the characteristic polynomial* (3. 2) *are precisely the eigenvalues of the matrix* A.

Definition 3. 5. *The reaction matrix has the following format* $A|B = [A|B]_{m\times(r+s)}$, *where* $A = [a_{ij}]_{m\times r}$ $(r < m)$ *and* $B = [b_{ij}]_{m\times s}$ $(s < m)$.

Definition 3. 6. *An extended matrix*

$$\mathrm{Ext}A|B = \begin{bmatrix} A & B \\ O & I \end{bmatrix} \tag{3.3}$$

of $(r+s)\times(r+s)$ *dimension, is singular if* $\det(\mathrm{Ext}A|B) = 0$. *where* $A = [a_{ij}]_{m\times r}$ $(r < m)$, $B = [b_{ij}]_{m\times s}$ $(s < m)$. I *is the identity matrix of* $(r - s - m)\times(r - s - m)$ *dimension and* O *is a zero matrix of* $(r + s - m)\times m$ *dimension.*

In[8] is treated the non-singular case.

Let $\sigma(\mathrm{Ext}A|B) = \{\lambda_i, 1 \leq i \leq k\}$ be the spectrum of $\mathrm{Ext}A|B$. Let $|\cdot|$ denotes a vector norm in $\mathbb{R}^k$.

Definition 3. 7. *The Lozinskiĭ measure* μ *on* $\mathbb{R}^n$ *with respect to* $|\cdot|$ *is defined by*

$$\mu(\mathrm{Ext}A|B) = \lim_{\rho\to 0} (|I + \rho\mathrm{Ext}A|B| - 1)/\rho. \tag{3.4}$$

Definition 3. 8. *The Lozinskiĭ measures of* $\mathrm{Ext}A|B = [a_{ij}]_{n\times n}$ *with respect to the three common norms*

$$\begin{aligned} |x|_\infty &= \sup_i |x_i|, \\ |x|_1 &= \Sigma_i |x_i|, \\ |x|_2 &= (\Sigma_i |x_i|^2)^{1/2}, \end{aligned} \tag{3.5}$$

are

$$\begin{aligned} \mu_\infty(\mathrm{Ext}A|B) &= \sup_i(a_{ii} + \Sigma_{k\neq i} |a_{ik}|), \\ \mu_1(\mathrm{Ext}A|B) &= \sup_k(a_{kk} + \Sigma_{i\neq k} |a_{ik}|), \\ \mu_2(\mathrm{Ext}A|B) &= \mathrm{stab}\{[(\mathrm{Ext}A|B) + (\mathrm{Ext}A|B)^{\mathrm{T}}]/2\}, \end{aligned} \tag{3.6}$$

where

$$\mathrm{stab}(\mathrm{Ext}A|B) = \max\{\lambda,\ \lambda \in \sigma(\mathrm{Ext}A|B)\}$$

is the stability modulus of $\mathrm{Ext}A|B$ *and* T *denoting transpose.*

Definition 3. 9. *The extended matrix* $\mathrm{Ext}A|B$ *is stable if* $\mathrm{stab}(\mathrm{Ext}A|B) < 0$.

4. MAIN RESULTS

In this section we will give a completely new method for balancing and stability of chemical equations. Given analysis is done for arbitrary chemical equation presented in its general form.

Proposition 4. 1. *Any chemical equation may be presented in this form*

$$\sum_{i=1}^{r} x_j \prod_{i=1}^{m} \Psi^i{}_{ai} = \sum_{i=1}^{s} y_j \prod_{i=1}^{n} \Omega^i{}_{bi} \tag{4.1}$$

where x_j $(1 \leq j \leq r)$ *and* y_j $(1 \leq j \leq s)$ *are unknown rational coefficients,* Ψ^i *and* Ω^i $(1 \leq i \leq m)$ *are chemical elements in reactants and products, respectively,* a_{ij} $(1 \leq i \leq m;\ 1 \leq j \leq r)$ *and* b_{ij} $(1 \leq i \leq m;\ 1 \leq j \leq s)$ *are numbers of atoms of elements* Ψ^i *and* Ω^i, *respectively, in j-th molecule.*

Proof. Let there exists an arbitrary chemical equation from s distinct elements and $m + n$ molecules

$$\sum_{i=1}^{r} x_j \Xi_j = \sum_{i=1}^{s} y_j \Theta_j \tag{4.2}$$

where $\Xi_j = \Psi^1{}_{a1}\Psi^2{}_{a2}\cdots\Psi^m{}_{am}$ $(1 \leq j \leq r)$ and $\Theta_j = \Omega^1{}_{b1}\Omega^2{}_{b2}\cdots\Omega^m{}_{bm}$ $(1 \leq j \leq s)$. Then previous expression becomes

$$\sum_{i=1}^{r} x_j \Psi^1{}_{a1}\Psi^2{}_{a2}\cdots\Psi^m{}_{am} = \sum_{i=1}^{s} y_j \Omega^1{}_{b1}\Omega^2{}_{b2}\cdots\Omega^m{}_{bm} \tag{4.3}$$

Now, if we write the above equation in a compact form, then immediately follows (4.1).

The coefficients x_j, y_j satisfy three basic principles (corresponding to a closed input-output static model[137,138])

· the low of conversation of atoms,

· the low of conversation of mass, and

· the time-independence of the reaction.

Theorem 4. 2. *The chemical equation* (4.1) *can be reduced to the following Diophantine matrix equation*

$$Ax = By, \tag{4.4}$$

where $A = [a_{ij}]_{m\times r}$ *is a matrix of the reactants,* $B = [b_{ij}]_{m\times s}$ *is a matrix of the products,* $x^{\mathrm{T}} = (x_1, x_2, \ldots, x_r)$

and $\mathbf{y}^T=(y_1, y_2,\ldots,y_s)$ *are column vectors of unknown coefficients and T denotes transpose.*

Proof. If we develop the molecules of the reaction (4.1) in an explicit form, then we obtain a matrix of the reactants A

$$\begin{array}{cccccc} & \Psi^1_{a11}\Psi^2_{a21}\cdots\Psi^m_{am1} & \Psi^1_{a12}\Psi^2_{a22}\cdots\Psi^m_{am2} & \cdots & \Psi^1_{a1r}\Psi^2_{a2r}\cdots\Psi^m_{amr} & \\ & a_{11} & a_{12} & \cdots & a_{1r} & \Psi^1 \\ \mathbf{A} & a_{21} & a_{22} & \cdots & a_{2r} & \Psi^2 \\ & a_{m1} & a_{m2} & \cdots & a_{mr} & \Psi^m \end{array}$$

and a matrix of the products B

$$\begin{array}{cccccc} & \Omega^1_{b11}\Omega^2_{b21}\cdots\Omega^m_{bm1} & \Omega^1_{b12}\Omega^2_{b22}\cdots\Omega^m_{bm2} & \cdots & \Omega^1_{b1s}\Omega^2_{b2s}\cdots\Omega^m_{bms} & \\ & b_{11} & b_{12} & \cdots & b_{1s} & \Omega^1 \\ \mathbf{B} & b_{21} & b_{22} & \cdots & b_{2s} & \Omega^2 \\ & b_{m1} & b_{m2} & \cdots & b_{ms} & \Omega^m \end{array}$$

From the above development we obtain that

$$\Xi_j = \sum_{i=1}^{m} a_{ij}\Psi^i (1 \le j \le r) \tag{4.5}$$

and

$$\Theta_j = \sum_{i=1}^{m} b_{ij}\Omega^i (1 \le j \le s) \tag{4.6}$$

If we substitute (4.5) and (4.6) into (4.2), follows

$$\sum_{j=1}^{r} x_j \sum_{i=1}^{m} a_{ij}\Psi^i = \sum_{j=1}^{s} y_j \sum_{i=1}^{m} b_{ij}\Psi^i \tag{4.7}$$

or

$$\sum_{i=1}^{m} \Psi^i \sum_{j=1}^{r} a_{ij}x_j = \sum_{i=1}^{m} \Omega^i \sum_{j=1}^{s} b_{ij}y_j \tag{4.8}$$

i.e., the above expression holds if only if

$$\sum_{j=1}^{r} a_{ij}x_j = \sum_{j=1}^{s} b_{ij}y_j (1 \le i \le m) \tag{4.9}$$

Last equation if we present in a matrix form, actually we obtain (4. 4).

Now we will prove the following result.

Theorem 4. 3. *Let* $A \in \mathbb{R}^{m\times r}$ *and* $B \in \mathbb{R}^{m\times s}$. *The matrix Diophantine equation* (4. 4) *is consistent if and only if for some* A^-

$$Gy = 0, \tag{4.10}$$

where

$$G = (I - AA^+)B. \tag{4.11}$$

If this is the case, a representation of the general solution is

$$y = (I - G^-G)u \tag{4.12}$$

and

$$x = A^-By + (I - A^-A)v, \tag{4.13}$$

with arbitrary vectors $u \in \mathbb{R}^{s\times 1}$ *and* $v \in \mathbb{R}^{r\times 1}$.

Proof. Matrix equation (4. 4) is consistent if and only if there exist vectors $x \in \mathbb{R}^{r\times 1}$ and $y \in \mathbb{R}^{s\times 1}$ such that

$$Ax - By = 0. \tag{4.14}$$

From (4. 13) we have $Ax = AA^-By$, since $A(I - A^-A) = 0$. Thus, from (4. 14) follows

$$Ax - By = AA^+By - By = -(I - AA^+)By = -Gy; \text{ since (4. 11).} \tag{4.15}$$

On the other hand, from (4. 12) it follows that

$$Gy = G(I - G^+G)u = 0, \text{ since } G = GG^-G. \tag{4.16}$$

Immediately, from (4.16) and (4.15) follows (4.4).

Remark 4. 4. *This theorem generalizes the theorem proved in*[7]. *By the above theorem, a century old chemistry problem of balancing chemical equations in a general form is completely solved.*

Definition 4. 5. *Chemical equation* (4.1) *is stable if* $\mathrm{stab}(\mathrm{Ext}A|B) < 0$.

Lemma 4. 6. *For any nonsingular matrix* U *and any vector norm* $|\cdot|$, *with the induced Lozinskiĭ measure* μ, $|Ux|$ *defines another vector norm and its induced matrix measure* μ_U *is given by*

$$\mu_U(\mathrm{Ext}A|B) = \mu[U(\mathrm{Ext}A|B)U^{-1}]. \tag{4.17}$$

Proof. The proof of this lemma follows directly from the definition 3.7.

Theorem 4. 7. *For any matrix* $\mathrm{Ext}A|B \in \mathbb{R}$ *it holds*

$$\mathrm{stab}(\mathrm{Ext}A|B) = \inf\{\mu(\mathrm{Ext}A|B),\ \mu \text{ is a Lozinskiĭ measure on } \mathbb{R}^n\}. \tag{4.18}$$

Proof. The relation (4.18) obviously holds for diagonizable matrices in view of (4.17) and the first two relations in (3.6). Furthermore, the infimum in (4.18) can be achieved if the matrix $\mathrm{Ext}A|B$ is diagonizable. The general case can be shown based on this observation, the fact that $\mathrm{Ext}A|B$ can be

approximated by diagonizable matrices in and the continuity of $\mu(\cdot)$, which is implied by the property

$$|\mu(\mathcal{J}) - \mu(\mathcal{N})| \le |\mathcal{J} - \mathcal{N}|.$$

Remark 4. 8. *From the above proof it follows that*

$$\text{stab}(\text{Ext}\boldsymbol{A}|\boldsymbol{B})=\inf\{\mu_{\infty}[\boldsymbol{U}(\text{Ext}\boldsymbol{A}|\boldsymbol{B})\boldsymbol{U}^{-1}], \det\boldsymbol{U}\neq 0\}.$$

The same relation holds if μ is replaced by μ_1.

Corollary 4. 9. *Let* $\text{Ext}\boldsymbol{A}|\boldsymbol{B}\in\mathbb{R}$. *Then* $\text{stab}(\text{Ext}\boldsymbol{A}|\boldsymbol{B}) < 0 \Leftrightarrow \mu(\text{Ext}\boldsymbol{A}|\boldsymbol{B}) < 0$ *for some Lozinskii measure* μ *on* $\mathbb{R}^n$.

There are more results for stability criteria obtained in the works.[139,140]

For the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ given in the theorem 4.2, let $rank\boldsymbol{A}|\boldsymbol{B}=\rho$ and $ker\boldsymbol{A}|\boldsymbol{B}=\kappa$, where *ker* denotes the *nullity* or *kernel* of matrix $\boldsymbol{A}|\boldsymbol{B}$. According to[141], the deterministic approach is important, since it enables us to classify the chemical reaction as:

1° *infeasible* when the nullity of the reaction matrix $\boldsymbol{A}|\boldsymbol{B}$ is zero;

2° *unique* (within relative proportions) when the nullity of the reaction matrix $\boldsymbol{A}|\boldsymbol{B}$ is one; or

3° *non-unique* when the nullity of the reaction matrix $\boldsymbol{A}|\boldsymbol{B}$ is bigger than one.

Possible cases of balancing chemical equations are the following

1. If $m \ge r + s = \rho$, then $\kappa = r + s - \rho = 0$, *i. e.*, trivial solution $\boldsymbol{x}, \boldsymbol{y} = \boldsymbol{0}$, the reaction is infeasible.

2. If $m \ge r + s, \rho = r + s - 1$, then $\kappa = r + s - \rho = 1$, unique solution $\boldsymbol{x}, \boldsymbol{y} > \boldsymbol{0}$, *i. e.*, the reaction is feasible and is unique.

In practical terms this means that the general procedure for obtaining these coefficients is to solve the system of linear equations derived from the principles of conservation of matter and charge, applied to the reaction element-by-element.

3. If $m \ge r + s, \rho < r + s - 1$, then $\kappa = r + s - \rho > 1$, $\kappa(>1)$ linearly independent solutions $\boldsymbol{x}, \boldsymbol{y} > \boldsymbol{0}$, *i. e.*, the reaction is feasible and is non-unique.

4. If $m < r + s, \rho \le m$, then $\kappa = r + s - \rho \ge 1$, $\kappa(\ge 1)$ linearly independent solutions $\boldsymbol{x}, \boldsymbol{y} > \boldsymbol{0}$, *i. e.*, the reaction is feasible and is non-unique.

Last two kinds of the reactions are puzzling in that they exhibit infinite linearly independent solution all of which satisfy the chemical balance, and yet they are not all chemically feasible solutions for a given set of experimental conditions. A unique solution is obtained by imposing a chemical constraint, namely, that reactants have to react only in certain proportions.

5. AN APPLICATION OF THE MAIN RESULTS

In this section will be applied above method on many chemical equations for their balancing. All chemical equations balanced here appear first time in professional literature and they are chosen with an intention to be avoided to date all well know chemical equations which were repeated many times in the chemical journals for explanation of certain particular techniques for balancing of some chemical equations using only atoms with integer oxidation numbers.

1° First we will consider an infeasible reaction, *i. e.*, the case when the nullity of the reaction matrix is zero.

Example 5. 1. Consider this equation

$$x_1\,Co(NH_3)_5SO_4I + x_2\,MgBr_2 + x_3\,MnO + x_4\,H_3PO_4 + x_5\,KBr = y_1\,K_3PO_4 + y_2\,Co(NH_3)_6Br_{2.98} + y_3\,MgI_2 + y_4\,H_2O + y_5\,NO_2 + y_6\,MnSO_4. \quad (5.\ 1)$$

By the schemes

	$Co(NH_3)_5SO_4I$	$MgBr_2$	MnO	H_3PO_4	KBr		K_3PO_4	$Co(NH_3)_6Br_{2.98}$	MgI_2	H_2O	NO_2	$MnSO_4$	
	1	0	0	0	0		0	1	0	0	0	0	Co
	5	0	0	0	0		0	6	0	0	1	0	N
	15	0	0	3	0		0	18	0	2	0	0	H
	1	0	0	0	0		0	0	0	0	0	1	S
	4	0	1	4	0		4	0	0	1	2	4	O
A	1	0	0	0	0	**B**	0	0	2	0	0	0	I
	0	1	0	0	0		0	0	1	0	0	0	Mg
	0	2	0	0	1		0	2.98	0	0	0	0	Br
	0	0	1	0	0		0	0	0	0	0	1	Mn
	0	0	0	1	0		1	0	0	0	0	0	P
	0	0	0	0	1		3	0	0	0	0	0	K

are determined the matrix of reactants $\boldsymbol{A}$ and the matrix of the products $\boldsymbol{B}$, an according to it the

reaction matrix $A|B$ obtains the form

$$A|B = \begin{bmatrix} 1.00 & 0 & 0 & 0 & 0 & 0 & 1.00 & 0 & 0 & 0 & 0 \\ 5.00 & 0 & 0 & 0 & 0 & 0 & 6.00 & 0 & 0 & 1 & 0 \\ 15.0 & 0 & 0 & 3 & 0 & 0 & 18.0 & 0 & 2 & 0 & 0 \\ 1.00 & 0 & 0 & 0 & 0 & 0 & 0.00 & 0 & 0 & 0 & 1 \\ 4.00 & 0 & 1 & 4 & 0 & 4 & 0.00 & 0 & 1 & 2 & 4 \\ 1.00 & 0 & 0 & 0 & 0 & 0 & 0.00 & 2 & 0 & 0 & 0 \\ 0.00 & 1 & 0 & 0 & 0 & 0 & 0.00 & 1 & 0 & 0 & 0 \\ 0.00 & 2 & 0 & 0 & 1 & 0 & 2.98 & 0 & 0 & 0 & 0 \\ 0.00 & 0 & 1 & 0 & 0 & 0 & 0.00 & 0 & 0 & 0 & 1 \\ 0.00 & 0 & 0 & 1 & 0 & 1 & 0.00 & 0 & 0 & 0 & 0 \\ 0.00 & 0 & 0 & 0 & 1 & 3 & 0.00 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The rank of the above matrix is $\rho = 11$. Since the nullity of the reaction matrix is $\kappa = r + s - \rho = 5 + 6 - 11 = 0$, then we have only a trivial solution $\boldsymbol{x}, \boldsymbol{y} = \boldsymbol{0}$, that means that the reaction is infeasible.

2° Next, we will consider the case when the chemical reaction is feasible and is unique, *i. e.*, the nullity of its reaction matrix is one. Here we will balance many special chemical equations with a goal to show the power of the offered mathematical method.

Example 5. 2. As a next example to illustrate above method, we will balance this chemical equation

$$x_1\ Pt(NH_3)BrNO_3 + x_2\ Cu(NH_3)KNO_3 - x_3\ BeCO_3 - y_1\ C_{44}H_{50}O_{14.98} + y_2\ Cu(NO_3)_2 + y_3\ PtO_3 + y_4\ Br_{1.99}NO_2 + y_5\ K_{1.97}O - y_6\ BeO + y_7\ HNO_3. \quad (5.\ 2)$$

From the schemes given below

	$Pt(NH_3)BrNO_3$	$Cu(NH_3)KNO$	$BeCO_3$		$C_{44}H_{50}O_{14.98}$	$Cu(NO_3)_2$	PtO_3	$Br_{1.88}NO_2$	$K_{1.97}O$	BeO	HNO_3	
	1	0	0		0	0	1	0	0	0	0	Pt
	2	2	0		0	2	0	1	0	0	1	N
	3	3	0		50	0	0	0	0	0	1	H
	1	0	0		0	0	0	1.99	0	0	0	Br
A	3	3	3	**B**	14.98	6	3	2	1	1	3	O
	0	1	0		0	1	0	0	0	0	0	Cu
	0	1	0		0	0	0	0	1.97	0	0	K
	0	0	1		0	0	0	0	0	1	0	Be
	0	0	1		44	0	0	0	0	0	0	C

are determined the matrix of reactants $\boldsymbol{A}$ and the matrix of the products $\boldsymbol{B}$, *i. e.*,

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 3 & 0 \\ 1 & 0 & 0 \\ 3 & 3 & 3 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \boldsymbol{B} = \begin{bmatrix} 0.000 & 0 & 1 & 0.00 & 0.00 & 0 & 0 \\ 0.000 & 2 & 0 & 1.00 & 0.00 & 0 & 1 \\ 50.00 & 0 & 0 & 0.00 & 0.00 & 0 & 1 \\ 0.000 & 0 & 0 & 1.99 & 0.00 & 0 & 0 \\ 14.98 & 6 & 3 & 2.00 & 1.00 & 1 & 3 \\ 0.000 & 1 & 0 & 0.00 & 0.00 & 0 & 0 \\ 0.000 & 0 & 0 & 0.00 & 1.97 & 0 & 0 \\ 0.000 & 0 & 0 & 0.00 & 0.00 & 1 & 0 \\ 44.00 & 0 & 0 & 0.00 & 0.00 & 0 & 0 \end{bmatrix}.$$

The reaction matrix $A|B$, according to the definition 3. 5 obtains the form

$$A|B = \begin{bmatrix} 1 & 0 & 0 & 0.000 & 0 & 1 & 0.00 & 0.00 & 0 & 0 \\ 2 & 2 & 0 & 0.000 & 2 & 0 & 1.00 & 0.00 & 0 & 1 \\ 3 & 3 & 0 & 50.00 & 0 & 0 & 0.00 & 0.00 & 0 & 1 \\ 1 & 0 & 0 & 0.000 & 0 & 0 & 1.99 & 0.00 & 0 & 0 \\ 3 & 3 & 3 & 14.98 & 6 & 3 & 2.00 & 1.00 & 1 & 3 \\ 0 & 1 & 0 & 0.000 & 1 & 0 & 0.00 & 0.00 & 0 & 0 \\ 0 & 1 & 0 & 0.000 & 0 & 0 & 0.00 & 1.97 & 0 & 0 \\ 0 & 0 & 1 & 0.000 & 0 & 0 & 0.00 & 0.00 & 1 & 0 \\ 0 & 0 & 1 & 44.00 & 0 & 0 & 0.00 & 0.00 & 0 & 0 \end{bmatrix}.$$

The rank of the above matrix is $\rho = 9$. Since the nullity of the reaction matrix is $\kappa = r + s - \rho = 3 + 7 - 9 = 1$, then we have only a unique solution, that means that the reaction is feasible. Next, we will determine the solution $\boldsymbol{x}, \boldsymbol{y}$.

The Moore-Penrose pseudoinverse $A^- = (A^{\mathrm{t}}A)^{-1}A^{\mathrm{t}}$ of the matrix $\boldsymbol{A}$ is

$$688\,A^- = \begin{bmatrix} 183 & 44 & 66 & 183 & 12 & -161 & -161 & -18 & -18 \\ -161 & 44 & 66 & -161 & 12 & 183 & 183 & -18 & -18 \\ -18 & -72 & -108 & -18 & 168 & -18 & -18 & 92 & 92 \end{bmatrix}.$$

and the matrix $\boldsymbol{G}$ has this form

$$688\,\boldsymbol{G} = (688\,\boldsymbol{I} - \boldsymbol{A}\boldsymbol{A}^-)\,\boldsymbol{B} =$$

$$\begin{bmatrix} -2687.76 & 1.00 & 469.00 & -432.17 & 305.17 & 6.00 & -146.00 \\ -10751.04 & 692.00 & -188.00 & 328.44 & -134.68 & 24.00 & 104.00 \\ 18273.44 & -1026.00 & -282.00 & -539.34 & -202.02 & 36.00 & -188.00 \\ -2687.76 & 1.00 & -219.00 & 936.95 & 305.17 & 6.00 & -146.00 \\ -9314.44 & 564.00 & 324.00 & 152.12 & 88.36 & -56.00 & 216.00 \\ -2687.76 & 345.00 & 125.00 & 252.39 & -372.51 & 6.00 & -146.00 \\ -2687.76 & -343.00 & 125.00 & 252.39 & 982.85 & 6.00 & -146.00 \\ -1164.64 & -846.00 & -486.00 & -228.18 & -132.54 & 428.00 & -324.00 \\ 29107.36 & -846.00 & -486.00 & -228.18 & -132.54 & -260.00 & -324.00 \end{bmatrix}.$$

Required vectors $\boldsymbol{y}$ and $\boldsymbol{x}$, according to (4.12) and (4.13) are

$$y = (I - G^{-}G)u = \begin{bmatrix} 0.0348595041 \\ 0.5130378112 \\ 0.1356805781 \\ 0.0681811950 \\ 0.2604252849 \\ 1.5338181820 \\ 0.2031799611 \end{bmatrix}$$

$$\text{and } x - A^{-}By + (I - A^{-}A)v = \begin{bmatrix} 0.1356805781 \\ 0.5130378112 \\ 1.5338181820 \end{bmatrix}.$$

Now immediately follows balanced equation in its conventional form

0.1356805781 $Pt(NH_3)BrNO_3$+0.5130378112 $Cu(NH_3)KNO_3$+1.5338181820 $BeCO_3$= 0.0348595041 $C_{44}H_{50}O_{1498}$+0.5130378112 $Cu(NO_3)_2$+0.1356805781 PtO_3+0.0681811950 $Br_{1.99}NO_2$+0.2604252849 $K_{1.97}O$+ 1.5338181820 BeO+0.2031799611 HNO_3.

Since the reaction matrix $A|B$ has a rectangular format, according to definition 3. 6, we will extend it to a square matrix by adding a new row. To do that, we will add the following vector-row (0, 0, 0, 0, 0, 0, 0, 0, 0, 1) as a 10th row, such that $Ext_1A|B$ obtains this form

$$Ext_1A|B = \begin{bmatrix} 1 & 0 & 0 & 0.000 & 0 & 1 & 0.00 & 0.00 & 0 & 0 \\ 2 & 2 & 0 & 0.000 & 2 & 0 & 1.00 & 0.00 & 0 & 1 \\ 3 & 3 & 0 & 50.00 & 0 & 0 & 0.00 & 0.00 & 0 & 1 \\ 1 & 0 & 0 & 0.000 & 0 & 0 & 1.99 & 0.00 & 0 & 0 \\ 3 & 3 & 3 & 14.98 & 6 & 3 & 2.00 & 1.00 & 1 & 3 \\ 0 & 1 & 0 & 0.000 & 1 & 0 & 0.00 & 0.00 & 0 & 0 \\ 0 & 1 & 0 & 0.000 & 0 & 0 & 0.00 & 1.97 & 0 & 0 \\ 0 & 0 & 1 & 0.000 & 0 & 0 & 0.00 & 0.00 & 1 & 0 \\ 0 & 0 & 1 & 44.00 & 0 & 0 & 0.00 & 0.00 & 0 & 0 \\ 0 & 0 & 0 & 0.000 & 0 & 0 & 0.00 & 0.00 & 0 & 1 \end{bmatrix}$$

The eigenvalues of the matrix $[Ext_1A|B - (Ext_1A|B)^{1}]/2$ are λ_1=35.111, λ_2=-33.595, λ_3=7.769, λ_4=1.630, λ_5= 1.140, λ_6=0.919, λ_7=0.205, λ_8=-1.285, λ_9=-1.087, λ_{10}= -0.807.

The Lozinskii measures of $Ext_1A|B$ given by (3.6) with respect to the three common norms (3.5) are

$\mu_1(Ext_1A|B)$ = max (2, 8, 57, 2.99, 36.98, 2, 2.97, 2, 45, 1) = 57,

$\mu_1(Ext_1A|B)$=max (10, 10, 5, 108.98, 9, 4, 4.99, 2.97, 2, 6)=108.98,

$\mu_2\{[Ext_1A|B + (Ext_1A|B)^{1}]/2\}$ = 35.111.

Since $\mu_2\{[Ext_1A|B+(Ext_1A|B)^{1}]/2\}>0$, according to the definition 4.5 the chemical equation (5.2) is unstable.

Example 5. 3. The chemical equation

x_1 $Pt(NH_3)BrNO_3$+x_2 $[4Fe(CN)_3{\cdot}3Fe(CN)_2]$+$x_3$ $C_{44}H_{50}O_{15}$ +x_4 $Cu(NO_3)_2$+x_5 K_2BeO_2=y_1 $Cu(NH_3)KNO_3$+y_2 $BeCO_3$ +y_3 $K_{3.97}Fe(CN)_6$+y_4 PtO_2+y_5 $Br_{1.96}NO_2$+y_6 HNO_3 (5.3)

like a characteristic case will be balanced too.

From the schemes given below

	$-Pt(NH_3)BrNO_3$	$=[4Fe(CN)_3{\cdot}3Fe(CN)_2]$	$C_{44}H_{50}O_{15}$	$=Cu(NO_3)_2$	$=K_2BeO_2$		$=Cu(NH_3)KNO_3$	$=BeCO_3$	$=K_{3.97}Fe(CN)_6$	$-PtO_2$	$=Br_{1.96}NO_2$	$=HNO_3$	
	1	0	0	0	0		0	0	0	0	0	0	Pt
	2	18	0	2	0		2	0	6	0	1	1	N
	3	0	50	0	0		3	0	0	0	0	1	H
	1	0	0	0	0		0	0	0	0	1.96	0	Br
A	3	0	15	6	2	B	3	3	0	2	2	3	O
	0	7	0	0	0		0	0	1	0	0	0	Fe
	0	18	44	0	0		0	1	6	0	0	0	C
	0	0	0	1	0		1	0	0	0	0	0	Cu
	0	0	0	0	2		1	0	3.97	0	0	0	K
	0	0	0	0	1		0	1	0	0	0	0	Be

are determined the matrix of reactants A and the matrix of the products B.

The reaction matrix $A|B$, according to the definition 3. 5 obtains this form

$$A|B = \begin{bmatrix} 1 & 0.00 & 0.00 & 0 & 0 & 0 & 0 & 0.00 & 1 & 0.00 & 0 \\ 2 & 18.0 & 0.00 & 2 & 0 & 2 & 0 & 6.00 & 0 & 1.00 & 1 \\ 3 & 0.00 & 50.0 & 0 & 0 & 3 & 0 & 0.00 & 0 & 0.00 & 1 \\ 1 & 0.00 & 0.00 & 0 & 0 & 0 & 0 & 0.00 & 0 & 1.96 & 0 \\ 3 & 0.00 & 15.0 & 6 & 2 & 3 & 3 & 0.00 & 2 & 2.00 & 3 \\ 0 & 7.00 & 0.00 & 0 & 0 & 0 & 0 & 1.00 & 0 & 0.00 & 0 \\ 0 & 18.0 & 44.0 & 0 & 0 & 0 & 1 & 6.00 & 0 & 0.00 & 0 \\ 0 & 0.00 & 0.00 & 1 & 0 & 1 & 0 & 0.00 & 0 & 0.00 & 0 \\ 0 & 0.00 & 0.00 & 0 & 2 & 1 & 0 & 3.97 & 0 & 0.00 & 0 \\ 0 & 0.00 & 0.00 & 0 & 1 & 0 & 1 & 0.00 & 0 & 0.00 & 0 \end{bmatrix}$$

The rank of the above matrix is $\rho = 10$. Since the

nullity of the reaction matrix is $\kappa - r + s - \rho - 5 + 6 - 10 - 1$, then we have only a unique solution, that means that the reaction is feasible. Next, we will determine the solution x, y.

The Moore-Penrose pseudoinverse $A^{+}=(A^{T}A)^{-1}A^{T}$ of the matrix A is

3821829615 A^{+} =

530923221	661977522	654140313	530923221	-175464270
7914996	123779502	-48555612	7914996	-39240720
-18772587	-45218394	29117439	-18772587	13543215
-271169424	-152766738	-354633372	-271169424	637257480
70185708	-86901384	-60307176	70185708	71450960

55404972	-683523900	-271169424	140371416	70185708
51402897	68554350	-12114684	31392576	15696288
-10122084	49154760	9177498	-10834572	-5417286
-84802788	185745600	303818211	-509805984	-254902992
109874016	44172600	-254902992	1471571078	735785539

and the matrix G has this form

3821829615 G = (3821829615 I - AA^{+})B =

-2629185165.00	1139731002.00	-483401225.52
1650524550.00	-803280696.00	1845272340.96
1783528350.00	-799162944.00	25653265.44
-2629185165.00	1139731002.00	-483401225.52
-1131087940.00	574862740.00	-933567342.40
-24135300.00	234300654.00	-5488412135.04
-1641138600.00	712163775.00	289110156.00
3485478540.00	-1842615048.00	1910859372.48
1669236275.00	-1988622038.00	3781386588.23
-1076296670.00	2827518596.00	-5695638491.66

3641834934.00	-1351658495.16	-789725025.00
-1096484442.00	-1156605852.12	198603765.00
-955676193.00	-1308663759.48	226220040.00
-179994681.00	6139127550.24	-789725025.00
532719070.00	483911309.20	167860800.00
493965108.00	-425680179.12	297487890.00
904386900.00	1322148144.00	314293500.00
-1003345536.00	-590256150.96	-1404372330.00
-426175256.00	-387129047.36	-134288640.00
-213087628.00	-193564523.68	-67144320.00

Required vectors y and x, according to (4.12) and (4.13) are

$$y = (I - G^{+}G)u = \begin{bmatrix} 1.2365580746 \\ 0.9079895101 \\ 0.1459498604 \\ 1.0653080872 \\ 0.5435245343 \\ 1.0866921189 \end{bmatrix}$$

and $x = A^{+}By - (I - A^{+}A)v = \begin{bmatrix} 1.0653080872 \\ 0.0208499800 \\ 0.0320088416 \\ 1.2365580746 \\ 0.9079895101 \end{bmatrix}$.

Now immediately follows balanced equation in its conventional form

1.0653080872 $Pt(NH_3)BrNO_3$−0.0208499800 $[4Fe(CN)_3{\cdot}3Fe(CN)_2]$ + 0.0320088416 $C_{44}H_{50}O_{15}$ + 1.2365580746 $Cu(NO_3)_2$ − 0.9079895101 K_2BeO_2 =1.2365580746 $Cu(NH_3)KNO_3$+0.9079895101 $BeCO_3$ + 0.1459498604 $K_{3.97}Fe(CN)_6$ + 1.0653080872 PtO_2 + 0.5435245343 $Br_{1.96}NO_2$ − 1.0866921189 HNO_3.

According to the definition 3. 6, extended matrix obtains the form

$$\mathrm{Ext}A|B - \begin{bmatrix} 1 & 0.00 & 0.00 & 0 & 0 & 0 & 0 & 0.00 & 1 & 0.00 & 0 \\ 2 & 18.0 & 0.00 & 2 & 0 & 2 & 0 & 6.00 & 0 & 1.00 & 1 \\ 3 & 0.00 & 50.0 & 0 & 0 & 3 & 0 & 0.00 & 0 & 0.00 & 1 \\ 1 & 0.00 & 0.00 & 0 & 0 & 0 & 0 & 0.00 & 0 & 1.96 & 0 \\ 3 & 0.00 & 15.0 & 6 & 2 & 3 & 3 & 0.00 & 2 & 2.00 & 3 \\ 0 & 7.00 & 0.00 & 0 & 0 & 0 & 0 & 1.00 & 0 & 0.00 & 0 \\ 0 & 18.0 & 44.0 & 0 & 0 & 0 & 1 & 6.00 & 0 & 0.00 & 0 \\ 0 & 0.00 & 0.00 & 1 & 0 & 1 & 0 & 0.00 & 0 & 0.00 & 0 \\ 0 & 0.00 & 0.00 & 0 & 2 & 1 & 0 & 3.97 & 0 & 0.00 & 0 \\ 0 & 0.00 & 0.00 & 0 & 1 & 0 & 1 & 0.00 & 0 & 0.00 & 0 \\ 0 & 0.00 & 0.00 & 0 & 0 & 0 & 0 & 0.00 & 0 & 0.00 & 1 \end{bmatrix}$$

The eigenvalues of the matrix $[\mathrm{Ext}A|B + (\mathrm{Ext}A|B)^{T}]/2$ are λ_1=59.824, λ_2=21.786, λ_3=-10.767, λ_4=5.648, λ_5=−4.030, λ_6=−2.045, λ_7=−0.940, λ_8=−0.044, λ_9=1.678, λ_{10}=0.831, λ_{11}=1.059.

The Lozinskiĭ measures of $\mathrm{Ext}A|B$ given by (3.6) with respect to the three common norms (3.5) are

μ $(\mathrm{Ext}A|B)$ = max (2, 32, 57, 2.96, 39, 8, 69, 2, 6.97, 2, 1) − 69,

$\mu_1(\mathrm{Ext}A|B)$ − max (10, 43, 109, 9, 5, 10, 5, 16.97, 3, 4.96, 6) − 109,

$\mu_2\{[\mathrm{Ext}A|B + (\mathrm{Ext}A|B)^{T}]/2\}$ = 59.824.

Since $\mu_2\{[\mathrm{Ext}A|B + (\mathrm{Ext}A|B)^{T}]/2\} > 0$, according to the definition 4.5 the chemical equation (5.3) is unstable.

Example 5. 4. We will now balance this equation

$$x_1\,KAu(CN)_2 - x_2\,AgRuAuTe_8 + x_3\,Fe_2(SO_4)_3 + x_4\,N_2Se_4 + x_5\,WO_3 - x_6\,Na_2CO_3 + x_7\,H_2CO_3 + x_8\,HCl - y_1\,[Ru(C_{10}H_8N_2)_3]Cl_2{\cdot}6H_2O + y_2\,C_nH_xAuNa_{1.96}OS_7 - y_3\,[WCl_4(NSeCl)]_2 + y_4\,K_{3.98}Fe(CN)_6 + y_5\,Au_2O_3 - y_6\,TeO_2 + y_7\,AgO + y_8\,NO_2. \quad (5.4)$$

The matrix of reactants A is

	$KAu(CN)_2$	$AgRuAuTe_8$	$Fe_2(SO_4)_3$	N_2Se_4	WO_3	Na_2CO_3	H_2CO_3	HCl	
	1	0	0	0	0	0	0	0	K
	1	1	0	0	0	0	0	0	Au
	2	0	0	0	0	1	1	0	C
	2	0	0	2	0	0	0	0	N
	0	1	0	0	0	0	0	0	Ag
	0	1	0	0	0	0	0	0	Ru
A	0	8	0	0	0	0	0	0	Te
	0	0	2	0	0	0	0	0	Fe
	0	0	3	0	0	0	0	0	S
	0	0	12	0	3	3	3	0	O
	0	0	0	4	0	0	0	0	Se
	0	0	0	0	1	0	0	0	W
	0	0	0	0	0	2	0	0	Na
	0	0	0	0	0	0	2	1	H
	0	0	0	0	0	0	0	1	Cl

and the matrix of the products B is

	$[Ru(C_{10}H_8N_2)_3]Cl_2{\cdot}6H_2O$	$C_nH_xAuNa_{1.96}OS_7$	$[WCl_4(NSeCl)]_2$	$K_{3.98}Fe(CN)_6$	Au_2O_3	TeO_2	AgO	NO_2	
	0	0	0	3.98	0	0	0	0	K
	0	1	0	0	2	0	0	0	Au
	30	4	0	6	0	0	0	0	C
	6	0	2	6	0	0	0	1	N
	0	0	0	0	0	0	1	0	Ag
	1	0	0	0	0	0	0	0	Ru
B	0	0	0	0	0	1	0	0	Te
	0	0	0	1	0	0	0	0	Fe
	0	7	0	0	0	0	0	0	S
	6	1	0	0	3	3	1	2	O
	0	0	2	0	0	0	0	0	Se
	0	0	2	0	0	0	0	0	W
	0	1.96	0	0	0	0	0	0	Na
	36	3	0	0	0	0	0	0	H
	2	0	10	0	0	0	0	0	Cl

The reaction matrix $A|B$, according to the definition 3. 5 obtains this form

$$A|B = \begin{bmatrix}
1&0&0.00&0&0&0&0&0&0.00&0.00&0.00&3.98&0&0&0&0\\
1&1&0.00&0&0&0&0&0&0.00&1.00&0.00&0.00&2&0&0&0\\
2&0&0.00&0&0&1&1&0&30.0&4.00&0.00&6.00&0&0&0&0\\
2&0&0.00&2&0&0&0&0&6.00&0.00&2.00&6.00&0&0&0&1\\
0&1&0.00&0&0&0&0&0&0.00&0.00&0.00&0.00&0&0&1&0\\
0&1&0.00&0&0&0&0&0&1.00&0.00&0.00&0.00&0&0&0&0\\
0&8&0.00&0&0&0&0&0&0.00&0.00&0.00&0.00&0&1&0&0\\
0&0&2.00&0&0&0&0&0&0.00&0.00&0.00&1.00&0&0&0&0\\
0&0&3.00&0&0&0&0&0&0.00&7.00&0.00&0.00&0&0&0&0\\
0&0&12.0&0&3&3&3&0&6.00&1.00&0.00&0.00&3&3&1&2\\
0&0&0.00&4&0&0&0&0&0.00&0.00&2.00&0.00&0&0&0&0\\
0&0&0.00&0&1&0&0&0&0.00&0.00&2.00&0.00&0&0&0&0\\
0&0&0.00&0&0&2&0&0&0.00&1.96&0.00&0.00&0&0&0&0\\
0&0&0.00&0&0&0&2&1&36.0&3.00&0.00&0.00&0&0&0&0\\
0&0&0.00&0&0&0&0&1&2.00&0.00&10.0&0.00&0&0&0&0
\end{bmatrix}$$

The rank of the above matrix is $\rho = 15$. Since the nullity of the reaction matrix is $\kappa = r + s - \rho = 8 + 8 - 15 = 1$, then we have only a unique solution, that means that the reaction is feasible. Next, we will determine the solution x, y.

The Moore-Penrose pseudoinverse $A^+ = (A^TA)^{-1}A^T$ of the matrix A is

$$94083728\,A^+ =$$

$$\begin{bmatrix}
12161840&11980320&15511840&19458944&-181520&-181520&-1452160&2315520\\
-181520&1225424&-231520&-290432&1406944&1406944&11255552&-34560\\
1157760&1140480&-3001536&1852416&-17280&-17280&[illegible]&8264608\\
-2432368&-2396064&-3102368&5516584&36304&36304&290432&-463104\\
3762720&3706560&-9754992&6020352&-56160&-56160&-449280&[illegible]\\
-2937280&-2893440&7615008&-4699648&43840&43840&350720&-3544704\\
-5874560&-5786880&15230016&-9399296&87680&87680&701440&-7089408\\
5874560&5786880&-15230016&9399296&-87680&-87680&-701440&7089408
\end{bmatrix}$$

$$\begin{bmatrix}
3473280&-1254240&-9729472&3762720&-5874560&-5874560&5874560\\
-51840&18720&145216&-56160&87680&87680&-87680\\
12396912&3363648&-926208&-10090944&-3544704&-3544704&3544704\\
-694656&250848&20762640&-752544&1174912&1174912&-1174912\\
-30272832&10931856&-3010176&61288160&-11520288&-11520288&11520288\\
-5317056&1920048&2349824&-5760144&-10354288&-6687576&6687576\\
10634112&3840096&4699648&11520288&13375152&33666712&-33666712\\
10634112&-3840096&-4699648&11520288&13375152&13375152&80708576
\end{bmatrix}$$

and the matrix G has this form

$$94083728\,G = (94083728\,I - AA^-)B =$$

$$\begin{bmatrix}
-374666864.00&-67948582.40&-85729984.00&113908890.24\\
-370479056.00&25745068.80&-84450432.00&-256655625.60\\
520438048.00&129819488.64&146991200.00&36642428.80\\
-147865088.00&-108717731.84&-61900992.00&34730938.88\\
4187808.00&-390076.80&1279552.00&3888721.60\\
98271536.00&-390076.80&1279552.00&3888721.60\\
33502464.00&-3120614.40&10236416.00&31109772.80\\
358573824.00&-123389272.32&-34235136.00&82128182.40\\
537860736.00&473502187.52&-51352704.00&-17933318.40\\
-194227488.00&-97810668.16&18544032.00&-9204700.80\\
73932544.00&54358865.92&30950496.00&-17365469.44\\
582682464.00&293432004.48&-55632096.00&27614102.40\\
31122208.00&81806257.92&-101311648.00&-4514163.20\\
31122208.00&81806257.92&-101311648.00&-4514163.20\\
-31122208.00&-81806257.92&101311648.00&4514163.20
\end{bmatrix}$$

$$\begin{bmatrix}
-20197920.00 & 5214880.00 & 1435760.00 & -16950464.00 \\
165462528.00 & -6096832.00 & 10096.00 & -16697472.00 \\
-40315632.00 & -7902832.00 & -3020144.00 & -31322272.00 \\
-32316672.00 & 8343808.00 & 2297216.00 & 48146240.00 \\
-2507008.00 & -11311712.00 & 92658064.00 & 252992.00 \\
-2507008.00 & -11311712.00 & -1425664.00 & 252992.00 \\
-20056064.00 & 3590032.00 & -11405312.00 & 2023936.00 \\
-24743808.00 & -19905408.00 & -6692736.00 & -17159424.00 \\
-37115712.00 & -29858112.00 & -10039104.00 & -25739136.00 \\
13402896.00 & 10782096.00 & 3625232.00 & 9294688.00 \\
16158336.00 & -4171904.00 & -1148608.00 & -24073120.00 \\
-40208688.00 & -32346288.00 & -10875696.00 & -27884064.00 \\
53472.00 & -12221728.00 & -3927776.00 & 1719104.00 \\
53472.00 & -12221728.00 & -3927776.00 & 1719104.00 \\
-53472.00 & 12221728.00 & 3927776.00 & -1719104.00
\end{bmatrix}$$

Required vectors $\boldsymbol{y}$ and $\boldsymbol{x}$, according to (4.12) and (4.13) are

$$\boldsymbol{y} = (\boldsymbol{I} - \boldsymbol{G}^{+}\boldsymbol{G})\boldsymbol{u} = \begin{bmatrix}
0.0710721984 \\
0.1656824131 \\
0.0679331915 \\
0.7731845945 \\
1.4913322357 \\
0.5685775876 \\
0.0710721984 \\
1.0210754230
\end{bmatrix}$$

$$\text{and } \boldsymbol{x} = \boldsymbol{A}^{+}\boldsymbol{B}\boldsymbol{y} + (\boldsymbol{I} - \boldsymbol{A}^{+}\boldsymbol{A})\boldsymbol{v} = \begin{bmatrix}
3.0772746861 \\
0.0710721984 \\
0.3865922972 \\
0.0339665957 \\
0.1358663829 \\
0.1623687648 \\
1.1170850360 \\
0.8214763117
\end{bmatrix}$$

Now immediately follows balanced equation in its conventional form

3.0772746861 $KAu(CN)_2$+0.0710721984 $AgRuAuTe_8$ + 0.3865922972 $Fe_2(SO_4)_3$ + 0.0339665957 N_2Sc_4 − 0.1358663829 WO_3 − 0.1623687648 Na_2CO_3 − 1.1170850360 H_2CO_3 + 0.8214763117 HCl − 0.0710721984 $[Ru(C_{10}H_8N_2)_3]Cl_2{\cdot}6H_2O$ + 0.1656824131 $C_4H_2AuNa_{1.96}OS_7$ + 0.0679331915 $[WCl_4(NScCl)]_2$ − 0.7731845945 $K_{3.98}Fe(CN)_6$ − 1.4913322357 Au_2O_3 + 0.5685775876 TeO_3 − 0.0710721984 AgO + 1.0210754230 NO_2.

According to the definition 3. 6, extended matrix obtains the form

$$\text{Ext}\boldsymbol{A}^{\dagger}\boldsymbol{B} = \begin{bmatrix}
1 & 0 & 0.00 & 0 & 0 & 0 & 0 & 0 & 0.00 & 0.00 & 0.00 & 3.98 & 0 & 0 & 0 & 0 \\
1 & 1 & 0.00 & 0 & 0 & 0 & 0 & 0 & 0.00 & 1.00 & 0.00 & 0.00 & 2 & 0 & 0 & 0 \\
2 & 0 & 0.00 & 0 & 0 & 1 & 1 & 0 & 30.0 & 4.00 & 0.00 & 6.00 & 0 & 0 & 0 & 0 \\
2 & 0 & 0.00 & 2 & 0 & 0 & 0 & 0 & 6.00 & 0.00 & 2.00 & 6.00 & 0 & 0 & 0 & 1 \\
0 & 1 & 0.00 & 0 & 0 & 0 & 0 & 0 & 0.00 & 0.00 & 0.00 & 0.00 & 0 & 0 & 1 & 0 \\
0 & 1 & 0.00 & 0 & 0 & 0 & 0 & 0 & 1.00 & 0.00 & 0.00 & 0.00 & 0 & 0 & 0 & 0 \\
0 & 8 & 0.00 & 0 & 0 & 0 & 0 & 0 & 0.00 & 0.00 & 0.00 & 0.00 & 0 & 1 & 0 & 0 \\
0 & 0 & 2.00 & 0 & 0 & 0 & 0 & 0 & 0.00 & 0.00 & 0.00 & 1.00 & 0 & 0 & 0 & 0 \\
0 & 0 & 3.00 & 0 & 0 & 0 & 0 & 0 & 0.00 & 7.00 & 0.00 & 0.00 & 0 & 0 & 0 & 0 \\
0 & 0 & 12.0 & 0 & 3 & 3 & 3 & 0 & 6.00 & 1.00 & 0.00 & 0.00 & 3 & 3 & 1 & 2 \\
0 & 0 & 0.00 & 4 & 0 & 0 & 0 & 0 & 0.00 & 0.00 & 2.00 & 0.00 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.00 & 0 & 1 & 0 & 0 & 0 & 0.00 & 0.00 & 2.00 & 0.00 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.00 & 0 & 0 & 2 & 0 & 0 & 0.00 & 1.96 & 0.00 & 0.00 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.00 & 0 & 0 & 0 & 2 & 1 & 36.0 & 3.00 & 0.00 & 0.00 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.00 & 0 & 0 & 0 & 0 & 1 & 2.00 & 0.00 & 10.0 & 0.00 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.00 & 0 & 0 & 0 & 0 & 0 & 0.00 & 0.00 & 00.0 & 0.00 & 0 & 0 & 0 & 1
\end{bmatrix}$$

The eigenvalues of the matrix $[\text{Ext}\boldsymbol{A}^{\dagger}\boldsymbol{B} + (\text{Ext}\boldsymbol{A}^{\dagger}\boldsymbol{B})^{+}]/2$ are λ_1=−28.656, λ_2=−24.826, λ_3=9.278, λ_4=−6.650, λ_5=−6.847, λ_6=5.301, λ_7=4.806, λ_8=3.495, λ_9=−3.389, λ_{10}=−2.805, λ_{11}=1.297, λ_{12}=−0.937, λ_{13}=0.759, λ_{14}=−0.312, λ_{15}=0.040, λ_{16}=0.134.

The Lozinskiĭ measures of $\text{Ext}\boldsymbol{A}^{\dagger}\boldsymbol{B}$ given by (3.6) with respect to the three common norms (3.5) are

$\mu_\infty(\text{Ext}\boldsymbol{A}^{\dagger}\boldsymbol{B})$ = max (4.98, 5, 44, 19, 2, 2, 9, 3, 10, 37, 6, 3, 3.96, 42, 13, 1) = 44,

$\mu_1(\text{Ext}\boldsymbol{A}^{\dagger}\boldsymbol{B})$ = max (6, 11, 17, 6, 4, 6, 6, 2, 81, 17.96, 16, 16.98, 5, 4, 2, 4) = 81,

$\mu_2\{[\text{Ext}\boldsymbol{A}^{\dagger}\boldsymbol{B} + (\text{Ext}\boldsymbol{A}^{\dagger}\boldsymbol{B})^{+}]/2\} - 28.656$.

Since $\mu_2\{[\text{Ext}\boldsymbol{A}^{\dagger}\boldsymbol{B} + (\text{Ext}\boldsymbol{A}^{\dagger}\boldsymbol{B})^{+}]/2\} > 0$, according to the definition 4. 5 the chemical equation (5.4) is unstable.

Example 5. 5. Also, we balanced this very interesting chemical equation

20666316 $C_{66}H_{30}Cl_6CuN_{27}O_{19}S_7$+5103865732 $C_{44}H_{34}Au_5MnO_3P_2$+1457092800 $C_{41}H_{14}O_3Si_3Ti$−1457092800 $C_{41}H_{31}FeBr_3ClO_2$+711775332 $C_{44}H_{32}O_8Os_2P_2Pt_2$−10333158 $As_8Co_2N_2O_{17}$+7117753320 ClF+9670698616 HNO_3 + 2152394100 H_2O−146703788 SO_2−4371278400 $C_{44}H_{34}BrN_2O_2P_2$+1423550664 $C_{41}H_{32}F_5NP_2Pt$+2967406141 $C_{44}H_{44}O_{10}$−20666316 $CuCoO_4$+41624000 $As_{1986}N_2PS_7$ +5103865732 MnO_2+4371278400 SiO_2−1457092800 $TiFeCl_{3.97}$+5103865732 Au_2O+711775332 Os_2O_3. (5.5)

According to the definition 3. 6, extended matrix

obtains the form

$$\mathrm{Ext}\boldsymbol{A}|\boldsymbol{B} =$$

69	44	44	44	44	0	0	0	0	0	44	44	44	0	0	0.000	0	0	0	0.00	0	0
39	34	44	34	32	0	0	1	2	0	34	32	44	0	0	0.000	0	0	0	0.00	0	0
0	0	0	3	0	0	0	0	0	0	1	0	0	0	0	0.000	0	0	0	0.00	0	0
27	0	0	0	0	2	0	1	0	0	2	1	0	0	2	0.000	0	0	0	0.00	0	0
19	3	3	5	8	17	0	3	1	2	2	0	10	4	0	0.000	2	2	0	0.00	1	3
0	2	0	0	2	0	0	0	0	0	2	2	0	0	1	0.000	0	0	0	0.00	0	0
0	0	0	0	0	0	1	0	0	0	0	5	0	0	0	0.000	0	0	0	0.00	0	0
0	0	0	0	2	0	0	0	0	0	0	1	0	0	0	0.000	0	0	0	0.00	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0.000	0	0	0	0.00	0	0
0	0	0	0	0	2	0	0	0	0	0	0	0	0	1	0.000	0	0	0	0.00	0	0
0	0	0	0	0	8	0	0	0	0	0	0	0	0	1	1.986	0	0	0	0.00	0	0
7	0	0	0	0	0	0	0	0	1	0	0	0	0	7	0.000	0	0	0	0.00	0	0
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0.000	1	0	0	0.00	0	0
0	0	3	0	0	0	0	0	0	0	0	0	0	0	0	0.000	0	1	0	0.00	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0.000	0	0	1	0.00	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0.000	0	0	1	0.00	0	0
6	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0.000	0	0	5	0.97	0	0
0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0.000	0	0	0	0.00	2	0
0	0	0	0	2	0	0	0	0	0	0	0	0	0	0	0.000	0	0	0	0.00	0	2
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.000	0	0	0	0.00	0	1

The eigenvalues of the matrix $[\mathrm{Ext}\boldsymbol{A}|\boldsymbol{B} + (\mathrm{Ext}\boldsymbol{A}|\boldsymbol{B})^{\mathrm{T}}]/2$ are λ_1=143.364, λ_2=-41.450, λ_3=14.630, λ_4=-11.150, λ_5=11.154, λ_6=-6.730, λ_7=4.060, λ_8=-3.424, λ_9=-3.357, λ_{10}=3.432, λ_{11}=-2.112, λ_{12}=2.481, λ_{13}=1.348, λ_{14}=1.453, λ_{15}=0.544, λ_{16}=0.614, λ_{17}=0.575, λ_{18}=0.137, λ_{19}=0.090, λ_{20}=0.006.

The Lozinskii measures of $\mathrm{Ext}\boldsymbol{A}|\boldsymbol{B}$ given by (3.6) with respect to the three common norms (3.5) are

$\mu_1(\mathrm{Ext}\boldsymbol{A}|\boldsymbol{B})$ = max (377, 296, 4, 35, 85, 9, 6, 3, 2, 3, 9.986, 15, 2, 4, 2, 2, 13.97, 4, 4, 1) = 377,

$\mu_1(\mathrm{Ext}\boldsymbol{A}|\boldsymbol{B})$ = max (168, 86, 95, 88, 90, 29, 2, 5, 3, 3, 85, 85, 98, 6, 11.986, 3, 3, 7.97, 3, 6) = 168,

$\mu_2\{[\mathrm{Ext}\boldsymbol{A}|\boldsymbol{B} + (\mathrm{Ext}\boldsymbol{A}|\boldsymbol{B})^{\mathrm{T}}]/2\}$ = 143.364.

Since $\mu_2\{[\mathrm{Ext}\boldsymbol{A}|\boldsymbol{B}+(\mathrm{Ext}\boldsymbol{A}|\boldsymbol{B})^{\mathrm{T}}]/2\}>0$, according to the definition 4. 5 the chemical equation (5.5) is unstable.

Example 5. 6. An other chemical equation which we balanced is the equation

103213240 $C_{60}I_{60}As_8B_4O_{16}$+5108088 $C_{44}H_{32}N_2O_6P_2Rh_2$+ 21220962 AgAuRuTlTe$_6$+8776350 $Fe_2(SCO)_6$+ 17552700 La(CN)$_3$+61083396 CaO+4388175 Sr_2O_3+ 17552700 SiO_2+20432352 ISCN+12444612 P_2O_5+ 141449244 CO_2=103213240 $C_{60}H_{60}As_8B_4O_{15}$+10216176 $C_{44}H_{32}I_2N_4Rh$+8776350 $Ca_{6.96}Fe_2La_8P_4Si_2SrO_{34}$+21220962 Au(CN)$_2$+21220962 AgO+10610481 Ru_2O+10663800 $Tl_{1.99}O_3$+190988658 TeO_3+73090452 H_2S. (5.6)

The reaction matrix $\boldsymbol{A}|\boldsymbol{B}$, according to the definition 3.5 has the form

$$\boldsymbol{A}|\boldsymbol{B} =$$

69	44	0	6	3	0	0	0	1	0	1	69	44	0	2	0	0	0	0	0
66	32	0	0	0	0	0	0	0	0	0	0	63	32	0	0	0	0	0	2
3	0	0	0	0	0	0	0	0	0	0	3	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	4	0	0	0	0	0	0	0	0
16	6	0	6	0	1	3	2	0	5	2	13	0	24	0	1	1	3	3	0
0	2	0	0	3	0	0	0	1	0	0	0	4	0	2	0	0	0	0	0
0	2	0	0	0	0	0	0	0	2	0	0	0	4	0	0	0	0	0	0
0	2	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1.99	0	0
0	0	9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
0	0	0	2	0	0	0	0	0	0	0	0	0	2	0	0	0	0	0	0
0	0	0	6	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1
0	0	0	0	1	0	0	0	0	0	0	0	0	2	0	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0	0	6.96	0	0	0	0	0	0
0	0	0	0	0	0	2	0	0	0	0	0	0	1	0	0	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0	2	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	2	0	0	0	0	0	0	0

The eigenvalues of the matrix $[\boldsymbol{A}|\boldsymbol{B}-(\boldsymbol{A}|\boldsymbol{B})^{\mathrm{T}}]/2$ are λ_1=131.840, λ_2=-24.264, λ_3=15.402, λ_4=-13.779, λ_5=-8.097, λ_6=4.189, λ_7=3.564, λ_8=-3.485, λ_9=-2.396, λ_{10}=1.459, λ_{11}=-1.347, λ_{12}=-1.023, λ_{13}=0.923, λ_{14}=0.748, λ_{15}=0.449, λ_{16}=0.722, λ_{17}=0.513, λ_{18}=0.116, λ_{19}=0.011, λ_{20}=-0.075.

The Lozinskii measures of $\boldsymbol{A}|\boldsymbol{B}$ with respect to the three common norms are

$\mu_1(\boldsymbol{A}|\boldsymbol{B})$ = max (239, 195, 6, 8, 86, 12, 8, 3, 2, 2, 3, 2.99, 10, 4, 8, 3, 7.96, 3, 3, 3) = 239,

$\mu_1(\boldsymbol{A}|\boldsymbol{B})$ = max (158, 88, 13, 20, 7, 2, 5, 3, 4, 7, 3, 152, 83, 41.96, 5, 2, 3, 4.99, 4, 3) =158,

$\mu_2\{[\boldsymbol{A}|\boldsymbol{B} - (\boldsymbol{A}|\boldsymbol{B})^{\mathrm{T}}]/2\}$ = 131.840.

Because $\mu_2\{[\boldsymbol{A}|\boldsymbol{B}-(\boldsymbol{A}|\boldsymbol{B})^{\mathrm{T}}]/2\}>0$, that means that the chemical equation (5. 6) is unstable.

Example 5. 7. It will be benefit for us if we balance some little bit bigger chemical equation, for instance like this

76152 $C_{55}H_{72}MgN_4O_8$+9528 $[PtCl_4(NSeCl)]_2$+ 1574253 HNO_3+ 1248508 $RuPO_4$ + 624290 $NiCl_2$ + 761917 $(NH_4)_3PO_4{\cdot}12MoO_3$ + 548066 $PdCl_4$ + 19056 AgCsCl$_2$ + 9528 $Cs_2Pb_2O_4$ + 19056 $K_4Fe(CN)_6$ + 14292 CrI_3 + 14292 $BeSiO_3$ + 6352 $BiAt_3$ + 624290 $Mg(SCO)_4$ + 624290 $\{[P(C_6H_5)_3]_2Ag\}_2$ = 6352 Bi + 624290 $Ni(SCO)_4{\cdot}\{[P(C_6H_5)_3]_2Ag\}_2$ + 14292 BeCSiCrKO$_7$ + 42876 KI(CN)$_2$ + 9143004 MoO_3 + 700442 RuMg(CN)$_6$ + 548066 RuPdCl$_6$ + 2010425 H_3PO_4 + 19056 $AtCs_2$ PbKPtAgFeCl$_{15}$ + 4800 $N_2Se_{3.97}$ + 5084463 H_2O.

(5. 7)

According to the definition 3.6, extended matrix obtains the form

$$\mathrm{Ext}A_1^1B =$$

$$\begin{bmatrix}
55 & 0000 & 000060004720761206000 & 00,000\\
72 & 0100120000000060060000003 & 00,002\\
1 & 0000 & 000000001 & 00 & 0000100 & 00,000\\
4 & 2100 & 300060000 & 00 & 0020600 & 02,000\\
5 & 0340400040030400 & 4703004 & 02,001\\
0 & 2000 & 000000000 & 00 & 0000000 & 10,000\\
0 & 10002 & 042000000 & 00 & 0000060 & 150,000\\
0 & 2000 & 000000000 & 00 & 0000000 & 03,970\\
0 & 0010 & 000000000 & 00 & 0000110 & 00,000\\
0 & 0010 & 100000000 & 40 & 4000001 & 00,000\\
0 & 0001 & 000000000 & 00 & 1000000 & 00,000\\
0 & 000012000000000 & 00 & 0001000 & 00,000\\
0 & 0000 & 010000000 & 00 & 0000010 & 00,000\\
0 & 0000 & 001000000 & 20 & 2000000 & 10,000\\
0 & 0000 & 001200000 & 00 & 0000000 & 20,000\\
0 & 0000 & 000200000 & 00 & 0000000 & 10,000\\
0 & 0000 & 000040000 & 00 & 0110000 & 10,000\\
0 & 0000 & 000010000 & 00 & 0000000 & 10,000\\
0 & 0000 & 000001000 & 00 & 0100000 & 00,000\\
0 & 0000 & 000003000 & 00 & 0010000 & 00,000\\
0 & 0000 & 000000100 & 00 & 0100000 & 00,000\\
0 & 0000 & 000000100 & 00 & 0100000 & 00,000\\
0 & 0000 & 000000010 & 01 & 0000000 & 00,000\\
0 & 0000 & 000000030 & 00 & 0000000 & 10,000\\
0 & 0000 & 000000004 & 00 & 4000000 & 00,000\\
0 & 0000 & 000000000 & 00 & 0000000 & 00,001
\end{bmatrix}$$

The eigenvalues of the matrix $[\mathrm{Ext}A_1^1B+(\mathrm{Ext}A_1^1B)^1]/2$ are λ_1=112.534, λ_2=-45.894, λ_3=22.466, λ_4=-22.572, λ_5=-11.987, λ_6=11.234, λ_7=-5.757, λ_8=-5.367, λ_9=5.011, λ_{10}=3.199, λ_{11}=-2.849, λ_{12}=-2.010, λ_{13}=1.948, λ_{14}=-1.549, λ_{15}=1.571, λ_{16}=1.269, λ_{17}=1.042, λ_{18}=-1.070, λ_{19}=0.814, λ_{20}=0.614, λ_{21}=-0.809, λ_{22}=-0.516, λ_{23}=-0.398, λ_{24}=0.154, λ_{25}=-0.091, λ_{36}=0.012.

The Lozinskiĭ measures of $\mathrm{Ext}A_1^1B$ given by (3.6) with respect to the three common norms (3.5) are

$\mu_1(\mathrm{Ext}A_1^1B)$ = max (222, 210, 3, 26, 84, 3, 39, 5.97, 3, 11, 2, 13, 2, 6, 5, 3, 7, 2, 2, 4, 2, 2, 2, 4, 8, 1) = 222,

$\mu_1(\mathrm{Ext}A_1^1B)$ = max (137, 16, 5, 6, 3, 68, 5, 4, 8, 17, 4, 5, 4, 13, 138, 1, 151, 12, 6, 4, 14, 8, 8, 23, 7.97, 4) = 151,

$\mu_2\{[\mathrm{Ext}A_1^1B + (\mathrm{Ext}A_1^1B)^1]/2\}$ = 112.534.

Since $\mu_2\{[\mathrm{Ext}A_1^1B+(\mathrm{Ext}A_1^1B)^1]/2\}>0$, according to the definition 4. 5 the chemical equation (5.7) is unstable.

Example 5. 8. Also we balanced this chemical equation

1044 $(NH_3)_3[P(Mo_3O_{10})]_1$ + 4356 $LaHgTlZrS_6$ + 10048 $AgRuAuTe_8$ − 8776 $C_4H_3AuNa_2OS_7$ + 252000 $KAu(CN)_2$ + 31500 $MgFe_2(SO_4)_4$ − 5024 $PbCrO_4$ − 22550 $Sn_3(AsO_4)_2$ + 67650 $BeSiO_3$ + 67650 $CuCsCl_3$ + 10947 N_2SiSe_6 + 248050 $CaAlF_5$ + 22550 $BiAt_3$ + 32841 W_2O − 140336 H_2CO_3 + 145556 HCl − 67650 $CaBeAsSAtCsF_{13}$ − 10048 $[Ru(C_{10}H_8N_2)_3]Cl_2{\cdot}6H_2O$ + 32841 $[WCl_4(NScCl)]_2$ + 12528 $(NH_4)_2MoO_2$ − 63000 $K_4Fe(CN)_6$ + 2512 $Na_2Cr_2O_7$ + 31500 MgS_2O_3 + 4356 $LaTlS_3$ + 4176 Na_3PO_4 − 5024 Ag_2PbO_2 − 67650 $SnSO_4$ + 180400 CaF_2 − 2200 $Hg_{1.98}S$ + 4356 ZrO_2 + 33825 Cu_2O − 124025 Al_2O_3 + 11275 Bi_2O_3 + 78597 SiO_2 + 135412 Au_2O − 80384 TeO_3. (5.8)

Also, the above equation is unstable.

Example 5. 9. This balanced chemical equation

17592 $(NH_4)_3[P(Mo_3O_{10})]_1$ + 339756 $PrHgTlZrS_6$ + 675522 $In_3ZnCeCl_{12}$ + 189456 $AgRuAuOs_8$ − 152916 $C_4H_3AuLi_2OS_7$ − 5714424 $KAu(CN)_2$ + 714303 $MgMn_2(SO_4)_4$ − 94728 $PbCrO_4$ − 558068 $Sn_3(WO_4)_2$ + 1674204 $BeSiO_3$ + 1674204 $CuCsCl_3$ + 337761 N_2SiSe_6 + 6138748 $CaAlF_5$ + 558068 $BiAt_3$ + 1013283 Te_2O + 2214804 H_2CO_3 − 3462564 HClO − 1674204 $CaBeWSAtCsF_{13}$ + 189456 $Ru(C_{10}H_8N_2)_3Cl_2H_{12}O_6$ + 1013283 $Te_2Cl_8(NSeInCl_3)_2$ + 211104 $(NH_4)_2MoO_4$ + 1428606 $K_4Mn(CN)_6$ − 675522 $CeCl_3$ + 47364 $Li_2Cr_2O_7$ + 714303 MgS_2O_3 + 339756 $PrTlS_3$ − 70368 Li_3PO_4 + 94728 Ag_2PbO_2 + 1674204 $SnSO_4$ + 4464544 CaF_2 + 169878 Hg_2S + 339756 ZrO_2 − 837102 Cu_2O + 3069374 Al_2O_3 + 293720 $Bi_{1.9}O_3$ − 2011965 SiO_2 + 3028398 Au_2O + 1515648 OsO_3 + 675522 ZnO, (5.9)

is unstable too.

3° Next, we will consider the case when the chemical reaction is non-unique, *i. e.*, when the nullity of its reaction matrix is bigger than one.

Example 5. 10. Last considered equation in this work will be this equation

$x_1\ NH_4ClO_4 + x_2\ HNO_3 + x_3\ HCl = y_1\ HClO_4 + y_2\ NOCl + y_3\ N_2O + y_4\ N_2O_5 + y_5\ H_2O - y_6\ Cl_2$. (5.10)

From (5. 10) follows this system of linear equations

$$x_1 - x_2 = y_2 + 2y_3 + 2y_4,$$

$$4x_1 + x_2 + x_3 = y_1 + 2y_5,$$
$$x_1 + x_3 = y_1 + y_2 + 2y_6,$$
$$4x_1 + 3x_2 = 4y_1 + y_2 + y_3 + 3y_4 + y_5.$$

The general solution of the above system is

$$y_1 = 4x_1 + x_2 + x_3 - 2y_5,\ y_2 = -3x_1 - x_2 + 2y_5 - 2y_6,$$
$$y_3 = 15x_1/2 + 3x_2/2 + 2x_3 - 4y_5 + y_6/2,$$
$$y_4 = -11x_1/2 - x_2/2 - 2x_3 + 3y_5 + y_6/2.$$

where x_1, x_2, x_3, y_5 and y_6 are arbitrary real numbers.

Now, the balanced equation has a form

$$x_1\ NH_4ClO_4 + x_2\ HNO_3 + x_3\ HCl = (4x_1 + x_2 + x_3 - 2y_5)\ HClO_4 + (-3x_1 - x_2 + 2y_5 - 2y_6)\ NOCl + (15x_1/2 + 3x_2/2 + 2x_3 - 4y_5 + y_6/2)\ N_2O + (-11x_1/2 - x_2/2 - 2x_3 + 3y_5 + y_6/2)\ N_2O_3 + y_5\ H_2O + y_6\ Cl_2.$$

where x_1, x_2, x_3, y_5 and y_6 are arbitrary real numbers.

A particular case of equation (5. 10) for $y_4 = 0$ is solved in[8], but there the author balanced this equation on completely different way by using of non-singular matrix method. So, it is the mathematical consideration of this equation, but for chemical purposes stoichiometric coefficients must be unique.

Now, if we employ here offered method we obtain that only possible chemical equation is this equation

$$30\ NH_4ClO_4 + 2\ NOCl + 12\ Cl_2 = 11\ HClO_4 + 6\ N_2O + 5\ N_2O_3 + 10\ HNO_3 + 45\ HCl + 27\ H_2O,$$

which actually is a particular case of the chemical equation (5.10) for x_1=1, x_2=-1/3, x_3=-3/2, y_5=9/10 and y_6=-2/5.

Also, in the article[78], is balanced a particular case for x_1=34, x_2=36, x_3=8, and y_5=y_6=0.

6. CONCLUSION

The practical power of the matrix procedure as the most general tool for balancing chemical equations is demonstrable. By this method are balanced completely new classes of chemical equations with atoms, which have fractional oxidation numbers. Research shown that employed pseudoinverse matrix method founded on virtue of the solution of the Diophantine matrix equation works perfectly for the all chemical equations.

Actually, this method is unique method both in mathematics and chemistry, which generalize to date the all known methods and particular techniques for balancing chemical equations cited in the bibliography given below. In other words speaking, the mathematical method given here is applicable for the all possible cases for balancing chemical equations, does not matter what kind of atoms they possess – fractional or integer oxidation numbers. Also, this method determine and $r + s - \rho$ linear independent solutions when $\kappa > 1$, *i.e.*, when the reaction is feasible and is non-unique. This is the main advantage of the method in relation of other known particular methods and techniques.

For all considered chemical equations which have a unique solution is made stability analysis, and as shown results all of them are unstable. This stability analysis is founded on virtue of the Lozinski measures of extended matrix.

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