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A New Approach to an Inventory with Constant Demand[1)]

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Abstract

An inventory with constant demand is studied. We adopt a renewal argument to obtain the transient and stationary distribution of the level of the inventory. We show that the stationary distribution can be also derived by making use of either the level crossing technique or the renewal reward theorem. After assigning several managing costs to the inventory, we calculate the long-run average cost per unit time. A numerical example is illustrated to show how we optimize the inventory.

Keywords : Inventory with constant demand; Level crossing technique; Long-run average cost; Poisson process; Renewal argument.

1. Introduction

In this paper, we consider an inventory model with constant demand, which was introduced by Lee and Park (1991). The level of the inventory is initially $\beta > 0$, thereafter decreases linearly at rate $\mu > 0$, and remains at 0 if the inventory becomes empty. A deliveryman arriving at the inventory according to a Poisson process of rate $\lambda > 0$ instantaneously increases the level of the inventory up to β only if the level of the inventory is below a threshold $\alpha (0 \le \alpha \le \beta)$ when he arrives. β may be considered as the capacity of the inventory.

Lee and Park (1991) obtained the transient and stationary distribution of the level of the inventory by establishing a partial differential equation for the distribution function. After assigning the cost of the inventory being empty and the keeping cost of the stock to the inventory, they also showed that there exists

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a unique optimal policy with respect to the threshold α, which minimizes the long-run average cost per unit time.

We, in this paper, adopt a different but simpler approach to extend the earlier results. A renewal argument is directly applied to the level of the inventory to obtain the transient and stationary distribution of the level of the inventory. We show that the stationary distribution can be also derived by making use of either the level crossing technique or the renewal reward theorem. After assigning two more costs to the inventory, the cost per visit of the deliveryman and the cost of replenishing the stock, we generalize the previous optimization result of Lee and Park (1991) to calculate the long-run average cost per unit time. A numerical example is illustrated to show how we optimize the inventory.

2. Transient and Stationary Distribution

Let $X(t)$ be the level of the inventory at time $t > 0$ and $F(x,t)$ be the transient distribution function of $X(t)$. After conditioning on whether the deliveryman arrives or not during a small interval $(t, t+\delta t)$ and on whether he makes a delivery or not, Lee and Park (1991) established the following partial differential equation for $F(x,t)$:

$$\frac{\partial}{\partial t}F(x,t) = \mu\frac{\partial}{\partial x}F(x,t) - \lambda F(x \wedge \alpha, t),$$

for $0 \leq x < \beta$, where $\wedge$ denotes the smaller value of the two. Since the level of the inventory can't exceed β, $F(\beta,t) = 1$ for all $t > 0$. Lee and Park (1991) solved analytically the above equation to obtain $F(x,t)$.

We, in this section, obtain the same $F(x,t)$ purely probabilistically. First, notice that the points where the stock of the inventory reaches α form an embedded renewal process. Let T be the period between two successive renewal points, then, due to the memoryless property,

$$T = E^{\lambda} + \frac{\beta - \alpha}{\mu},$$

where E^{λ} is an exponential random variable with parameter λ. The probability density function of T, $g(t)$ is given by

$$g(t) = \lambda e^{-\lambda(t - \frac{\beta-\alpha}{\mu})},$$

for $t > \frac{\beta - \alpha}{\mu}$. Let $h(t)$ be the renewal density function of the embedded renewal process, then $h(t) = \sum_{n=1}^{\infty} g^{(n)}(t)$, where (n) denotes n-fold recursive convolution. We are now ready to obtain $F(x,t)$.

Theorem 1. (i) If we ignore the first passage time to α from the origin, $\frac{\beta - \alpha}{\mu}$, then for $0 \leq x < \alpha$,

$$F(x,t) = e^{-\lambda t} I_{\left\{t > \frac{\alpha - x}{\mu}\right\}} + \int_0^{t - \frac{\alpha - x}{\mu}} e^{-\lambda(t-u)} h(u) du. \tag{1}$$

(ii) Let $\overline{F}(x,t) = 1 - F(x,t)$, then for $\alpha \leq x \leq \beta$,

$$\overline{F}(x,t) = I_{\left\{t < \frac{\beta - x}{\mu}\right\}} + \int_{t - \frac{\beta - x}{\mu}}^{t} h(u) du, \tag{2}$$

where I_A denotes the indicator of event A.

Proof. (i) Recall that the points where the stock of the inventory reaches α form an embedded renewal process. Hence, $X(t) \leq x$ for $0 \leq x < \alpha$ if and only if either that the deliveryman does not arrived by time t and t is at least larger than $\frac{\alpha - x}{\mu}$ or that there is a renewal in the embedded renewal process at $u \in (0, t]$, the deliveryman does not arrived during the interval $[u, t]$ and $t - u$ is at least larger than $\frac{\alpha - x}{\mu}$. Therefore,

$$F(x,t) = e^{-\lambda t} I_{\left\{t > \frac{\alpha - x}{\mu}\right\}} + \int_0^t e^{-\lambda(t-u)} I_{\left\{t - u > \frac{\alpha - x}{\mu}\right\}} h(u) du,$$

which gives the result.

(ii) Consider the points where the replenishments occur, that is, the points where the level of the inventory is increased to β. These points also form an embedded renewal process with renewal density function $h(t) = \sum_{n=1}^{\infty} g^{(n)}(t)$. Hence, $X(t) > x$ for $\alpha \leq x \leq \beta$ if and only if either that t is smaller than $\frac{\beta - x}{\mu}$ or that there is a renewal in the embedded renewal process at $u \in (0, t]$ and $t-u$ is smaller than $\frac{\beta - x}{\mu}$. Therefore,

$$\overline{F}(x,t) = I_{\left\{t < \frac{\beta - x}{\mu}\right\}} + \int_0^t I_{\left\{t-u < \frac{\beta - x}{\mu}\right\}} h(u)du,$$

which gives the result. ■

Let $F(x) = \lim_{t \to \infty} F(x,t)$ be the stationary distribution of the level of the inventory. By making use of the key renewal theorem [Ross (1996) p. 112], we can obtain $F(x)$.

Theorem 2. For $0 \le x < \alpha$,

$$F(x) = \frac{\mu e^{\lambda(x-\alpha)/\mu}}{\mu + \lambda(\beta - \alpha)}, \tag{3}$$

and for $\alpha \le x \le \beta$,

$$F(x) = \frac{\mu - \alpha\lambda + \lambda x}{\mu + \lambda(\beta - \alpha)}. \tag{4}$$

Proof. Applying the key renewal theorem to equation (1) gives

$$F(x) = \lim_{t \to \infty} F(x,t) = \frac{1}{E(T)} \int_0^\infty e^{-\lambda t} I_{\left\{t > \frac{\alpha - x}{\mu}\right\}} dt,$$

which is equal to equation (3), since $E(T) = \frac{1}{\lambda} + \frac{\beta - \alpha}{\mu}$. Similarly, Applying the key renewal theorem to equation (2) gives

$$F(x) = 1 - \lim_{t \to \infty} \overline{F}(x,t) = 1 - \frac{1}{E(T)} \int_0^\infty I_{\left\{t < \frac{\beta - x}{\mu}\right\}} dt,$$

which is equal to equation (4). ■

Remark 1. It is possible to obtain $F(x)$ by making use of the level crossing technique of Brill and Posner (1977). Let $f(x) = \frac{d}{dx} F(x)$ be the stationary density function of the level of the inventory. Recall that the points where the stock of the inventory reaches α form an embedded renewal process with inter-renewal time T. We define the inter-renewal time as a cycle. Let $D(x)$ be the number of down-crossings to level x during a cycle. For $0 \leq x < \alpha$, $D(x)$ is at most 1 and is equal to 1 if and only if $E^{\lambda} > \frac{\alpha - x}{\mu}$. Hence, $f(x) = \frac{E[D(x)]}{E(T)} \frac{1}{\mu}$, which is equal to the derivative of $F(x)$ in equation (3). For $\alpha \leq x \leq \beta$, $D(x)$ is always equal to 1. Hence, $f(x) = \frac{1}{E(T)} \frac{1}{\mu}$, which is equal to the derivative of $F(x)$ in equation (4).

Remark 2. It is also possible to derive $F(x)$ through the renewal reward theorem [Ross (1996) p. 133]. Define, again, as a cycle, the time interval between two successive points where the stock of the inventory reaches α. Assume that we receive a unit of reward per unit time during a cycle if and only if the level of the inventory is over x. For $0 \leq x < \alpha$,

$$E(\text{reward during a cycle}) = \frac{\beta - \alpha}{\mu} + \int_0^{\frac{\alpha - x}{\mu}} x\lambda e^{-\lambda x} dx + \frac{\alpha - x}{\mu} e^{-\lambda(\frac{\alpha - x}{\mu})}$$

$$= \frac{1}{\lambda} - \frac{1}{\lambda} e^{-(\frac{\alpha - x}{\mu})\lambda} + \frac{\beta - \alpha}{\mu}.$$

Hence, dividing the above equation by $E(T)$ gives $\overline{F}(x)$, which is equivalent to equation (3). For $\alpha \leq x \leq \beta$, E(reward during a cycle) is just $\frac{\beta - x}{\mu}$, and hence, dividing it by $E(T)$ gives $\overline{F}(x)$, which is equivalent to equation (4).

3. Optimization

Lee and Park (1991) studied an optimal policy of the inventory. They assign two costs to the inventory, the cost per unit time of the inventory being empty, C_1 say, and the cost of keeping a unit of the stock per unit time, C_2 say. They showed that the expected total cost during a cycle, the time interval between two successive points where the stock of the inventory reaches α, is given by

$$\hat{C}(\alpha) = C_1 e^{-\alpha\lambda/\mu} + C_2 \left(\frac{\beta^2}{2\mu} - \frac{\alpha^2}{2\mu} + \frac{\alpha}{\lambda} + \frac{\mu}{\lambda^2} e^{-\alpha\lambda/\mu} - \frac{\mu}{\lambda^2}\right).$$

Dividing the above equation by $E(T)$, Lee and Park (1991) calculated the long-run average cost per unit time and showed that there exists a unique value of α which minimizes the long-run average cost.

In this section, we assign two more costs to the inventory. One is the cost per visit of the deliveryman, C_3 say, and the other is the cost of replenishing the stock by a unit amount, C_4 say. After calculating the generalized long-run average cost (including C_3 and C_4) per unit time as a function of not only the threshold α but also the arrival rate λ of the deliveryman, we study an optimal policy with respect to both α and λ.

The expected cost related to the visits of the deliveryman during a cycle is

$$C_3\lambda E(T) = C_3\lambda(\frac{1}{\lambda}+\frac{\beta-\alpha}{\mu}).$$

The expected cost of replenishing the stock during a cycle is $C_4E(\beta-X')$, where X' is the level of the inventory just before the replenishment. Notice that

$$X' = \begin{cases} 0, & \text{if } E^{\lambda} > \frac{\alpha}{\mu}, \\ \alpha-\mu E^{\lambda}, & \text{if } E^{\lambda} \le \frac{\alpha}{\mu}. \end{cases}$$

Hence,

$$E(X') = \int_0^{\frac{\alpha}{\mu}} (\alpha-\mu x)\lambda e^{-\lambda x}dx$$

$$= \alpha + \frac{\mu}{\lambda}(e^{-\alpha\lambda/\mu}-1).$$

Summarizing the foregoing, we can see that the expected total cost during a cycle is given by

$$\hat{C}(\alpha,\lambda) = C_1e^{-\alpha\lambda/\mu}/\lambda + C_2(\frac{\beta^2}{2\mu}-\frac{\alpha^2}{2\mu}+\frac{\alpha}{\lambda}+\frac{\mu}{\lambda^2}e^{-\alpha\lambda/\mu}-\frac{\mu}{\lambda^2})$$

$$+ C_3\lambda(\frac{1}{\lambda}+\frac{\beta-\alpha}{\mu}) + C_4(\beta-\alpha+\frac{\mu}{\lambda}-\frac{\mu}{\lambda}e^{-\alpha\lambda/\mu}).$$

Dividing $\hat{C}(\alpha,\lambda)$ by the expected length of a cycle, $E(T) = \frac{1}{\lambda}+\frac{\beta-\alpha}{\mu}$, we finally obtain the long-run average cost per unit time, $C(\alpha,\lambda) = \dfrac{\hat{C}(\alpha,\lambda)}{E(T)}$.

We, now, illustrate a numerical example to derive an optimal policy of managing the inventory with respect to α and λ.

Example. It is not hard to find numerically the values of α and λ which minimizes $C(\alpha,\lambda)$. Assume that $C_1 = 10$, $C_2 = 0.01$, $C_3 = 0.5$ and $C_4 = 0.1$. In Figure 1, $C(\alpha,\lambda)$ is drawn with respect to α, when $\beta = 100$, $\mu = 10$ and $\lambda = 1$. We can see that $C(\alpha,\lambda)$ is minimized at $\alpha = 30$. In Figure 2, $C(\alpha,\lambda)$ is drawn with respect to λ, when $\beta = 100$, $\mu = 10$, $\alpha = 30$. It can be seen that $C(\alpha,\lambda)$ is minimized at $\lambda = 0.8$. All figures are drawn by MATLAB.

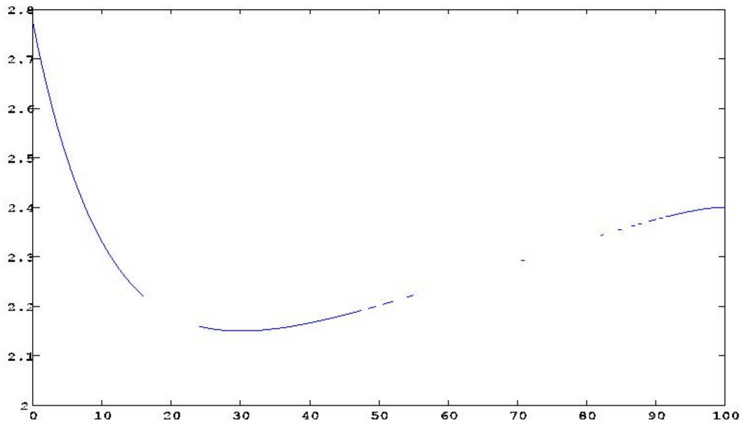

Figure 1. $C(\alpha,\lambda)$ with respect to α

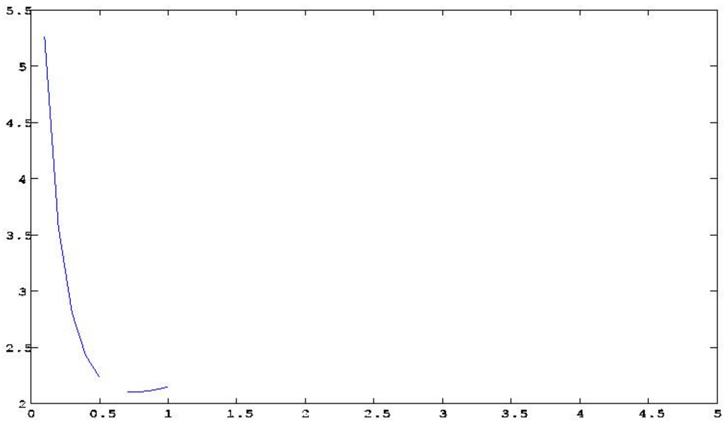

Figure 2. $C(\alpha,\lambda)$ with respect to λ

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