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MYLLER CONFIGURATIONS IN FINSLER SPACES. APPLICATIONS TO THE STUDY OF SUBSPACES AND OF TORSE FORMING VECTOR FIELDS

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ABSTRACT. In this paper we define a Myller configuration in a Finsler space and use some special configurations to obtain results about Finsler subspaces. Let $F^n = (M, F)$ be a Finsler space, with M a real, differentiable manifold of dimension n. Using the pull back bundle $(\pi^*TM, \tilde{\pi}, \widetilde{TM})$ of the tangent bundle (TM, π, M) by the mapping $\tilde{\pi} = \pi/\widetilde{TM}$ and the Cartan Finsler connection of a Finsler space, we obtain an orthonormal frame of sections of π^*TM along a regular curve in $\widetilde{TM}$ and a system of invariants, geometrically associated to the Myller configuration. The fundamental equations are written in a very simple form and we prove a fundamental theorem. Important lines in a Finsler subspace are defined like special lines in a Myller configuration, geometrically associated to the subspace: auto parallels, lines of curvature, asymptotes. Torse forming vector fields with respect to the Cartan Finsler connection are characterized by means of the invariants of the Frenet frame of a versor field along a curve, and the new notion of torse forming vector fields in the sense of Myller is introduced. The particular cases of concurrence and parallelism in the sense of Myller are completely studied, for vector fields from the distribution T^m of the Myller configuration and also from the normal distribution T^p.

1. Introduction

Myller configurations appeared for the first time in the Euclidean space E^3, in the papers of A. Myller [16, 17]. The theory was extended by R. Miron in [13]. A Myller configuration in E^3 is a triplet formed by a regular curve C in E^3, a family of planes $\Pi : C(s) \to \Pi(C(s))$ and a versor field along C, $C(s) \to \xi(C(s)) \in \Pi(C(s))$, differentiable of class C^∞. The fundamental equations of ξ in this configuration are determined and a fundamental theorem is formulated. The main applications are to the study of torse forming versor and vector fields along C, in the sense of Myller (with the particular cases of

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concurrent and parallel versor fields) and to the theory of surfaces. R. Miron generalized the notion of Myller configuration for Riemann spaces and applied it to Riemann subspaces [12]. Khu Quoc Anh was the first who introduced Myller configurations in Finsler spaces [8].

In this paper we indicate a simpler way to construct Myller configurations in a Finsler space and we present their principal applications .

Section 2 is devoted to some preliminary aspects of Finsler geometry. We define all the geometric objects needed for our study, objects from the geometry of the pull back bundle $(\pi^*TM, \tilde{\pi}, \widetilde{TM})$, [1, 2, 20]. From all the known Finsler connections we choose the Cartan Finsler connection.

In Section 3 we construct the Frenet frame of a Finsler versor field along a curve in $\widetilde{TM}$, introduce torse forming versor fields with respect to the Cartan connection and characterize them by means of the invariants of this versor field. Ricci identities are written for all the particular cases (concircular, concurrent, parallel versor fields).

In Section 4 we define Myller configurations in Finsler spaces: $\mathcal{M}(\tilde{C}, \bar{\xi}_1, \bar{T}^m)$, formed by $\tilde{C}$, a regular curve on $\widetilde{TM}$, differentiable of class C^∞, a regular distribution of class C^∞ and dimension m, restricted to $\tilde{C}$: $\bar{T}^m$: $\tilde{C}(s) \to \bar{T}^m(\tilde{C}(s)) \subset \pi^*_{\tilde{C}(s)}TM$ and $(\tilde{C}, \bar{\xi}_1)$ a $\tilde{\pi}$-versor field from the given distribution: $\bar{\xi}_1(\tilde{C}(s)) \in \bar{T}^m(\tilde{C}(s))$, $\forall s \in I$. We obtain an orthonormal frame of sections of π^*TM along C,

$$\Re_{\mathcal{M}} = \{\bar{\eta}_1, \ldots, \bar{\eta}_m, \bar{n}_1, \ldots, \bar{n}_p\}, \ \bar{\eta}_1 = \bar{\xi}_1$$

and a system of invariants, geometrically associated to the Myller configuration. The fundamental equations are written in a very simple form and we prove a fundamental theorem. In Section 4.3 is presented the theory of tangent Myller configurations. For a more detailed study, see [5].

In Section 4.2 we introduce a new notion, torse forming vector fields in the sense of Myller, with respect to the Cartan connection. Some particular type of torse forming versor and vector fields are studied using the geometric invariants of the Myller configurations. Their principal applications are presented in the Section 5.4, where torse forming vector fields tangent or normal to a Finsler subspace, with respect to the induced connections are torse forming vector fields in the sense of Myller in a special Myller configuration, geometrically associated to the subspace.

Section 5 is dedicated to the study of Finsler subspaces. For the case of Finsler hypersurfaces, please see [6]. For any subspace of a Finsler space one can construct a tangent Myller configuration, and all the important lines in this configuration become important lines in the Finsler subspace: autoparallels, lines of curvature, asymptotes. We reobtain known results of the theory of Finsler subspaces in a surprising knew way. We present in an invariant forme all the results, but the study in local coordinates can also be developed using [1, 2, 14].

2. Preliminaries

Let M be a n dimensional, real, differentiable manifold of class C^∞ and $(U, \varphi, \mathbb{R}^n)$ a local chart on M. We denote by $(x^1, \dots, x^n) = \varphi(x)$, $x \in U$, the local coordinates of a point $x \in U$.

Let (TM, π, M) be the tangent bundle of M. The local coordinates induced on TM by those on M are denoted by (x^i, y^i).

Let $d\pi : TTM \to TM$ be the tangent application of the projection $\pi : TM \to M$. We consider $\mathrm{Ker}(d\pi_u) = \{z \in T_uTM \mid d\pi_u(z) = 0\}$ and $VTM = \bigcup_{u \in TM} \mathrm{Ker}(d\pi_u)$. $(VTM, d\pi \mid_{VTM}, TM)$ is a vector subbundle of the tangent bundle, called the vertical bundle, with the local fibers isomorphic to $\mathbb{R}^n$.

Let $(\widetilde{TM}, \tilde{\pi}, M)$ be the subbundle of non vanishing tangent vectors on M, with $\widetilde{TM} = TM \setminus \{0\}$, $\tilde{\pi} = \pi \mid_{\widetilde{TM}}$ and $(\pi^*TM, \pi^*, \widetilde{TM})$ the pull back bundle of (TM, π, M) by the mapping $\tilde{\pi} : \widetilde{TM} \to M$:

$$(1) \qquad \pi^*TM = \{(\tilde{x}, u) \in \widetilde{TM} \times TM \mid \tilde{\pi}(\tilde{x}) = \pi(u)\},$$

$$\pi^* : \pi^*TM \to \widetilde{TM},\ \pi^*(\tilde{x}, u) = \tilde{x},\ \forall (\tilde{x}, u) \in \pi^*TM.$$

The fiber $(\pi^*TM)_{\tilde{x}} = \{\tilde{x}\} \times T_{\tilde{\pi}(\tilde{x})}M$, $\tilde{x} \in \widetilde{TM}$, is isomorphic to $T_{\tilde{\pi}(\tilde{x})}M$.

The coordinate functions induced on π^*TM by those on M are denoted by $(\bar{x}^i)_{i \in \overline{1,n}}$.

We also denote by $\Gamma(U, E)$ the $\mathcal{F}(U)$ module of differentiable sections (of class C^∞) of a vector bundle (E, p, M), ($U \subset M$ an open set), and particularly $\Gamma(U, TM) = \chi(U)$ is the $\mathcal{F}(U)$ module of differentiable vector fields on U. $\mathcal{F}(U)$ is the ring of real functions, differentiable of class C^∞ on $U \subset M$.

Any section of the tangent bundle determines a section of the pull back bundle [20]: for any $X \in \chi(U)$, let $\bar{X} \in \Gamma(\tilde{\pi}^{-1}(U), \pi^*TM)$ be defined by:

$$(2) \qquad \bar{X}(\tilde{x}) = (\tilde{x}, X(\tilde{\pi}(\tilde{x}))),\ \forall \tilde{x} \in \tilde{\pi}^{-1}(U).$$

$\bar{X}$ is the lift of the vector field X on M to a local section of π^*TM and is called a $\tilde{\pi}$-vector field. In particular,

$$\overline{\frac{\partial}{\partial x^i}}(\tilde{x}) = (\tilde{x}, \frac{\partial}{\partial x^i}(\tilde{\pi}(\tilde{x}))),$$

in the local system of coordinates (U, x^i) on M. So, $\{\overline{\frac{\partial}{\partial x^i}}\}_{i \in \overline{1,n}}$ is a local basis in $\Gamma(\tilde{\pi}^{-1}(U), \pi^*TM)$.

We suppose that $F^n = (M, F(x, y))$ is a Finsler space.

Definition 2.1 ([1]). A Finsler space is a pair $F^n = (M, F(x, y))$, where $F : TM \to \mathbb{R}$ is a scalar function such that:

1) $F(x, y)$ is differentiable on $\widetilde{TM}$ and continuous on the null section;
2) $F(x, y) > 0$ on TM;
3) F is positively homogeneous of order 1 on the fibers of the tangent bundle:

$$F(x, \lambda y) = \lambda F(x, y),\ \forall \lambda > 0.$$

4) The distinguished tensor field

$$g_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}(x,y)$$

is positively defined. It implies that $\mathrm{Ker}[g_{ij}(x,y)] = n$ on $\widetilde{TM}$.

The fundamental tensor $g_{ij}(x,y)$ determines a Riemannian natural metric on π^*TM, denoted by $\bar{g}$,

$$\bar{g} = g_{ij} d\overline{x}^i \otimes d\overline{x}^j \in \Gamma(\otimes^2 \pi^* T^* M),$$

where $\{d\overline{x}^i\}$ is the basis in π^*T^*M, dual to $\{\overline{\frac{\partial}{\partial x^i}}\}$.

Let $\bar{C} = C_{ijk} d\overline{x}^i \otimes d\overline{x}^j \otimes d\overline{x}^k \in \Gamma(\otimes^3 \pi^* T^* M)$ be the Cartan tensor field, $C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}$, and $\bar{\alpha}$ the Liouville $\tilde{\pi}$-vector field (the intrinsic Finsler vector field): $\bar{\alpha}(\tilde{x}) = (\tilde{x}, \tilde{x})$, $\forall \tilde{x} \in \tilde{\pi}^{-1}(U)$, $\bar{\alpha} = y^i \overline{\frac{\partial}{\partial x^i}} \in \Gamma(\pi^*TM)$, $\bar{g}(\bar{\alpha}, \bar{\alpha}) = F^2$.

Finsler connections (vectorial connection on $(\pi^*TM, \pi^*, \widetilde{TM})$) play a significant role in this paper. A Finsler connection is a pair $FC = (HTM, \nabla)$ formed by a nonlinear connection on TM and a linear connection on $(\pi^*TM, \pi^*, \widetilde{TM})$.

Remark 2.2. A. Bejancu [2] introduced Finsler connection like vectorial connection on the vertical subbundle.

The nonlinear connection on TM represents a distribution HTM complementary to VTM in TTM:

$$TTM = HTM \oplus VTM\,. \tag{3}$$

A local basis in $\Gamma(HTM)$ is $\{\frac{\delta}{\delta x^i}\}_{i \in \overline{1,n}}$, with $\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j(x,y)\frac{\partial}{\partial y^j}$, where N_j^i are the local coefficients of the nonlinear connection. $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}_{i \in \overline{1,n}}$ is a local basis in $\chi(TM)$, adapted to the decomposition (3). The Sasaki lift of the fundamental d-tensor field to $\otimes^2 T^*(\widetilde{TM})$ is:

$$G(x,y) = g_{ij}(x,y) dx^i \otimes dx^j + g_{ij}(x,y) \delta y^i \otimes \delta y^j\,. \tag{4}$$

Let $h : \chi(TM) \to \Gamma(HTM)$, $v : \chi(TM) \to \Gamma(VTM)$ be the horizontal and vertical projectors defined by the given nonlinear connection.

We consider the next morphism of vector bundle [20]: $\rho : T(\widetilde{TM}) \to \pi^*TM$,

$$\rho(\tilde{X}) = (\pi_{\widetilde{TM}}(\tilde{X}),\, d\tilde{\pi}(\tilde{X}))\,. \tag{5}$$

It induces a morphism between the $\mathcal{F}(\widetilde{TM})$-modules of sections, denoted also by ρ: $\rho : \chi(\widetilde{TM}) \to \Gamma(\pi^*TM)$, $\rho(\tilde{X})(u) = \rho(\tilde{X}(u))$, $\forall u \in \widetilde{TM}$, called the lift of a vector field on $\widetilde{TM}$ to a $\tilde{\pi}$-vector field. We observe that $\mathrm{Ker}\rho = V\widetilde{TM}$ and ρ is an epimorphism of vector bundles.

Also, $\rho/_{H\widetilde{TM}}$ is an isomorphism of vector bundles and we define

$$\beta = (\rho/_{H\widetilde{TM}})^{-1}, \quad \beta : \pi^*TM \to H\widetilde{TM}\,.$$

We denote by the same symbol the induced morphism between the modules of sections and we call it the horizontal lift of a $\tilde{\pi}$-vector field.

Let $\gamma : \pi^*TM \to T(\widetilde{TM})$ be defined by $\gamma(\tilde{x}, u)$ = the vector tangent at $\tilde{x}$ to the curve $\sigma(t) = \tilde{x} + tu$ on $\widetilde{TM}$, $\forall(\tilde{x}, u) \in \pi^*TM$. We have $\mathrm{Im}\gamma = V\widetilde{TM}$. We denote also by γ the morphism between the modules of sections: $\gamma : \Gamma(\pi^*TM) \to \Gamma(V\widetilde{TM})$. It is named the vertical lift of a $\tilde{\pi}$-vector field and $\gamma : \pi^*TM \to V\widetilde{TM}$ is an isomorphism of vector bundles.

The next sequence of vector bundle is an exact sequence:

$$0 \to \pi^*TM \xrightarrow{\gamma} T(\widetilde{TM}) \xrightarrow{\rho} \pi^*TM \to 0. \tag{6}$$

We introduce a new morphism of vector bundle: $l : T(\widetilde{TM}) \to \pi^*TM$, $l = \gamma^{-1} \circ v$. It induces a morphism between the modules of sections, denoted also by l.

For any $X \in \chi(\widetilde{TM})$, the operator

$$\begin{aligned} \nabla_X : \Gamma(\pi^*TM) \quad &\to \quad \Gamma(\pi^*TM) \\ Y \quad &\to \quad \nabla_X Y, \forall Y \in \Gamma(\pi^*TM), \end{aligned} \tag{7}$$

named the covariant derivative with respect to X, has the following properties:

$$\nabla_{fX+Y} Z = f\nabla_X Z + \nabla_Y Z,$$

$$\nabla_X (fZ + W) = f\nabla_X Z + X(f)Z + \nabla_X W$$

for any $X, Y \in \chi(\widetilde{TM})$, $Z, W \in \Gamma(\pi^*TM)$, $f \in \mathcal{F}(\widetilde{TM})$.

A Finsler connection on M induces the operators of h and v covariant derivative:

$$S : \Gamma((\pi^*TM)^*) \times \cdots \times \Gamma((\pi^*TM)^*) \times \Gamma(\pi^*TM) \times \cdots \times \Gamma(\pi^*TM) \to \mathcal{F}(TM)$$

$$\nabla^H_{\tilde{X}} S = \nabla_{h\tilde{X}} S, \quad \nabla^V_{\tilde{X}} S = \nabla_{v\tilde{X}} S.$$

We consider the torsion forms defined by the Finsler connection $FC = (HTM, \nabla)$:

$$\tilde{T}(\tilde{X}, \tilde{Y}) = \nabla_{\tilde{X}} \rho(\tilde{Y}) - \nabla_{\tilde{Y}} \rho(\tilde{X}) - \rho[\tilde{X}, \tilde{Y}], \forall \tilde{X}, \tilde{Y} \in \chi(\widetilde{TM}),$$

$$\tilde{T}_1(\tilde{X}, \tilde{Y}) = \nabla_{\tilde{X}} l(\tilde{Y}) - \nabla_{\tilde{Y}} l(\tilde{X}) - l[\tilde{X}, \tilde{Y}], \forall \tilde{X}, \tilde{Y} \in \chi(\widetilde{TM}),$$

and define different types of torsion:

$$T(\bar{X}, \bar{Y}) = \tilde{T}(\gamma\bar{X}, \beta\bar{Y}), \quad A(\bar{X}, \bar{Y}) = \tilde{T}(\beta\bar{X}, \beta\bar{Y}),$$

$$R_1(\bar{X}, \bar{Y}) = \tilde{T}_1(\beta\bar{X}, \beta\bar{Y}), \quad P_1(\bar{X}, \bar{Y}) = \tilde{T}_1(\gamma\bar{X}, \beta\bar{Y}), \quad S_1(\bar{X}, \bar{Y}) = \tilde{T}_1(\gamma\bar{X}, \gamma\bar{Y}).$$

The curvature form of $FC = (HTM, \nabla)$ is the 2-form of curvature of the linear connection ∇:

$$\Omega(\tilde{X}, \tilde{Y}) = \nabla_{\tilde{X}} \circ \nabla_{\tilde{Y}} - \nabla_{\tilde{Y}} \circ \nabla_{\tilde{X}} - \nabla_{[\tilde{X}, \tilde{Y}]}, \forall \tilde{X}, \tilde{Y} \in \chi(\widetilde{TM}).$$

The induced curvature tensor fields are denoted by:

$$R(\bar{X},\bar{Y})\bar{Z} = \Omega(\beta\bar{X},\beta\bar{Y})\bar{Z}, \quad P(\bar{X},\bar{Y})\bar{Z} = \Omega(\gamma\bar{X},\beta\bar{Y})\bar{Z},$$
$$S(\bar{X},\bar{Y})\bar{Z} = \Omega(\gamma\bar{X},\gamma\bar{Y})\bar{Z},$$

and are named the horizontal, the mix and respectively the vertical curvature tensor.

Theorem 2.3. *If $F^n = (M,F)$ is a Finsler space, there is an unique Finsler connection on M, $FC = (HTM,\nabla) = (N^i_j, F^i_{jk}, C^i_{jk})$, which satisfies the next system of axioms:*

(C1) $\nabla^H \bar{\alpha} = 0$;

(C2) $\nabla\bar{g} = 0 \Leftrightarrow \tilde{X}\bar{g}(\bar{Y},\bar{Z}) = \bar{g}(\nabla_{\tilde{X}}\bar{Y},\bar{Z}) + \bar{g}(\bar{Y},\nabla_{\tilde{X}}\bar{Z})$, $\forall\tilde{X} \in \chi(\widetilde{TM})$, $\forall\bar{Y},\bar{Z} \in \Gamma(\pi^*TM)$;

(C3) $A(\bar{X},\bar{Y}) = S_1(\bar{X},\bar{Y}) = 0$, $\forall\bar{X},\bar{Y} \in \Gamma(\pi^*TM)$.

This is the Cartan Finsler connection of $F^n = (M,F)$. We note that HTM is the Cartan nonlinear connection of F^n, with the coefficients $N^i_j = \gamma^i_{j0} - \gamma^k_{00}C^i_{kj}$, where $\gamma^i_{j0} = \gamma^i_{jk}y^k$ and γ^i_{jk} are the symbols of Christoffel of g_{ij}.

Definition 2.4 ([14]). A differentiable curve on M, locally given by $x^i = x^i(t)$, $t \in I$, is a geodesic of the Finsler space F^n if it satisfies the Euler-Lagrange equations. When the curve is parametrised by the arc length parameter, their equation are given by:

$$\frac{d^2x^i}{dt^2} + \gamma^i_{jk}(x,\frac{dx}{dt})\frac{dx^j}{dt}\frac{dx^k}{dt} = 0, \tag{8}$$

where γ^i_{jk} are the Christoffel Symbols of the tensor field g_{ij}. A differentiable curve on M, locally given by $x^i = x^i(t)$, $t \in I$, is a geodesic of the Finsler connection $FC = (N^i_j, F^i_{jk}, C^i_{jk})$ on F^n if its lift to $\widetilde{TM}$,

$$\tilde{C} : x^i = x^i(t),\ y^i = \frac{dx^i}{dt}(t),$$

satisfies

$$\nabla_{\frac{d\tilde{C}}{dt}}\rho(\frac{d\tilde{C}}{dt}) = 0. \tag{9}$$

A differentiable curve on M, locally given by $x^i = x^i(t)$, $t \in I$, is an auto parallel of the nonlinear connection $N = (N^i_j)$ on TM if its lift to $\widetilde{TM}$ is a horizontal curve:

$$\frac{\delta y^i}{dt} = 0,\ y^i = \frac{dx^i}{dt}(t).$$

Theorem 2.5 ([14]). *The geodesics of the Cartan Finsler connection of a Finsler space coincide with the auto parallels of the Cartan nonlinear connection* *and with the geodesics of the Finsler space, parametrised by arc length. Their equations are*

$$\frac{d^2x^i}{dt^2} + N^i_j(x, \frac{dx}{dt})\frac{dx^j}{dt} = 0\,, \tag{10}$$

where (N^i_j) *are the coefficients of the Cartan nonlinear connection.*

Proposition 2.6 ([14]). *The Cartan Finsler connection satisfies the next Ricci identities*:

$$\begin{aligned} X^i{}_{|k|h} - X^i{}_{|h|k} &= X^r R^{\;i}_{r\,kh} - X^i{}_{|r}\, R^r_{kh}\,, \\ X^i{}_{|k}|_h - X^i|_{h|k} &= X^r P^{\;i}_{r\,kh} - X^i{}_{|r}\, C^r_{kh} - X^i|_r\, P^r_{kh}\,, \\ X^i|_k|_h - X^i|_h|_k &= X^r S^{\;i}_{r\,kh}, \end{aligned} \tag{11}$$

where

$$\begin{aligned} R(\overline{\frac{\partial}{\partial x^i}}, \overline{\frac{\partial}{\partial x^j}})\overline{\frac{\partial}{\partial x^k}} &= R^{\;h}_{k\,ji}\overline{\frac{\partial}{\partial x^h}}\,, \\ P(\overline{\frac{\partial}{\partial x^i}}, \overline{\frac{\partial}{\partial x^j}})\overline{\frac{\partial}{\partial x^k}} &= P^{\;h}_{k\,ji}\overline{\frac{\partial}{\partial x^h}}\,, \\ S(\overline{\frac{\partial}{\partial x^i}}, \overline{\frac{\partial}{\partial x^j}})\overline{\frac{\partial}{\partial x^k}} &= S^{\;h}_{k\,ji}\overline{\frac{\partial}{\partial x^h}}\,, \end{aligned} \tag{12}$$

and

$$\begin{aligned} R^{\;h}_{k\,ji} &= \frac{\delta F^h_{kj}}{\delta x^i} - \frac{\delta F^h_{ki}}{\delta x^j} + F^m_{kj}F^h_{mi} - F^m_{ki}F^h_{mj} + C^h_{km}R^m_{ji}\,, \\ P^{\;h}_{k\,ji} &= \frac{\partial F^h_{kj}}{\partial y^i} - \frac{\delta C^h_{ki}}{\delta x^j} + \frac{\partial N^m_j}{\partial y^i}C^h_{km} + F^m_{kj}C^h_{mi} - C^m_{ki}F^h_{mj}\,, \\ S^{\;h}_{k\,ji} &= \frac{\partial C^h_{kj}}{\partial y^i} - \frac{\partial C^h_{ki}}{\partial y^j} + C^m_{kj}C^h_{mi} - C^m_{ki}C^h_{mj}\,. \end{aligned} \tag{13}$$

Remark 2.7. The four known Finsler connections of a Finsler space (Chern, Berwald, Hashiguchi, Cartan) have the same Ricci identities.

3. Torse forming vector fields in F^n

We consider a Finsler space $F^n = (M, F(x,y))$ and the Cartan Finsler connection $FC = (HTM, \nabla)$ of F^n.

Let $\tilde{C} : s \to \tilde{C}(s)$ be a regular curve in $\widetilde{TM}$, locally given by

$$\begin{cases} x^i = x^i(s)\,, \\ y^i = y^i(s)\,,\; i \in \overline{1,n}\,, \end{cases} \tag{14}$$

where s is the arc-length of the projection $\pi \circ \tilde{C} = C \, : \, x^i = x^i(s)$. In other words, $F(x, \frac{dx}{ds}) = 1$, or $\bar{g}(\rho(\frac{d\tilde{C}}{ds}), \rho(\frac{d\tilde{C}}{ds})) = g_{ij}(x, \frac{dx}{ds})\frac{dx^i}{ds}\frac{dx^j}{ds} = 1$. We take a local section $\bar{\xi}_1 \in \Gamma(\pi^* TM)$ and $\bar{\xi}_1 \circ \tilde{C}$ its restriction to $\tilde{C}$, denoted also by $\bar{\xi}_1$. We suppose that

$$\bar{g}(\bar{\xi}_1, \bar{\xi}_1) = 1 \Leftrightarrow g_{ij}\xi_1^i \xi_1^j = 1 \text{ for } \bar{\xi}_1 = \xi_1^i \overline{\frac{\partial}{\partial x^i}} \,. \tag{15}$$

Definition 3.1. A pair $(\tilde{C}, \bar{\xi}_1(s))$, given by (14) and $\bar{\xi}_1 \in \Gamma(\pi^* TM)$ with the property (15) is called a $\tilde{\pi}$-versor field in F^n.

Let $\frac{\nabla}{ds}$ be the operator of covariant differentiation on $\tilde{C}$ induced by the Cartan Finsler connection of F^n. We use the fact that the Cartan Finsler connection is metrical with respect to $\bar{g}$ and that $\bar{\xi}_1$ is a $\tilde{\pi}$-versor field. By induction, we prove:

Theorem 3.2. *Let $(\tilde{C}, \bar{\xi}_1(s))$ be a $\tilde{\pi}$-versor field in F^n. One can associate an orthonormal frame, positively orientated,*

$$\Re = \{\tilde{C}(s), \bar{\xi}_1(s), \bar{\xi}_2(s), \ldots, \bar{\xi}_n(s)\}$$

and a system of invariants

$$k_1(s) > 0\,,\ k_2(s) > 0\,, \ldots, k_{n-2} > 0\,,\ k_{n-1}\,,$$

satisfying the formulae:

$$\begin{aligned} \frac{\nabla \bar{\xi}_1}{ds}(s) &= k_1(s)\bar{\xi}_2(s)\,, \\ \frac{\nabla \bar{\xi}_a}{ds}(s) &= -k_{a-1}(s)\bar{\xi}_{a-1}(s) + k_a(s)\bar{\xi}_{a+1}(s)\,,\ \forall a \in \overline{2, n-1}\,, \\ \frac{\nabla \bar{\xi}_n}{ds}(s) &= -k_{n-1}(s)\bar{\xi}_{n-1}(s)\,. \end{aligned} \tag{16}$$

All of these in the assumptions $k_a > 0\,, \forall a \in \overline{1, n-2}$. The sign of k_{n-1} is chosen such that the frame

$$\Re = \{\bar{\xi}_1, \bar{\xi}_2, \ldots, \bar{\xi}_n\}$$

be positively orientated.

The $\tilde{\pi}$-versor fields of this frame are called the *$\tilde{\pi}$-principal versor fields of* $\bar{\xi}_1$ in F^n, $k_1, \ldots, k_{n-2}$ *the curvatures* of $\bar{\xi}_1$ and k_{n-1} *the torsion* of $\bar{\xi}_1$. $\Re$ is the Frenet frame of the $\tilde{\pi}$-versor field $(\tilde{C}, \bar{\xi}_1)$.

We get some other invariants expressing the tangent vector field along $\tilde{C}$ with respect to $\Re$:

$$\frac{d\tilde{C}}{ds} = \frac{dx^i}{ds}\frac{\delta}{\delta x^i} + \frac{\delta y^i}{ds}\frac{\partial}{\partial y^i} \Rightarrow \rho(\frac{d\tilde{C}}{ds}) = \frac{dx^i}{ds}\overline{\frac{\partial}{\partial x^i}} \,.$$

$$\rho(\frac{d\tilde{C}}{ds}) = a^1(s)\bar{\xi}_1(s) + a^2(s)\bar{\xi}_2(s) + \cdots + a^n(s)\bar{\xi}_n(s)\,, \tag{17}$$

with

$$\sum_{i=1}^{n}(a^i(s))^2 = 1\,.$$

Theorem 3.3. *If we consider some continuous functions of parameter s,*

$$k_a\,,\ a \in \overline{1,n-1},\ k_a > 0\,,\ \forall a \in \overline{1,n-2}\,,\ a^i,\ i \in \overline{1,n}\,,\ \sum_{i=1}^{n}(a^i(s))^2 = 1$$

and

$$\{\bar{\xi}_{01}, \bar{\xi}_{02}, \ldots, \bar{\xi}_{0n}\}$$

an orthonormal frame in $\tilde{x}_0 = (x_0, y_0) \in \widetilde{TM}$, positively orientated, then there exist an unique curve C on M, an unique horizontal curve $\tilde{C}$ on $\widetilde{TM}$, with $\pi \circ \tilde{C} = C$ and an unique $\bar{\pi}$-versor field $\bar{\xi}_1$ along $\tilde{C}$, such that s is the natural parameter of C, the functions k_a, $a \in \overline{1,n-1}$, a^i, $i \in \overline{1,n}$, are the invariants of $(\tilde{C}, \bar{\xi}_1)$ and $\bar{\xi}_a(s_0) = \bar{\xi}_{0a}$, $a \in \overline{1,n}$, $\tilde{C}(s_0) = \tilde{x}_0$, where s_0 is a point of the domain of definition of $\bar{\xi}_1$.

Proof. The system of differential equations (16) has unique solution when initial conditions are given. Let $\Re = \{\bar{\xi}_1, \bar{\xi}_2, \ldots, \bar{\xi}_n\}$ be this solution. Since this frame is orthonormal, positively orientated in $\tilde{x}_0$, we prove that it has the same properties on all its domain of definition. We also verify that $k_1, \ldots, k_{n-1}$ are the invariants of $(\tilde{C}, \bar{\xi}_1(s))$ in F^n.

From the equations (17), in witch we put $\bar{\xi}_1, \bar{\xi}_2, \ldots, \bar{\xi}_n$, the solution of the given system, we determine the curve $C\,:\,x^i = x^i(s)$ such that $C(s_0) = x_0 = (x_0^i)$. Indeed:

$$x^i(s) = \int_{s_0}^{s}(a^1(s)\xi_1^i(s) + \cdots + a^n(s)\xi_n^i(s))ds + x_0^i\,. \tag{18}$$

Since $\bar{g}(\rho(\frac{d\tilde{C}}{ds}), \rho(\frac{d\tilde{C}}{ds})) = \sum_{i=1}^{n}(a^i(s))^2 = 1$, it results that s is the arc-length parameter of C. From $\frac{\delta y^i}{ds} = 0$ we obtain $y^i = y^i(s)$, $y^i(s_0) = y_0$. □

Next we'll define *torse forming vector fields along a curve* in $\widetilde{TM}$.

Let $\tilde{C}\,:\,t \to \tilde{C}(t)$, be a regular curve in $\widetilde{TM}$, locally expressed by

$$\begin{cases} x^i &= x^i(t)\,, \\ y^i &= y^i(t)\,,\ i \in \overline{1,n}\,,\ t \in I\,, \end{cases} \tag{19}$$

and $\bar{X} \in \Gamma(\pi^*TM)$ a $\bar{\pi}$-vector field restricted to $\tilde{C}$.

Let $FC = (HTM, \nabla)$ be a Finsler connection on F^n (not necessarily the Cartan connection $C\Gamma(N)$).

Definition 3.4. $\bar{X}$ is a $\tilde{\pi}$-torse forming vector field along $\tilde{C}$, with respect to the Finsler connection $FC = (HTM, \nabla)$ on F^n, if the $\tilde{\pi}$-vector fields $\frac{\nabla \bar{X}}{dt}$, $\rho(\frac{d\tilde{C}}{dt})$ and $\bar{X}$ are linear dependent along $\tilde{C}$: $\exists\ \alpha, \beta : \widetilde{TM} \to \mathbb{R}$ such that

$$\frac{\nabla \bar{X}}{dt}(t) = \alpha(t)\rho(\frac{d\tilde{C}}{dt}(t)) + \beta(t)\bar{X}(t)\,,\ t \in I\,. \tag{20}$$

We denoted by $\frac{\nabla}{dt}$ the operator of covariant differentiation along $\tilde{C}$ induced by the Finsler connection FC. This notion was studied by many geometers like Alexandru Myller, J. Scouten, K. Yano, [16, 19, 22, 23] and others [3, 11, 18]. We have the next particular cases:

$\bar{X}$ is called

1) a concircular $\tilde{\pi}$-vector field if $\beta(s) = \omega(s)(\frac{d\tilde{C}}{ds})$, with $\omega \in \Lambda^1(\widetilde{TM})$ the gradient of a function on $\widetilde{TM}$;

2) a special concircular $\tilde{\pi}$-vector field if $\beta = 0$;

3) a concurrent $\tilde{\pi}$-vector field if $\alpha = \mathrm{cst}\,, \alpha \neq 0\,, \beta = 0$;

4) a recurrent $\tilde{\pi}$-vector field if $\alpha = 0$;

5) a parallel $\tilde{\pi}$-vector field if $\alpha = 0$ and $\beta = 0$.

Locally, we consider $\bar{X}$ the restriction to $\tilde{C}$ of a $\tilde{\pi}$-vector fields on $\widetilde{TM}$. We need to introduce the next definition:

Definition 3.5. $\bar{X} \in \Gamma(\pi^* TM)$ is a torse forming $\tilde{\pi}$-vector field with respect to the Finsler connection $FC = (HTM, \nabla)$ on F^n if $\exists\, \alpha : \widetilde{TM} \to \mathbb{R}\,, \omega \in \Lambda^1(\widetilde{TM})$ such that

$$\nabla_Y \bar{X} = \alpha\rho(Y) + \omega(Y)\bar{X}\,,\ \ \forall Y \in \chi(\widetilde{TM})\,. \tag{21}$$

Remark 3.6. 1) If $\bar{X}$ is a torse forming $\tilde{\pi}$-vector field, then $f\bar{X}$ is a torse forming vector field, $\forall f : \widetilde{TM} \to \mathbb{R}$. Indeed,

$$\nabla_Y f\bar{X} = (f\alpha)\rho(Y) + (df + f\omega)(Y)\bar{X}\,,\ \ \forall Y \in \chi(\widetilde{TM})\,.$$

So, for $\alpha \neq 0$ one can always choose $\alpha = 1 = \mathrm{cst}$.

2) In [15] R. Misra consider the situation $\bar{X} = X^i(x)\frac{\bar{\partial}}{\partial x^i}$, $\omega = \omega_i(x, y)dx^i$ and prove for the case of the Berwald connection that α is a function of position only and $\frac{\partial \omega_i}{\partial y^j} = \frac{\partial \omega_i}{\partial y^j}$. The result is based on the totally symmetry of the horizontal curvature tensor of the Berwald connection and it is not true for the Cartan connection.

3) M. Matsumoto [9] obtained that the existence of a concurrent global vector field on certain particular Finsler spaces (like Landsberg spaces, 2 dimensional Finsler spaces, C-reductible Finsler spaces) is a sufficient condition for this space to be Riemannian.

4) It has been pointed out in several publications that ω should be the gradient of a function depending only of x^i. In the present paper we'll make the same assumption.

First, we will study the particular case of a torse forming $\tilde{\pi}$-vector field along a given curve on $\widetilde{TM}$, with respect to an arbitrary Finsler connection.

$\bar{X}$ is a torse forming $\tilde{\pi}$-vector field on $\tilde{C}$, with respect to the Finsler connection $FC = (HTM, \nabla)$ of F^n if there exist a function α and a 1-form ω defined along $\tilde{C}$ such that the following relation holds good:

$$\frac{\nabla \bar{X}}{dt}(t) = \alpha(t)\rho(\frac{d\tilde{C}}{dt}(t)) + \omega(\frac{d\tilde{C}}{dt}(t))\bar{X}(t), \forall t \in I. \tag{22}$$

Locally, we have:

Theorem 3.7. *Let $FC = (HTM, \nabla)$ be a Finsler connection in F^n, with*

$$\omega^i_j = F^i_{jk}dx^k + C^i_{jk}\delta y^k \tag{23}$$

the connection 1-forms. Then, $\bar{X} = X^i(x,y)\overline{\frac{\partial}{\partial x^i}}$ is a torse forming vector fields on a given curve $\tilde{C}$, locally expressed by (19), if there exist a real function α and a 1-form ω on $\tilde{C}$, such that the components X^i verify the next system of differential equations:

$$\frac{dX^i}{dt} + X^j\frac{\omega^i_j}{dt} = \alpha(t)\frac{dx^i}{dt} + (\omega_i\frac{dx^i}{dt} + \dot{\omega}_i\frac{\delta y^i}{dt})X^i, \ i \in \overline{1,n}, \tag{24}$$

with

$$\frac{dX^i}{dt} = \frac{dx^k}{dt}\frac{\delta}{\delta x^k} + \frac{\delta y^k}{dt}\frac{\partial}{\partial y^k}$$

and

$$\omega = \omega_i dx^i + \dot{\omega}_i \delta y^i.$$

We obtain immediately a theorem of existence and uniqueness:

Theorem 3.8. *Consider a regular curve $\tilde{C}$ on $\widetilde{TM}$, $\bar{X}_0$ a vector in a point of $\tilde{C}$, a real function α on $\widetilde{TM}$ and a 1-form $\omega \in \Lambda^1(\widetilde{TM})$, restricted to $\tilde{C}$, both differentiable of class C^∞. Locally, there exists an unique $\tilde{\pi}$-torse forming vector field $\bar{X}$ on $\tilde{C}$ which verifies the initial conditions $\bar{X}(s_0) = \bar{X}_0$, with s_0 a point from the domain of definition of $\bar{X}$.*

Remark 3.9. For the case of concurrent $\tilde{\pi}$-vector fields, the system (24) become:

$$\frac{dX^i}{dt} + X^j\frac{\omega^i_j}{dt} = \alpha(t)\frac{dx^i}{dt}, \ i \in \overline{1,n},$$

and for parallel $\tilde{\pi}$-vector fields:

$$\frac{dX^i}{dt} + X^j\frac{\omega^i_j}{dt} = 0, \ i \in \overline{1,n}.$$

Proposition 3.10. *A $\tilde{\pi}$-vector field $\bar{X}$ on $\tilde{C}$ with $\bar{X} = X^i(x,y)\overline{\frac{\partial}{\partial x^i}}$ is a torse forming vector field with respect to a Finsler connection with the coefficients $(N^i_j, F^i_{jk}, C^i_{jk})$, if and only if*

$$X^i{}_{|k} = \alpha\delta^i_k + \omega_k X^i, \ X^i|_k = \dot{\omega}_k X^i, \tag{25}$$

where $\alpha : \widetilde{TM} \to \mathbb{R}$ *and* $\omega = \omega_i dx^i + \dot{\omega}^i \delta y^i$ *is an 1-form on* $\widetilde{TM}$, *both restricted to* $\tilde{C}$.

Proof. The conditions (24) are equivalent with:

$$\frac{dx^k}{dt}\frac{\delta X^i}{\delta x^k} + \frac{\delta y^k}{dt}\frac{\partial X^i}{\partial y^k} + X^j(F^i_{jk}\frac{dx^k}{dt} + C^i_{jk}\frac{\delta y^k}{dt}) = \alpha\frac{dx^i}{dt} + \omega_k X^i \frac{dx^k}{dt} + \dot{\omega}_k X^i \frac{\delta y^k}{dt}$$

$$\Rightarrow \begin{cases} \frac{\delta X^i}{\delta x^k} + F^i_{jk}X^j & = \alpha\delta^i_k + \omega_k X^i\,, \\ \\ \frac{\partial X^i}{\partial y^k} + C^i_{jk}X^j & = \dot{\omega}_k X^i\,. \end{cases}$$

□

We replace the conditions (25) in Ricci's identities of the Finsler connection and obtain:

Theorem 3.11. *A torse forming* $\tilde{\pi}$*-vector field satisfies the next Ricci identities*:

$$(\alpha \mid_h \delta^i_k - \alpha \mid_k \delta^i_h) - (\alpha\omega_h\delta^i_k - \alpha\omega_k\delta^i_h) = (\omega_h \mid_k - \omega_k \mid_h - \dot{\omega}_r R^r_{kh})X^i + X^r R^{\;i}_{r\;kh},$$

$$(\alpha \mid_h - \alpha\dot{\omega}_h)\delta^i_k + \alpha C^i_{kh} = (\dot{\omega}_h \mid_k - \omega_k \mid_h - \omega_r C^r_{kh} - \dot{\omega}_r P^r_{kh})X^i + X^r P^{\;i}_{r\;kh}\,,$$

$$(\dot{\omega}_h \mid_k - \dot{\omega}_k \mid_h)X^i + X^r S^{\;i}_{r\;kh} = 0\,.$$

Let's particularize this result for concircular and respectively concurrent $\tilde{\pi}$-vector fields.

We suppose that $\bar{X} = X^i(x,y)\overline{\frac{\partial}{\partial x^i}}$ is a concircular $\tilde{\pi}$-vector field along $\tilde{C}$. In this case, $\beta(s) = \omega(s)(\frac{d\tilde{C}}{ds})$ with ω the gradient of a function depending only on x^i. Since we have

$$\omega_h \mid_k - \omega_k \mid_h = 0 \text{ and } \dot{\omega}_i = 0\,,\ \forall i,h,k \in \overline{1,n}\,,$$

the Ricci equations become:

Theorem 3.12. *A concircular* $\tilde{\pi}$*-vector field satisfies the next Ricci identities*:

$$\begin{array}{rcl} (\alpha \mid_h \delta^i_k - \alpha \mid_k \delta^i_h) - (\alpha\omega_h\delta^i_k - \alpha\omega_k\delta^i_h) & = & X^r R^{\;i}_{r\;kh}\,, \\ \alpha \mid_h \delta^i_k + \alpha C^i_{kh} & = & (\omega_k \mid_h - \omega_r C^r_{kh})X^i + X^r P^{\;i}_{r\;kh}\,, \\ S^{\;i}_{r\;kh} & = & 0\,. \end{array} \tag{26}$$

For a concurrent vector field one gets:

Theorem 3.13. *A concurrent* $\tilde{\pi}$*-vector field satisfies the next Ricci identities*:

$$\begin{array}{rcl} X^r R^{\;i}_{r\;kh} & = & 0\,, \\ X^r P^{\;i}_{r\;kh} & = & \alpha C^i_{kh}\,, \\ S^{\;i}_{r\;kh} & = & 0\,, \end{array} \tag{27}$$

with α *a constant, nonvanishing function defined on* $\tilde{C}$.

For concurrent vector fields, we reobtain a result of M. Matsumoto:

Theorem 3.14 ([9]). *If $\bar{X} = X^i(x,y)\overline{\frac{\partial}{\partial x^i}}$ is a concurrent $\tilde{\pi}$-vector field on $\tilde{C}$ with respect to the Cartan Finsler connection $C\Gamma(N)$ of a Finsler space F^n, then the next equations hold true:*

(28) $$X^i{}_{|k} = \delta^i_k\,, \quad X^i\,|_k = 0\,,$$

and the components X^i and $X_i = g_{ij}X^j$ are functions only of the variables x^i.

Proof. In the formulae (27) we replace $\alpha = 1, \omega = 0$ and obtain the first two relations like a particular case of the former theorem. We'll use the second Ricci identity to prove that X^i and $X_i = g_{ij}X^j$ depend only of x^i :

$X^r P^{\ i}_{r\,kh} = C^i_{kh} \Rightarrow X^r X^s P_{rskh} = X^s C_{ksh}$. But P_{rskh} is skew-symmetric in r and s, so $X^s C_{ksh} = 0 \Rightarrow X_i C^i_{kh} = 0$.

$g_{ij}\,|_k = 0 \Rightarrow X_i\,|_k = 0 \Leftrightarrow \frac{\partial X_i}{\partial y^k} + X_h C^h_{ik} = 0 \Rightarrow \frac{\partial X_i}{\partial y^k} = 0$.

$X^i\,|_k = 0$ and $X^s C_{ksh} = 0$ imply $\frac{\partial X^i}{\partial y^k} = 0$. □

Next, we'll find necessary and sufficient conditions for a $\tilde{\pi}$-vector fields to be a torse forming vector field with respect to the Cartan connection, using the invariants $k_1, \dots, k_n$. It is a generalization of the work of R. Miron for the Euclidian case ([13]).

Definition 3.15. A $\tilde{\pi}$-versor field $(\tilde{C}, \bar{\xi}_1)$ is a torse forming versor field in F^n, with respect to the Cartan Finsler connection, if there exists a torse forming vector field $(\tilde{C}, \bar{X})$ with $\bar{X}(s) = \lambda(s)\bar{\xi}_1(s)$, $\lambda : \widetilde{TM} \to \mathbb{R}$.

Theorem 3.16. *Let $(\tilde{C}, \bar{\xi}_1)$ be a $\tilde{\pi}$-versor field with $k_1 > 0$.*

a) *$(\tilde{C}, \bar{\xi}_1)$ is a torse forming versor field in F^n $(\alpha \neq 0)$, with respect to the Cartan Finsler connection if and only if the invariants $a^3, a^4, \dots, a^n$ are vanishing and the functions $\lambda(s)$ (from $\bar{X}(s) = \lambda(s)\bar{\xi}_1(s)$) and $\beta(s)$ satisfy the equations:*

$$\frac{d\lambda}{ds} = a^1 + \lambda\beta\,, \quad \lambda k_1 = a^2\,.$$

b) *$(\tilde{C}, \bar{\xi}_1)$ is a concurrent $\tilde{\pi}$-versor field in F^n if and only if*

(29) $$\begin{cases} a^3 & = a^4 = \cdots = a^n = 0\,, \\ \frac{d}{ds}\left(\frac{a^2}{k_1}\right) & = a^1\,. \end{cases}$$

c) *$(\tilde{C}, \bar{\xi}_1)$ is parallel in $F^n \Leftrightarrow k_1(s) = 0$, $\forall s \in I$.*

Proof. a) $(\tilde{C}, \bar{\xi}_1)$ is a torse forming versor field $\Leftrightarrow \exists\,\lambda : \widetilde{TM} \to \mathbb{R}$ and $\omega \in \Lambda^1(\widetilde{TM})$ such that

$$\begin{aligned} \frac{\nabla(\lambda\bar{\xi}_1)}{dt} &= \frac{d\lambda}{dt}\bar{\xi}_1 + \lambda\frac{\nabla\bar{\xi}_1}{dt} \\ &= \rho\left(\frac{d\tilde{C}}{dt}\right) + \omega\left(\frac{d\tilde{C}}{dt}\right)\lambda\bar{\xi}_1\,, \ (\alpha = 1)\,. \end{aligned}$$

Replacing the relations (16) and (17) in this formulae we get that $(\tilde{C}, \bar{\xi}_1)$ is a torse forming versor field $\Leftrightarrow$

$$\begin{cases} a^3 & = a^4 = \cdots = a^n = 0\,, \\ \frac{d\lambda}{ds} & = a^1 + \lambda\beta\,, \\ \lambda k_1 & = a^2\,, \end{cases}$$

where we denoted $\omega(\frac{d\tilde{C}}{dt})(s) = \beta(s)$. Using $a^2 k_1 \neq 0$, we obtain

$$\lambda = \frac{a^2}{k_1} \quad \text{and} \quad \beta = \frac{k_1}{a^2}(\frac{d}{ds}(\frac{a^2}{k_1}) - a^1)\,.$$

So, if $a^3 = a^4 = \cdots = a^n = 0$, one can find the functions λ and β which verify the definition of torse forming vector field. The reciprocal is an evidence. b) c) are proved by analogy. $\square$

The results of Theorem 3.16 and the fundamental Theorem 3.3 can be summarized in the following way:

Theorem 3.17. *Let*

$$k_i\,,\, i \in \overline{1, n-1},\ k_i > 0\,,\ \forall i \in \overline{1, n-2}\,,\ a^1,\, a^2,\, a^3 = a^4 = \cdots = a^n = 0\,,$$
$$(a^1(s))^2 + (a^2(s))^2 = 1\,,$$

be some continuous functions of parameter s, $\{\bar{\xi}_{01}, \bar{\xi}_{02}, \ldots, \bar{\xi}_{0n}\}$ a $\bar{g}$-orthonormal, positive orientated frame and $\lambda_0 \in \mathbb{R}$, then there locally exists an unique torse forming vector field $(\tilde{C}\,, \bar{X} = \lambda(s)\bar{\xi}_1(s))$, such that $\tilde{C}$ is a horizontal curve with s the natural parameter of its projection $d\pi \circ \tilde{C}$, the functions $k_i\,,\, i \in \overline{1, n-1}$, a^i, $i \in \overline{1, n}$ are exactly the invariants of $(\tilde{C}, \bar{\xi}_1)$ and $\bar{\xi}_a(s_0) = \bar{\xi}_{0a}\,,\ a \in \overline{1, n}\,,\ \tilde{C}(s_0) = \tilde{x}_0\,,\ \lambda(s_0) = \lambda_0\,,\ \bar{X}(s_0) = \lambda_0\bar{\xi}_{10}\,.$

One can formulate another theorem of existence and uniqueness:

Theorem 3.18. *Let $(\tilde{C}, \bar{\xi}_1)$ be a $\tilde{\pi}$-versor field in F^n with the invariants verifying the conditions $k_1 > 0$ and $a^3 = a^4 = \cdots = a^n = 0$. We consider $\bar{X}_0 = \lambda_0\bar{\xi}_0$, with $\bar{\xi}_0$ a versor, in a point of $\tilde{C}$, corresponding to $s = s_0 \in I$. Then there exists an unique torse forming vector field $(\tilde{C}, \bar{X})$, with $\bar{X}(s) = \lambda(s)\bar{\xi}_1(s)$ such that $\bar{X}(s) = \bar{X}_0$.*

Similar theorems can be formulate for concurrent/parallel versor fields in F^n.

Next the $\tilde{\pi}$-torse forming vector fields in F^n are characterized by means of the above invariants. We consider a $\tilde{\pi}$-vector field along $\tilde{C}$ and we'll decompose it in the frame $\Re$. We'll find necessary and sufficient conditions for it to be of torse forming type.

Theorem 3.19. *A $\tilde{\pi}$-vector field along $\tilde{C}$, $\bar{X} = X^1\bar{\xi}_1 + X^2\bar{\xi}_2 + \cdots + X^n\bar{\xi}_n$, is torse forming in F^n with respect to the Cartan Finsler connection $\Leftrightarrow$ its components in the frame $\Re$ verify the next system of differential equations:*

$$\frac{dX^i}{ds} - \beta X^i + k_{i-1}X^{i-1} - k_i X^{i+1} = \alpha a^i\,,\ i \in \overline{1,n}\,,\ k_0 = k_n = 0\,, \tag{30}$$

with $\alpha, \beta : \widetilde{TM} \to \mathbb{R}$ differentiable of class C^∞.

If initial conditions are given, the former system has unique solution. Then, one can formulate a theorem of existence and uniqueness:

Theorem 3.20. *Let $\tilde{C}$ be a curve on $\widetilde{TM}$, $\alpha, \beta : \widetilde{TM} \to \mathbb{R}$ some differentiable function restricted to $\tilde{C}$ and $\bar{X}_0$ a vector in a point of the curve corresponding to $s = s_0$. Then there exists an unique $\tilde{\pi}$-torse forming vector field $\bar{X}$, such that the functions from the definition* (3.4) *are precisely α and β and that $\bar{X}(s_0) = \bar{X}_0$.*

In particular, we get:

Theorem 3.21. a) $\bar{X} = X^1\bar{\xi}_1 + X^2\bar{\xi}_2 + \cdots + X^n\bar{\xi}_n$ *is concurrent in F^n* ($\alpha = 1$) $\Leftrightarrow$

$$\frac{dX^i}{ds} + k_{i-1}X^{i-1} - k_i X^{i+1} = a^i\,,\ i \in \overline{1,n}\,,\ k_0 = k_n = 0\,. \tag{31}$$

b) $\bar{X} = X^1\bar{\xi}_1 + X^2\bar{\xi}_2 + \cdots + X^n\bar{\xi}_n$, *is parallel in F^n* $\Leftrightarrow$

$$\frac{dX^i}{ds} + k_{i-1}X^{i-1} - k_i X^{i+1} = 0\,,\ i \in \overline{1,n}\,,\ k_0 = k_n = 0\,. \tag{32}$$

One can formulate theorems of existence and uniqueness for these cases.

4. Myller configurations

Important tools for studying surfaces and torse forming vector fields in the Euclidian space, Myller configurations seem to be suitable for Finsler spaces too.

So, we introduce in a new way the Myller configuration in Finsler spaces. In [4] the author tried to generalize this theory also for the case of $\widetilde{TM}$, with M endowed with a Finsler structure.

A similar theory for maximal Myller configurations $\mathcal{M}(\tilde{C}, \bar{\xi}_1, \bar{T}^{n-1})$ is presented in [6].

4.1. The fundamental equations of a Myller configuration

Let $\tilde{C}$ be a regular curve on $\widetilde{TM}$, differentiable of class C^∞, locally given by

$$\begin{cases} x^i = x^i(s)\,, \\ y^i = y^i(s)\,,\ s \in I\,, \end{cases} \tag{33}$$

with s the arc-length parameter of the projection $C = \pi \circ \tilde{C}$, expressed by $C : x^i = x^i(s)$.

We also consider a regular distribution of class C^∞ and dimension m, $1 < m < n-1$, restricted to $\tilde{C}$:

$$\bar{T}^m : \tilde{C}(s) \to \bar{T}^m(\tilde{C}(s)) \subset \pi^*_{\tilde{C}(s)}TM \tag{34}$$

and $(\tilde{C}, \bar{\xi}_1)$ a $\bar{\pi}$-versor field from the given distribution:

$$\bar{\xi}_1(\tilde{C}(s)) \in \bar{T}^m(\tilde{C}(s)),\ \forall s \in I,$$

$$\bar{\xi}_1 = \xi_1^i \frac{\bar{\partial}}{\partial x^i}, \quad \bar{g}(\bar{\xi}_1, \bar{\xi}_1) = g_{ij}\xi_1^i \xi_1^j = 1. \tag{35}$$

So, we obtain the triplet $\mathcal{M}(\tilde{C}, \bar{\xi}_1, \bar{T}^m)$ and we call it a *Myller configuration* in the Finsler space F^n.

Will determine a complete set of invariants and a moving frame geometrically associated to this Myller configuration, and will formulate a fundamental theorem.

Let $FC = (HTM, \nabla) = (N^i_j, F^i_{jk}, C^i_{jk})$ be the Cartan Finsler connection of F^n and $\frac{\nabla}{ds}$ the operator of covariant differentiation along $\tilde{C}$.

For any $s \in I$, we consider the orthogonal complement (with respect to $\bar{g}(\tilde{C}(s))$) of the linear subspace $\bar{T}^m(\tilde{C}(s))$ in $\pi^*_{\tilde{C}(s)}TM$, and we denote it

$$\bar{T}^p(\tilde{C}(s)), \quad p = n - m.$$

We decompose $\frac{\nabla \bar{\xi}_1}{ds}(\tilde{C}(s))$ in $\bar{T}^m(\tilde{C}(s)) \oplus \bar{T}^p(\tilde{C}(s))$, putting in evidence the lengths and the versors of the sections:

$$\frac{\nabla \bar{\xi}_1}{ds}(\tilde{C}(s)) = G_1(\tilde{C}(s))\bar{\eta}_2(\tilde{C}(s)) + N_{11}(\tilde{C}(s))\bar{n}_1(\tilde{C}(s)),\ s \in I, \tag{36}$$

$$\bar{g}(\bar{\eta}_2, \bar{\eta}_2) = \bar{g}(\bar{n}_1, \bar{n}_1) = 1,\ \bar{g}(\bar{\xi}_1, \bar{\eta}_2) = \bar{g}(\bar{\xi}_1, \bar{n}_1) = \bar{g}(\bar{\eta}_2, \bar{n}_1) = 0,$$
$$G_1 \geqslant 0 \text{ for } m \geqslant 3 \text{ and } N_{11} \geqslant 0 \text{ for } p \geqslant 2.$$

We remark that $\bar{\eta}_2, \bar{n}_1$ have geometric character and G_1, N_{11} are invariants. If $G_1 \neq 0$, we decompose $\frac{\nabla \bar{\eta}_2}{ds}(\tilde{C}(s))$ with respect to $\bar{\xi}_1(\tilde{C}(s))$, $\bar{\eta}_2(\tilde{C}(s))$ and to their orthogonal complement in $\bar{T}^m(\tilde{C}(s))$, also with respect to $\bar{n}_1(\tilde{C}(s))$ and its orthogonal complement in $\bar{T}^p(\tilde{C}(s))$. Using the fact that $\bar{\xi}_1, \bar{\eta}_2, \bar{n}_1$ are orthogonal unitary sections and that the Cartan connection is metrical with respect to $\bar{g}$, we obtain the unique decomposition:

$$\frac{\nabla \bar{\eta}_2}{ds} = -G_1\bar{\xi}_1 + G_2\bar{\eta}_3 + N_{21}\bar{n}_1 + N_{22}\bar{n}_2, \tag{37}$$

$$G_2 \geqslant 0\,(m \geqslant 4),\ N_{22} \geqslant 0\,(p \geqslant 3),$$

$(\bar{\xi}_1, \bar{\eta}_2, \bar{\eta}_3)$ an orthogonal triplet in $\bar{T}^m$ and $(\bar{n}_1, \bar{n}_2)$ an orthogonal pair in $\bar{T}^p$.

If $G_2 \neq 0$ the method can be continued and, by induction, one can prove the next result:

Theorem 4.1. *The fundamental formulae of the $\bar{\pi}$-versor field $(\tilde{C}, \bar{\xi}_1)$ in the Myller configuration $\mathcal{M}(\tilde{C}, \bar{\xi}_1, \bar{T}^m)$ of the Finsler space F^n are:*

$$\frac{\nabla \bar{\eta}_a}{ds} = -G_{a-1}\bar{\eta}_{a-1} + G_a\bar{\eta}_{a+1} + \sum_{\beta=1}^{p} N_{a\beta}\bar{n}_\beta\,, \quad a \in \overline{1,m}\,,\ \bar{\eta}_1 = \bar{\xi}_1\,, \tag{38}$$

$$\frac{\nabla \bar{n}_\alpha}{ds} = -\sum_{b=1}^{m} N_{b\alpha}\bar{\eta}_b + \sum_{\beta=1}^{p} M_{\alpha\beta}\bar{n}_\beta\,, \quad \alpha \in \overline{1,p}\,, \tag{39}$$

and its invariants verify:
for $p \leqslant m$:

$$\begin{aligned} G_a > 0\,, &\quad a \in \overline{1,m-2}\,, \quad G_0 = G_m = 0\,,\\ N_{\alpha\alpha} > 0\,, &\quad \alpha \in \overline{1,p-1}\,, \quad N_{a\beta} = 0\,,\ a < \beta\,,\\ M_{\alpha\beta} + M_{\beta\alpha} = 0\,, & \end{aligned} \tag{40}$$

for $p > m$:

$$\begin{array}{lll} G_a > 0\,, & a \in \overline{1,m-1}\,, & G_0 = G_m = 0\,,\\ N_{\alpha\alpha} > 0\,, & \alpha \in \overline{1,m}\,, & N_{a\beta} = 0\,,\ a < \beta\,,\\ M_{\alpha\beta} + M_{\beta\alpha} = 0\,, & M_{i,m+i} > 0\,,\ i \in \overline{1,p-m-1}\,, & M_{i,m+j} = 0\,,\ j > i\,. \end{array} \tag{41}$$

One obtaines an orthonormal, positively orientated frame along $\tilde{C}$,

$$\Re_{\mathcal{M}} = \{\bar{\eta}_1, \dots, \bar{\eta}_m, \bar{n}_1, \dots, \bar{n}_p\},\ \bar{\eta}_1 = \bar{\xi}_1\,,$$

and some functions on π^*TM,$G_a, N_{a\beta}, M_{\alpha\beta}$, invariant under coordinates transformations on π^*TM and under changes of natural parameter $s \longrightarrow s + a$ of C. The sections of the frame $\Re_{\mathcal{M}}$ are geometrically associated to $\bar{\xi}_1$ in $\mathcal{M}(\tilde{C}, \bar{\xi}_1, \bar{T}^m)$.

Another invariants are given by the coordinates of $\rho(\frac{d\tilde{C}}{ds})$ in $\Re_{\mathcal{M}}$:

$$\rho(\frac{d\tilde{C}}{ds}) = \bar{\alpha}/_{\tilde{C}} = b^1\bar{\xi}_1 + \cdots + b^n\bar{n}_p\,, \tag{42}$$

$$\sum_{i=1}^{n}(b^i)^2 = 1\,. \tag{43}$$

Definition 4.2. We call $\bar{\eta}_a$ the geodesic $\bar{\pi}$-versor field of rank a of $\bar{\xi}_1$ in $\mathcal{M}(\tilde{C}, \bar{\xi}_1, \bar{T}^m)$, span$\{\bar{\eta}_1, \dots, \bar{\eta}_a\}$-the geodesic space of rank a, $\bar{n}_\alpha$-the $\bar{\pi}$-normal versor field of rank α, span$\{\bar{n}_1, \dots, \bar{n}_\alpha\}$-the normal space of rank α.

The invariants of the Myller configurations are called: G_a-the geodesic curvature of rank a, N_{11}-the normal curvature of the $\bar{\pi}$-versor field $\bar{\xi}_1$ in $\mathcal{M}(\tilde{C}, \bar{\xi}_1, \bar{T}^m)$.

Theorem 4.3 (fundamental). *Let*

$$G_a, N_{a\beta}, M_{\alpha\beta}\,,\; a \in \overline{1,m}\,,\; \alpha,\beta \in \overline{1,p}\,,\; b^1,\dots,b^n,$$

be some continuous functions of parameter $s \in I$, satisfying the conditions (43), (40) or (41) and let

$$\Re_0 = \{\bar{\eta}_{01},\dots,\bar{\eta}_{0m},\bar{n}_{01},\dots,\bar{n}_{0p}\}$$

be a $\bar{g}$-orthonormal, positively orientated frame in $\pi^*_{\tilde{x}_0}TM$, $\tilde{x}_0 \in \widetilde{TM}$. Then, there exists in a neighborhood of $\tilde{x}_0$ an unique curve $C : x^i = x^i(s)$ on M, such that s is its arc-length parameter, there exists an unique horizontal curve $\tilde{C}$ on $\widetilde{TM}$ with $\pi \circ \tilde{C} = C$, there exist an unique regular distribution $\bar{T}^m$ of dimension m restricted to $\tilde{C}$ and an unique $\bar{\pi}$-versor field $\bar{\xi}_1$ from this distribution, such that the invariants of $\bar{\xi}_1$ in $\mathcal{M}(\tilde{C},\bar{\xi}_1,\bar{T}^m)$ are exactly the given functions $G_a, N_{a\beta}, M_{\alpha\beta}$ and the following initial conditions are satisfied:

$$\tilde{C}(s_0) = \tilde{x}_0\,,\; \bar{\eta}_a(s_0) = \bar{\eta}_{0a}\,,\; a \in \overline{1,m}\,,\; \bar{n}_\alpha(s_0) = \bar{n}_{0\alpha}\,, \alpha \in \overline{1,p}\,.$$

Proof. We consider the system of differentiable equations (38) and (39) and suppose that the quantities $G_a, N_{a\beta}, M_{\alpha\beta}$ satisfies the conditions (40) or (41). If initial conditions are given, this system has unique solution. Let

$$\{\bar{\eta}_1,\dots,\bar{\eta}_m,\bar{n}_1,\dots,\bar{n}_p\},\; \bar{\eta}_1 = \bar{\xi}_1$$

be this one. Since in $\tilde{x}_0$ this frame is orthonormal and positive orientated, we can prove that it has these properties on the entire neighborhood of $\tilde{x}_0$.

We define $\bar{T}^m$ by $\bar{T}^m(\tilde{C}(s)) = \operatorname{span}\{\bar{\eta}_1(s),\dots,\bar{\eta}_m(s)\}$.

We introduce the solutions of the system (38) and (39) in the relation (42) and determine the unique curve C that verifies the conditions

$$C(s_0) = (x_0)\,,\; s_0 \in I\,,\; \tilde{x}_0 = (x_0, y_0)\,.$$

From $\frac{\delta y^i}{ds} = 0$ we obtain the unique functions $y^i = y^i(s)$ with $y^i(s_0) = y_0$.

Then we verify that s is the arc-length parameter of C, and prove that the invariants of $\bar{\xi}_1$ in $\mathcal{M}(\tilde{C}\,,\,\bar{\xi}_1\,,\,\bar{T}^m)$ are exactly the functions $G_a, N_{a\beta}, M_{\alpha\beta}$. □

Remark 4.4. If $\bar{\xi}_1 = \bar{\alpha}/_{\tilde{C}} = \rho(\frac{d\tilde{C}}{ds})$, the equations (38) and (39) are the fundamental equations of the curve $\tilde{C}$ in the Myller configuration $\mathcal{M}(\tilde{C}\,,\,\bar{\xi}_1\,,\,\bar{T}^m)$.

Starting from the fundamental equations of a Myller configuration, it is easy to check that the next proposition is true:

Proposition 4.5. *The geodesic curvatures of a $\bar{\pi}$-vector field in a Myller configuration have the next geometrical interpretations:*

$$
\begin{aligned}
&\frac{\nabla \bar{\xi}_1}{ds} \wedge \bar{n}_1 = G_1 \bar{\eta}_2 \wedge \bar{n}_1 ,\\
&\frac{\nabla}{ds}(\bar{\xi}_1 \wedge \bar{\eta}_2 \wedge \cdots \wedge \bar{\eta}_a) \wedge \bar{n}_1 \wedge \cdots \wedge \bar{n}_a\\
&= G_a \bar{\xi}_1 \wedge \bar{\eta}_2 \wedge \cdots \wedge \bar{\eta}_{a-1} \wedge \bar{\eta}_{a+1} \wedge \bar{n}_1 \wedge \cdots \wedge \bar{n}_a ,\ a \leqslant p,\\
&\frac{\nabla}{ds}(\bar{\xi}_1 \wedge \bar{\eta}_2 \wedge \cdots \wedge \bar{\eta}_a) \wedge \bar{n}_1 \wedge \cdots \wedge \bar{n}_p\\
&= G_a \bar{\xi}_1 \wedge \bar{\eta}_2 \wedge \cdots \wedge \bar{\eta}_{a-1} \wedge \bar{\eta}_{a+1} \wedge \bar{n}_1 \wedge \cdots \wedge \bar{n}_p ,\ p+1 \leqslant a \leqslant m-2 .
\end{aligned}
\tag{44}
$$

Next, we determine the relations between different invariants of the same Myller configuration. The results are used to prove theorems of existence and uniqueness for torse forming vector fields in the sense of Myller.

Theorem 4.6 (The relations between two $\bar{\pi}$-versor fields of $\mathcal{M}(\tilde{C}, \bar{\xi}_1, \bar{T}^m)$). *Let $\bar{\xi}_1^*$ be another $\bar{\pi}$-versor field of $\mathcal{M}(\tilde{C}, \bar{\xi}_1, \bar{T}^m)$. It determines an invariant frame*

$$\Re^*_{\mathcal{M}} = \{\bar{\eta}_1^*, \ldots, \bar{\eta}_m^*, \bar{n}_1^*, \ldots, \bar{n}_p^*\},\ \ \bar{\eta}_1^* = \bar{\xi}_1^* ,$$

and the invariants

$$G_a^* ,\ N_{a\beta}^* ,\ M_{\alpha\beta}^* ,\ \ a \in \overline{1,m} ,\ \ \alpha, \beta \in \overline{1,p} ,\ b^{*1}, \ldots, b^{*n},$$

with

$$G_a^* > 0\ \ \forall a \in \overline{1,m} ,\ \ \sum_{i=1}^{n} (b^{*i})^2 = 1 .$$

We suppose that

$$
\begin{aligned}
\bar{\eta}_a &= \sum_{b=1}^{m} \cos\theta_{ab} \bar{\eta}_b^* ,\ \ a \in \overline{1,m} ,\ \ \bar{\eta}_1 = \bar{\xi}_1 ,\ \bar{\eta}_1^* = \bar{\xi}_1^* ,\\
\bar{n}_\alpha &= \sum_{\beta=1}^{p} \cos\omega_{\alpha\beta} \bar{n}_\beta^* ,\ \ \alpha \in \overline{1,p} ,
\end{aligned}
\tag{45}
$$

where the matrices $(\cos\theta_{ab})$, $(\cos\omega_{\alpha\beta})$ are orthogonal. Then:

$$
\begin{aligned}
&- G_{a-1} \cos\theta_{a-1,b} + G_a \cos\theta_{a+1,b}\\
&= \frac{d\cos\theta_{ab}}{ds} - G_b^* \cos\theta_{a,b+1} + G_{b-1}^* \cos\theta_{a,b-1} ,\ a, b \in \overline{1,m} ,\ G_0 = 0,\\
&\sum_{\beta=1}^{p} N_{a\beta} \cos\omega_{\beta\alpha} = \sum_{b=1}^{m} N_{b\alpha}^* \cos\theta_{ab} ,\ \ a \in \overline{1,m} ,\\
&\sum_{\beta=1}^{p} M_{\alpha\beta} \cos\omega_{\beta\gamma} = \frac{d\cos\omega_{\alpha\gamma}}{ds} + \sum_{\beta=1}^{p} M_{\beta\gamma}^* \cos\omega_{\alpha\beta} ,\ \ \alpha, \gamma \in \overline{1,p} .
\end{aligned}
\tag{46}
$$

$$b^{*i} = \sum_{a=1}^{m} b^a \cos\theta_{ai}\,,\ i\in\overline{1,m}\,,$$

$$b^{*m+i} = \sum_{\alpha=1}^{p} b^{m+\alpha} \cos\omega_{\alpha i}\,,\ i\in\overline{1,p}\,.$$

Proof. We use the formulae (45), (38), and (39). □

Corollary 4.7 (a generalization of Bortolotti's result). *The expression*

$$\sum_{a=1}^{m}\sum_{\alpha=1}^{p} N_{a\alpha}^2$$

depends only of the distribution $\bar{T}^m$ and not of the choice of $\bar{\xi}_1$ in $\bar{T}^m$.

Proof. From (46) we get:

$$\sum_{a=1}^{m}\sum_{\alpha=1}^{p} N_{a\alpha}^2 = \sum_{a=1}^{m}\sum_{\alpha=1}^{p} N_{a\alpha}^{*2}\,.$$

□

Corollary 4.8. *The invariant $|N_{a\alpha}|$ has extreme value if*

$$N_{b\beta}=0\ \ \forall b\neq a\,,\ b\in\overline{1,m}\,,\ \forall\beta\neq\alpha\,,\ \beta\in\overline{1,p}\,.$$

Theorem 4.9 (Relations between the invariants k_i, $i\in\overline{1,n-1}$ and G_a, $N_{a\beta}$, $M_{\alpha\beta}$). *Let $\Re_{\mathcal{M}}=\{\bar{\eta}_1,\dots,\bar{\eta}_m,\bar{n}_1,\dots,\bar{n}_p\}$ be the principal versors of $(\tilde{C},\bar{\xi}_1)$ in $\mathcal{M}(\tilde{C}\,,\bar{\xi}_1\,,\bar{T}^m)$ and G_a, $N_{a\beta}$, $M_{\alpha\beta}$ its invariants . We also consider the frame $\Re=\{\bar{\xi}_1,\dots,\bar{\xi}_n\}$ associated to the $\bar{\pi}$-versor field $(\tilde{C},\bar{\xi}_1)$ in F^n and k_i, $i\in\overline{1,n-1}$ its curvatures and torsion. If*

$$\bar{\xi}_i=\sum_{b=1}^{m}\cos\varphi_{ib}\bar{\eta}_b+\sum_{\alpha=1}^{p}\cos\phi_{i\alpha}\bar{n}_\alpha\,, \tag{47}$$

$$\cos\varphi_{1b}=\cos\phi_{1\alpha}=0\,,\ \ \forall b\in\overline{2,m}\,,\ \ \alpha\in\overline{1,p}\,,$$

$$\cos\varphi_{2b}=\cos\phi_{2\alpha}=0\,,\ \ \forall b\in\overline{3,m}\,,\ \ \alpha\in\overline{2,p}\,,\ \ (\bar{\xi}_2\in[\bar{\eta}_2,\bar{n}_1])\,,$$

with the matrix $(\cos\varphi_{ib},\cos\phi_{i\alpha})$ orthogonal and with $\det(\cos\varphi_{ib},\cos\phi_{i\alpha})=1$. *Then, the following relations hold good:*

$$\begin{aligned}&-k_{i-1}\cos\varphi_{i-1,b}+k_i\cos\varphi_{i+1,b}\\ &=\frac{d\cos\varphi_{i,b}}{ds}-G_b\cos\varphi_{i,b+1}+G_{b-1}\cos\varphi_{i,b-1}-\sum_{\alpha=1}^{p}N_{b\alpha}\cos\phi_{i\alpha},\\ &-k_{i-1}\cos\phi_{i-1,\alpha}+k_i\cos\phi_{i+1,\alpha}\\ &=\frac{d\cos\phi_{i\alpha}}{ds}+\sum_{b=1}^{m}N_{b\alpha}\cos\varphi_{i,b}+\sum_{\beta=1}^{p}M_{\beta\alpha}\cos\phi_{i\beta},\\ &i\in\overline{1,n}\,,\ b\in\overline{1,m}\,,\ \alpha\in\overline{1,p}\,,\ k_0=k_n=G_0=G_m=0\,.\end{aligned} \tag{48}$$

Particularly,

$$
(49) \qquad
\begin{aligned}
k_1 \cos\varphi_{22} &= G_1 \,, \\
k_1 \cos\phi_{21} &= N_{11} \,, \\
\cos^2\varphi_{22} + \cos^2\phi_{21} &= 1 \,.
\end{aligned}
$$

The following relations are also verified:

$$
\begin{aligned}
b^i &= a^1 \cos\varphi_{1i} + \cdots + a^n \cos\varphi_{ni} \,,\, i \in \overline{1,m} \,, \\
b^{m+i} &= a^1 \cos\phi_{1i} + \cdots + a^n \cos\phi_{ni} \,,\, i \in \overline{3,p} \,,
\end{aligned}
$$

$$
\begin{aligned}
&\cos\varphi_{1b} = \cos\phi_{1\alpha} = 0 \,,\ \ \forall b \in \overline{2,m} \,,\ \ \alpha \in \overline{1,p} \,, \\
&\cos\varphi_{2b} = \cos\phi_{2\alpha} = 0 \,,\ \ \forall b \in \overline{3,m} \,,\ \ \alpha \in \overline{2,p} \,.
\end{aligned}
$$

The next results are necessary to the study of some special curves in a Myller configuration, in Section 4.3.

Definition 4.10. $\bar{\xi}_1$ is a $\tilde{\pi}$-versor field conjugate of rank 1 with $\rho(\frac{d\tilde{C}}{ds})$ in $\mathcal{M}(\tilde{C}\,,\,\bar{\xi}_1\,,\,\bar{T}^m)$ if

$$
(50) \qquad \bar{g}(\bar{\xi}_1, \frac{\nabla\bar{n}}{ds}) = 0 \,,\ \forall \bar{n} \in \bar{T}^p,\ \bar{g}(\bar{n},\bar{n}) = 1 \Leftrightarrow \frac{\nabla\bar{\xi}_1}{ds} \in \bar{T}^m \,.
$$

$\bar{\xi}_1$ is a $\tilde{\pi}$-versor field conjugate of rank k with $\rho(\frac{d\tilde{C}}{ds})$ in $\mathcal{M}(\tilde{C}\,,\,\bar{\xi}_1\,,\,\bar{T}^m)$ if the geodesic spaces of rank k of $\bar{\xi}_1$ in $\mathcal{M}(\tilde{C}\,,\,\bar{\xi}_1\,,\,\bar{T}^m)$ are orthogonal to $\frac{\nabla\bar{n}}{ds}$, for any normal $\tilde{\pi}$-versor field $\bar{n} \in \bar{T}^p \ \Leftrightarrow \frac{\nabla\bar{\eta}_a}{ds} \in \bar{T}^m$, $\forall a \in \overline{1,k}$.

Proposition 4.11. *$\bar{\xi}_1$ is a $\tilde{\pi}$-versor field conjugate of rank 1 with $\rho(\frac{d\tilde{C}}{ds})$ in $\mathcal{M}(\tilde{C}\,,\,\bar{\xi}_1\,,\,\bar{T}^m)$ if and only if one of the next equivalent relations is satisfied:*

1) $N_{11} = 0$;
2) $k_1 = G_1$ *along* $\tilde{C}$;
3) $\bar{\xi}_2 = \bar{\eta}_2$.

Proof. 1) Supposing

$$
\bar{n} = \sum_{\alpha=1}^{p} \cos\sigma_\alpha \bar{n}_\alpha \,,\ \ \sum_{\alpha=1}^{p} \cos^2\sigma_\alpha = 1 \,,
$$

and using the fundamental equations, we obtain:

$$
\frac{\nabla\bar{n}}{ds} = -\sum_{b=1}^{m}(\sum_{\beta=1}^{p} \cos\sigma_\beta N_{b\beta})\bar{\eta}_b + \sum_{\alpha=1}^{p}(\frac{d\cos\sigma_\alpha}{ds} + \sum_{\beta=1}^{p} \cos\sigma_\beta M_{\beta\alpha})\bar{n}_\alpha \Rightarrow
$$

$$
\bar{g}(\bar{\xi}_1, \frac{\nabla\bar{n}}{ds}) = N_{11}\cos\sigma_1 \,.
$$

2) We use the formulae (49). 3) $\bar{\xi}_2 = \frac{1}{k_1}(G_1\bar{\eta}_2 + N_{11}\bar{n}_1) \Rightarrow k_1 = G_1 \Leftrightarrow N_{11} = 0 \Leftrightarrow \bar{\xi}_2 = \bar{\eta}_2$.

$\bar{\xi}_1$ is a $\bar{\pi}$-versor field conjugate of rank $k > 1$ with $\rho(\frac{d\tilde{C}}{ds})$ in $\mathcal{M}(\tilde{C}, \bar{\xi}_1, \bar{T}^m)$ if and only if one of the next equivalent relations hold good:

1) $N_{i\beta} = 0$, $\forall i \in \overline{1,k}$, $\forall \beta \in \overline{1,p}$;

2) $k_i = G_i \, \forall i \in \overline{1,k}$;

3) $\bar{\xi}_i = \bar{\eta}_i \, \forall i \in \overline{2,k+1}$.

$\bar{g}(\bar{\eta}_i, \frac{\nabla\bar{n}}{ds}) = -\sum_{\beta=1}^{p} N_{i\beta}\cos\sigma_\beta$, $\forall i \in \overline{1,k} \Rightarrow$span$\{\bar{\xi}_1, \ldots, \bar{\eta}_k\}$ is orthogonal on $\frac{\nabla\bar{n}}{ds} \Leftrightarrow N_{i\beta} = 0$, $\forall i \in \overline{1,k}$, $\forall \beta \in \overline{1,p}$.

2) 3) From (48) by induction. □

Remark 4.12. For a more detailed theory see [5].

4.2. Torse forming vector fields in the sense of Myller

Let $\mathcal{M}(\tilde{C}, \bar{\xi}_1, \bar{T}^m)$ be a Myller configuration,

$\Re = \{\bar{\xi}_1, \bar{\eta}_2, \ldots, \bar{\eta}_m, \bar{n}_1, \ldots, \bar{n}_p\}$ the orthonormal frame, positively orientated, geometrically associated to this configuration. We recall that we are working with the Cartan Finsler connection.

The next definitions are natural generalizations of those from the Euclidean case. R. Miron extended the notion of vector field parallel in the sense of Myller to Riemann spaces [12] and Khu Quoc Anh to Finsler spaces [8]. But torse forming vector fields in the sense of Myller (with the particular case of concurrence), appear for the first time in this material.

Definition 4.13. A $\bar{\pi}$-versor field $(\tilde{C}, \bar{\xi}_1)$ is called a torse forming versor field in the sense of Myller if there exist a vector field $\bar{X}(s) = \lambda(s)\bar{\xi}_1(s)$ and α, β some differential, real functions defined on $\tilde{C}$, such that

$$\frac{\nabla\bar{X}}{ds}(s) - \alpha(s)\rho(\frac{d\tilde{C}}{ds}) - \beta(s)\bar{X}(s) \in \bar{T}^p(s), \ s \in I. \tag{51}$$

$(\tilde{C}, \bar{\xi}_1)$ is:

- concurrent in the sense of Myller if $\alpha = \text{cst} \neq 0$ and $\beta = 0$;
- parallel in the sense of Myller if $\alpha = 0$ and $\beta = 0$.

We will find characterizations of torse forming $\bar{\pi}$-versor fields, in the sense of Myller, by means of the invariants of the Myller configuration, and also we'll formulate and prove some theorems of existence and uniqueness.

By some easy calculations we get:

Theorem 4.14. a) $(\tilde{C}, \bar{\xi}_1)$ *with* $G_1 > 0$ *is torse forming in the sense of Myller* $(\alpha \neq 0)$ *if and only if*

$$b^3 = \cdots = b^m = 0. \tag{52}$$

b) $(\tilde{C}, \bar{\xi}_1)$, $G_1 > 0$ *is concurrent in the sense of Myller if and only if*

$$\begin{cases} b^3 & = b^4 = \cdots = b^m = 0\,, \\ \frac{d}{ds}\left(\frac{b^2}{G_1}\right) & = b^1\,. \end{cases} \tag{53}$$

c) $(\tilde{C}, \bar{\xi}_1)$ *este parallel in the sense of Myller if and only if* $G_1 = 0$.

From the fundamental theorem and Theorem 4.15 one obtain:

Theorem 4.15. *Let* G_a, $N_{a\beta}$, $M_{\alpha\beta}$, $a \in \overline{1,m}$, $\alpha, \beta \in \overline{1,p}$, $b^1, \ldots, b^n$, *some continuous functions of parameter s, that satisfies the conditions* $\sum_{i=1}^n (b^i)^2 = 1$, $b^3 = \cdots = b^m = b^{m+2} = \cdots = b^n = 0$, (40) *or* (41) *and*

$$\Re_0 = \{\bar{\eta}_{01}, \ldots, \bar{\eta}_{0m}, \bar{n}_{01}, \ldots, \bar{n}_{0p}\}$$

a $\bar{g}$*-orthonormal, positively orientated frame in* $\pi^*_{\tilde{x}_0}TM$, $\tilde{x}_0 \in \widetilde{TM}$, *and* $\bar{X}_0 = \lambda_0 \bar{\eta}_{01}$ *a vector in* $\tilde{x}_0$. *Then there exists an unique Myller configuration with* $\tilde{C}$ *horizontal curve, such that s is the arc length parameter of the projection of* $\tilde{C}$ *on* M, *there exists an unique* $\bar{\pi}$*-vector field* $\bar{X} = \lambda \bar{\xi}_1$, *torse forming in the sense of Myller in* $\mathcal{M}(\tilde{C}, \bar{\xi}_1, \bar{T}^m)$ $(\alpha \neq 0)$, *which has like invariants in* $\mathcal{M}(\tilde{C}, \bar{\xi}_1, \bar{T}^m)$ *the given functions* G_a, $N_{a\beta}$, $M_{\alpha\beta}$ *and verify the next initial conditions*:

$$\tilde{C}(s_0) = \tilde{x}_0\,,\ \ \bar{\eta}_a(s_0) = \bar{\eta}_{0a}\,,\ \ a \in \overline{1,m}\,,\ \ \bar{n}_\alpha(s_0) = \bar{n}_{0\alpha}\,,\ \ \alpha \in \overline{1,p}\,,\ \ \lambda(s_0) = \lambda_0\,.$$

We can particularize the former results for concurrent and parallel versor fields, but we prefer to formulate different types of theorems of existence and uniqueness.

Theorem 4.16. *Let* $\tilde{C}$ *be a curve on* $\widetilde{TM}$, $\bar{T}^m$ *a regular, differentiable distribution of dimension m, restricted to* $\tilde{C}$, G_{01} *a strictly positive real number and* $\bar{X}_0 \in \bar{T}^m(\tilde{C}(s_0))$, $s_0 \in I$. *Then there exists an unique* $\bar{\pi}$*-vector field* $\bar{X} = \lambda \bar{\xi}_1$, *concurrent in the sense of Myller, such that* $\bar{X}(s_0) = \bar{X}_0$, $\frac{\nabla \bar{X}}{ds}(s_0) = \frac{\nabla \bar{X}_0}{ds}$ *and* $G_1(s_0) = G_{01}$, *where* G_1 *is the geodesic curvature of rank 1 of* $\bar{\xi}_1$.

Proof. Let's take $\bar{\xi}^*_1 \in \bar{T}^m$ and G^*_a, $N^*_{a\alpha}$, $M^*_{\alpha\beta}$ its invariants in $\mathcal{M}'(\tilde{C}, \bar{\xi}^*_1, \bar{T}^m)$. We suppose that $G^*_a > 0$, $a \in \overline{1,m}$ and let $\{\bar{\xi}^*_1, \bar{\eta}^*_2, \ldots, \bar{\eta}^*_m, \bar{n}^*_1, \ldots, \bar{n}^*_p\}$ be the intrinsic orthonormal frame associated to $\bar{\xi}^*_1$. We consider that

$$\bar{\eta}_a = \sum_{b=1}^m \cos\theta_{ab} \bar{\eta}^*_b,\ \ a \in \overline{1,m}\,,\ \ \bar{\eta}_1 = \bar{\xi}_1,\ \bar{\eta}^*_1 = \bar{\xi}^*_1.$$

To determine the $\bar{\pi}$-versor field $\bar{\xi}_1$ means to determine the unknown functions θ_{1a}, $a \in \overline{1,m}$, From (46) we obtain:

$$\begin{aligned} G_1 \cos\theta_{2,b} \ &= \ \frac{d}{ds}(\cos\theta_{1,b}) - G^*_b \cos\theta_{1,b+1} + G^*_{b-1}\cos\theta_{1,b-1}\,, \\ & \quad\ b \in \overline{1,m}\,,\ G_0 = 0\,, \end{aligned} \tag{54}$$

$$b^{*i} = \sum_{\alpha=1}^{p} b^{\alpha} \cos\theta_{\alpha i}\,,\ i \in \overline{1,m}\,. \tag{55}$$

We already saw that $\bar{\xi}_1$ is concurrent in the sense of Myller $\Leftrightarrow$

$$\begin{cases} b^3 & = b^4 = \cdots = b^m = 0\,, \\ \frac{d}{ds}(\frac{b^2}{G_1}) & = b^1\,. \end{cases} \tag{56}$$

In addition we have

$$(b^1)^2 + (b^2)^2 + (b^{m+1})^2 + \cdots + (b^n)^2 = 1\,.$$

$$\frac{d}{ds}(\frac{b^2}{G_1}) = b^1 \Leftrightarrow (G_1)' = \frac{(b^2)'}{b^2}G_1 - \frac{b^1}{b^2}G_1^2\,. \tag{57}$$

Using (57), (54) and

$$\begin{cases} (b^1)^* = & b^1\cos\theta_{11} + b^2\cos\theta_{21}\,, \\ \vdots & \vdots \\ (b^m)^* = & b^1\cos\theta_{1m} + b^2\cos\theta_{2m}\,, \end{cases} \tag{58}$$

we obtain a system of second order differential equations in the unknown functions $\theta_{1,b}\,,\ \theta_{2,b}\,,\ b \in \overline{1,m}$. This system has unique solutions when initial conditions are given. □

Theorem 4.17. *Let $\tilde{C}$ be a curve on $\widetilde{TM}$, $\bar{T}^m$ a regular, differentiable distribution of dimension m, restricted to $\tilde{C}$ and $\bar{X}_0 \in \bar{T}^m(\tilde{C}(s_0))$, $s_0 \in I$. Then there exists an unique $\bar{\pi}$-vector field $\bar{X} = \lambda\bar{\xi}_1$, parallel in the sense of Myller, such that $\bar{X}(s_0) = \bar{X}_0$.*

Proof. We decompose the searched $\bar{\pi}$-vector field with respect to the principal versors of a $\bar{\pi}$-versor field with non vanishing invariants G^*_a. The parallelism of $\bar{X}$ reduces to the vanishing of its first geodesic curvature. Using (54) we get:

$$\frac{d}{ds}(\cos\theta_{1,b}) - G^*_b\cos\theta_{1,b+1} + G^*_{b-1}\cos\theta_{1,b-1} = 0\,,\ b \in \overline{1,m}\,. \tag{59}$$

This system has unique solutions when initial conditions are given. These are assured by $\bar{\eta}_{10} = \sum_{b=1}^{m}\cos\theta_{01b}\bar{\eta}^*_{b0}$. □

Another characterization of the parallelism of a $\bar{\pi}$-versor field in the sense of Myller is obtained using the relations (49):

Proposition 4.18. *If $K_1 > 0$, the $\bar{\pi}$-versor field $\bar{\xi}_1$ is parallel in the sense of Myller $\Leftrightarrow$ $N_{11} = K_1$ along $\tilde{C}$ $\Leftrightarrow$ $\bar{\xi}_2 \in \bar{T}^p$.*

Proof. From $\begin{cases} K_1 \cos\varphi_{22} & = G_1\,, \\ K_1 \cos\phi_{21} & = N_{11}\,, \\ \cos^2\varphi_{22} + \cos^2\phi_{21} & = 1\,, \end{cases}$ it results that $G_1 = 0 \Leftrightarrow N_{11} = K_1$.

$\bar{\xi}_2 = \frac{1}{K_1}\frac{\nabla\bar{\xi}_1}{ds} = \frac{1}{K_1}(G_1\bar{\eta}_2 + N_{11}\bar{n}_1) \Rightarrow G_1 = 0 \Leftrightarrow \bar{\xi}_2 \in \bar{T}^p$. □

Next, we'll study the torse forming $\tilde{\pi}$-vector fields from $\bar{T}^m$.

Definition 4.19. A $\tilde{\pi}$-vector field $\bar{X} \in \bar{T}^m$ is torse forming in the sense of Myller if

$$\frac{\nabla\bar{X}}{ds}(s) = \alpha(s)\rho(\frac{d\tilde{C}}{ds}) + \beta(s)\bar{X}(s) + \gamma(s)\bar{n}(s)\,,\ s \in I\,,$$

where α, β, γ are functions of class C^∞ on the manifold $\widetilde{TM}$, restricted to $\tilde{C}$ and $\bar{n} \in \bar{T}^p$ is a vector field normal to $\mathcal{M}(\tilde{C}, \bar{\xi}_1, \bar{T}^m)$.

Theorem 4.20. *A $\tilde{\pi}$-vector field from the distribution of a Myller configuration, $\bar{X} = X^1\bar{\xi}_1 + \cdots + X^m\bar{\eta}_m \in \bar{T}^m$, is a torse forming vector field in the sense of Myller $\Leftrightarrow$ its components in the intrinsic frame $\Re_{\mathcal{M}}$ associated to the configuration, are the solutions of the next system of differential equations*:

$$\frac{dX^a}{ds} - G_aX^{a+1} + G_{a-1}X^{a-1} = \alpha b^a + \beta X^k\,,\ \ a \in \overline{1,m}\,,\ \ G_0 = G_m = 0\,, \tag{60}$$

with α, β functions of class C^∞ on $\widetilde{TM}$ restricted to $\tilde{C}$.

An immediate consequence (which can be particularized also for concurrent and parallel vector fields) is the next theorem:

Theorem 4.21. *Let $\mathcal{M}(\tilde{C}, \bar{\xi}_1, \bar{T}^m)$ be a Myller configuration, α, β some differentiable functions along $\tilde{C}$ and $\bar{X}_0 \in \bar{T}^m(\tilde{C}(s_0))$. Then there exists an unique $\tilde{\pi}$-vector field $\bar{X} \in \bar{T}^m$, torse forming in the sense of Myller, with $\bar{X}(s_0) = \bar{X}_0$.*

This theorem, for the case of parallelism in the sense of Myller, allow us to introduce *the parallel transport* of $\tilde{\pi}$-vector fields from $\bar{T}^m$ and prove:

Theorem 4.22. *The parallel transport in the sense of Myller of $\tilde{\pi}$-vector fields from $\bar{T}^m$ preserves the length of vectors and the angle between them.*

Proof. $\bar{X} = X^1\bar{\xi}_1 + \cdots + X^m\bar{\eta}_m \in \bar{T}^m$ is parallel in the sense of Myller $\Leftrightarrow$

$$\frac{dX^a}{ds} - G_aX^{a+1} + G_{a-1}X^{a-1} = 0\,,\ \ a \in \overline{1,m}\,,\ \ G_0 = G_m = 0\,.$$

Multiplying these relations respectively with X^a, $a \in \overline{1,m}$, we obtain:

$$\frac{d}{ds}((X^1)^2 + \cdots (X^m)^2) = 0 \ \Leftrightarrow \ \frac{d}{ds}\bar{g}(\bar{X}, \bar{X}) = 0,$$

so $\bar{g}(\bar{X}, \bar{X}) = \text{cst}$. □

We introduce now a more general notion, for the first time in Finsler geometry.

Definition 4.23. The geodesic space of rank k of $\bar{\xi}_1$ in $\mathcal{M}(\tilde{C}, \bar{\xi}_1, \bar{T}^m)$ is parallel in the sense of Myller if

$$\frac{\nabla}{ds}(\bar{\xi}_1 \wedge \bar{\eta}_2 \wedge \cdots \wedge \bar{\eta}_k) \wedge \bar{n}_1 \wedge \cdots \wedge \bar{n}_k = 0 . \tag{61}$$

From the fundamental theorem, a straightforward calculation implies

Proposition 4.24. a) *The parallelism in the sense of Myller of the geodesic space of rank k of $\bar{\xi}_1$ in $\mathcal{M}(\tilde{C}, \bar{\xi}_1, \bar{T}^m)$ is characterized by $G_k = 0$.*

b) *The geodesic space of rank k of $\bar{\xi}_1$ in $\mathcal{M}(\tilde{C}, \bar{\xi}_1, \bar{T}^m)$ is parallel in the sense of Myller if and only if*

$$\begin{aligned} & - G_{a-1} \cos\theta_{a-1,b} + G_a \cos\theta_{a+1,b} \\ &= \frac{d\cos\theta_{ab}}{ds} - G_b^* \cos\theta_{a,b+1} + G_{b-1}^* \cos\theta_{a,b-1},\ b \in \overline{1,m},\ a \in \overline{1,k},\ G_0 = G_k = 0 , \end{aligned}$$

where

$$\bar{\eta}_a = \sum_{b=1}^{m} \cos\theta_{ab} \bar{\eta}_b^*,\ \ a \in \overline{1,m}\,,\ \ \bar{\eta}_1 = \bar{\xi}_1,\ \bar{\eta}_1^* = \bar{\xi}_1^*$$

and $\bar{\xi}_1^* \in \bar{T}^m$ has non vanishing invariants.

This system has unique solution when initial conditions are given, so we can prove a theorem of existence and uniqueness.

Next, we continue with the study of $\bar{\pi}$-torse forming versor/vector fields in the sense of Myller from $\bar{T}^p$.

Definition 4.25. a) A $\bar{\pi}$-vector field $\bar{N} \in \bar{T}^p$, normal to $\mathcal{M}(\tilde{C}, \bar{\xi}_1, \bar{T}^m)$ is torse forming in the sense of Myller if there is some differentiable functions δ, ϵ on $\tilde{C}$, such that

$$\frac{\nabla \bar{N}}{ds} - \delta(s)\rho(\frac{d\tilde{C}}{ds}) - \epsilon(s)\bar{N}(s) \in \bar{T}^m(s)\,,\ \ s \in I\,. \tag{62}$$

b) A $\bar{\pi}$-versor field $\bar{n}_\alpha \in \bar{T}^p$, ($\alpha \in \overline{1,p}$ fixed) normal to $\mathcal{M}(\tilde{C}, \bar{\xi}_1, \bar{T}^m)$, is torse forming in the sense of Myller if there is a vector field $\bar{N}(s) = \lambda(s)\bar{n}_\alpha$ torse forming in the sense of Myller.

The definition is particularized in the same way:

If $\delta = \mathrm{cst} \neq 0$ and $\epsilon = 0$, then $\bar{n}_\alpha \in \bar{T}^p$ is called concurrent in the sense of Myller;

If $\delta = \epsilon = 0$, then $\bar{n}_\alpha \in \bar{T}^p$ is parallel in the sense of Myller.

Theorem 4.26. *Let $\bar{n}_\alpha$ be a principal versor of $\bar{\xi}_1$ in $\mathcal{M}(\tilde{C}, \bar{\xi}_1, \bar{T}^m)$, normal to $\bar{T}^m$, with $\alpha \in \overline{1,p}$ fixed. Then $\bar{n}_\alpha$ is torse forming in the sense of Myller if and only if there exist some functions of class C^∞ on $\widetilde{TM}$, λ, δ, ϵ, which verify the next system of differentiable equations:*

$$\begin{cases} \frac{d\lambda}{ds} - \epsilon\lambda = \delta b^{m+\alpha}\,, \\ \\ \lambda M_{\alpha\beta} = \delta b^{m+\beta}\,,\ \forall \beta \neq \alpha,\ \beta \in \overline{1,p}\,. \end{cases} \tag{63}$$

Proof. We use the fundamental equations (38). □

Theorem 4.27. *We consider a Myller configuration $\mathcal{M}(\tilde{C},\bar{\xi}_1,\bar{T}^m)$, some functions of class C^∞ ϵ, δ on $\tilde{C}$ and a normal $\tilde{\pi}$-vector field $\bar{N}_0 = \lambda_0\bar{n}_0 \in \tilde{C}(s_0)$, with $\bar{g}(\bar{n}_0, \bar{n}_0) = 1$. We suppose that the invariants $M_{\alpha\beta}$ are non vanishing for all $\alpha \neq \beta$ and that*

$$\frac{b^{m+\beta}}{M_{\alpha\beta}} = \frac{b^{m+\gamma}}{M_{\alpha\gamma}}\,,\ \forall\beta,\gamma \in \{1,\ldots,\alpha-1,\alpha+1,\ldots,p\}\,,\ \beta\neq\gamma\,.$$

Then there exists an unique $\tilde{\pi}$-vector field $\bar{N} = \lambda\bar{n}_\alpha \in \bar{T}^p$, torse forming in the sense of Myller, which verifies the initial condition $\bar{N}(s_0) = \bar{N}_0$.

Similar results can be found for concurrent and parallel versor fields normal to the Myller configuration.

Theorem 4.28. *Let $\bar{n}_\alpha$ be a principal versor of $\bar{\xi}_1$ in $\mathcal{M}(\tilde{C},\bar{\xi}_1,\bar{T}^m)$, normal to $\bar{T}^m$, with $\alpha \in \overline{1,p}$ fixed. Then*

a) *$\bar{n}_\alpha$ is concurrent in the sense of Myller if and only if there exists a function of class C^∞ on $\widetilde{TM}$, λ, which verifies the next system of differentiable equations:*

$$\text{(64)}\qquad \begin{cases} \frac{d\lambda}{ds} & = b^{m+\alpha}\,, \\ \lambda M_{\alpha\beta} & = b^{m+\beta}\,,\ \forall\beta\neq\alpha,\ \beta\in\overline{1,p}\,. \end{cases}$$

b) *$\bar{n}_\alpha$ is parallel in the sense of Myller if and only if $M_{\alpha\beta} = 0$, $\forall\beta\in\overline{1,p}$.*

For $\tilde{\pi}$-vector fields $\bar{N}$ we have:

Theorem 4.29. *In the Myller configuration $\mathcal{M}(\tilde{C},\bar{\xi}_1,\bar{T}^m)$ a normal vector field*

$$\bar{N} = Y^1\bar{n}_1 + \cdots + Y^p\bar{n}_p \in \bar{T}^p$$

is torse forming in the sense of Myller if and only if there exist the functions δ, ϵ on $\tilde{C}$ such that $Y^1,\ldots,Y^p$, verify the next system:

$$\text{(65)}\qquad \frac{dY^\alpha}{ds} + \sum_{\beta=1}^{p} M_{\beta\alpha}Y^\beta - \epsilon Y^\alpha = \delta b^{m+\alpha}\,,\ \alpha\in\overline{1,p}\,.$$

A natural consequence is the next result:

Theorem 4.30. *Let $\mathcal{M}(\tilde{C},\bar{\xi}_1,\bar{T}^m)$ be a Myller configuration, δ, ϵ some function on $\tilde{C}$ and $\bar{N}_0 \in \tilde{C}(s_0)$ a normal vector field. Then there exists an unique vector field $\bar{N}\in\bar{T}^p$, torse forming in the sense of Myller, witch verifies $\bar{N}(s_0) = \bar{N}_0$.*

The results is true also for concurrent and parallel normal vector fields. The parallel transport can be introduce also for normal vector fields and it preserves the length of vectors and the angle between them.

4.3. Tangent Myller configurations $\mathcal{M}_t(\tilde{C}, \bar{\xi}_1, \bar{T}^m)$

Tangent Myller configurations are used to the study of Finsler subspaces. A Myller configuration with the property $\rho(\frac{d\tilde{C}}{ds}) \in \bar{T}^m$ is denoted by $\mathcal{M}_t(\tilde{C}, \bar{\xi}_1, \bar{T}^m)$. It is characterized by

$$b^{m+1} = \cdots = b^n = 0\,.$$

We consider even more that $\bar{\xi}_1 = \bar{\alpha}/_{\tilde{C}}$ and apply the whole theory to this particular case.

Indeed, let $C : s \to x^i(s)$ be a regular curve on M, with s the arc-length parameter and let $\tilde{C}$ be the canonic lift of C to $\widetilde{TM}$:

$$\tilde{C} : \begin{cases} x^i &= x^i(s)\,, \\ y^i &= \frac{dx^i}{ds}(s)\,. \end{cases}$$

We consider a regular distribution $\bar{T}^m$ on $\widetilde{TM}$, of class C^∞ and dimension m, restricted to $\tilde{C}$, with the property $\bar{\alpha}/_{\tilde{C}} \in \bar{T}^m$.

We obtain an orthonormal frame, geometrically associated to C,

$$\Re_{\mathcal{M}} = \{\bar{\mu}_1 = \bar{\alpha}\,, \bar{\mu}_2\,, \ldots\,, \bar{\mu}_m\,, \bar{\nu}_1\,, \ldots\,, \bar{\nu}_p\}$$

and a system of invariants of $\bar{\alpha}$ in $\mathcal{M}_t(\tilde{C}, \bar{\alpha}, \bar{T}^m)$: $\chi^g_a\,, \chi_{a\alpha}\,, \chi^n_{\alpha\beta}$. We call them the geodesic curvatures, the normal curvature and the geodesic torsions of the curve C in $\mathcal{M}_t(\tilde{C}, \bar{\alpha}, \bar{T}^m)$.

The fundamental formulae of C in $\mathcal{M}_t(\tilde{C}, \bar{\alpha}, \bar{T}^m)$ are:

$$\begin{aligned} \rho(\frac{d\tilde{C}}{ds}) &= \bar{\alpha}\,, \\ \frac{\nabla\bar{\mu}_a}{ds} &= \chi^g_{a-1}\bar{\mu}_{a-1} + \chi^g_a\bar{\mu}_{a+1} + \sum_{\beta=1}^{p}\chi_{a\beta}\bar{\nu}_\beta\,, \quad a \in \overline{1,m}\,, \quad \bar{\mu}_1 = \bar{\alpha}\,, \\ \frac{\nabla\bar{\nu}_\alpha}{ds} &= -\sum_{b=1}^{m}\chi_{b\alpha}\bar{\mu}_\beta + \sum_{\beta=1}^{p}\chi^n_{\alpha\beta}\bar{\nu}_\beta\,, \quad \alpha \in \overline{1,p}\,, \end{aligned} \tag{66}$$

and the invariants of C satisfy:
for $p \leqslant m$:

$$\begin{aligned} &\chi^g_a > 0, \quad a \in \overline{1, m-2}, \quad \chi^g_0 = \chi^g_m = 0, \\ &\chi_{\alpha\alpha} > 0, \quad \alpha \in \overline{1, p-1}, \quad \chi_{a\beta} = 0, \quad a < \beta, \quad \chi^n_{\alpha\beta} + \chi^n_{\beta\alpha} = 0, \end{aligned} \tag{67}$$

for $p > m$:

$$\begin{aligned} &\chi^g_a > 0, \quad a \in \overline{1, m-1}, \quad \chi^g_0 = \chi^g_m = 0, \\ &\chi_{\alpha\alpha} > 0, \quad \alpha \in \overline{1, m}, \quad \chi_{a\beta} = 0, \quad a < \beta, \\ &\chi^n_{\alpha\beta} + \chi^n_{\beta\alpha} = 0, \quad \chi^n_{i,m+i} > 0, \quad i \in \overline{1, p-m-1}, \quad \chi^n_{i,m+j} = 0, \; j > i. \end{aligned} \tag{68}$$

Theorem 4.31 (fundamental). *Let*

$$\chi_a^g\,,\ \chi_{a\alpha}\,,\ \chi_{\alpha\beta}^n\,,\ \ a\in\overline{1,m}\,,\ \alpha,\beta\in\overline{1,p}\,,$$

be continuous functions of parameter $s\in I$, *satisfying the conditions* (67) *or* (68) *and*

$$\{\bar{\mu}_{01},\ldots,\bar{\mu}_{0m},\bar{\nu}_{01},\ldots,\bar{\nu}_{0p}\}$$

an orthonormal, positively orientated frame in $\tilde{x}_0\in\widetilde{TM}$. *Then there exists an unique curve* C *on* M *with* s *its arc-length parameter and there exists an unique distribution* $\bar{T}^m$ *tangent to* $\tilde{C}$, *the canonic lift of* C *to* $\widetilde{TM}$, *such that the invariants of* C *in the Myller configuration* $\mathcal{M}_t(\tilde{C}\,,\,\bar{\alpha}=\rho(\frac{d\tilde{C}}{ds})\,,\,\bar{T}^m)$ *are the given functions* $\chi_a^g\,,\ \chi_{a\alpha}\,,\ \chi_{\alpha\beta}^n$ *and the next initial conditions are satisfied:*

$$\tilde{C}(s_0)=\tilde{x}_0\,,\ \bar{\mu}_a(s_0)=\bar{\mu}_{0a}\,,\ \bar{\nu}_\alpha(s_0)=\bar{\nu}_{0\alpha}\,,\ \ a\in\overline{1,m}\,,\ \alpha\in\overline{1,p}\,.$$

Definition 4.32. The curve $\tilde{C}$ is an asymptote of rank k for $\mathcal{M}_t(\tilde{C}\,,\,\bar{\xi}_1\,,\,\bar{T}^m)$ if $\bar{\alpha}$ is a $\tilde{\pi}$-versor field auto conjugate of rank k in $\mathcal{M}_t(\tilde{C}\,,\,\bar{\xi}_1\,,\,\bar{T}^m)$.

$\tilde{C}$ is a line of curvature in $\mathcal{M}_t(\tilde{C}\,,\,\bar{\xi}_1\,,\,\bar{T}^m)$ if the linear space span$\{\bar{\mu}_a\}_{a\in\overline{2,m}}$ is orthogonal to $\frac{\nabla\bar{n}}{ds}$, $\forall\bar{n}\in\bar{T}^p\ \Leftrightarrow\frac{\nabla\bar{\mu}_a}{ds}\in\bar{T}^m$, $\forall a\in\overline{2,m}$.

The $\tilde{\pi}$-versor field $(\tilde{C},\bar{\xi}_1)$ is parallel in the sense of Myller in $\mathcal{M}_t(\tilde{C}\,,\,\bar{\xi}_1\,,\,\bar{T}^m)$ if there exists a $\tilde{\pi}$-vector field $\bar{X}(s)=X^1(s)\bar{\xi}_1(s)$ such that $\frac{\nabla}{ds}\bar{X}(s)\in\bar{T}^p(s),\ s\in I$.

The curve $\tilde{C}$ is an auto parallel of $\mathcal{M}_t(\tilde{C}\,,\,\bar{\xi}_1\,,\,\bar{T}^m)$ if $\bar{\alpha}=\rho(\frac{d\tilde{C}}{ds})$ is parallel in the sense of Myller in $\mathcal{M}_t(\tilde{C}\,,\,\bar{\xi}_1\,,\,\bar{T}^m)$.

From the fundamental equations (66) we have:

Theorem 4.33. *The curve* $\tilde{C}$ *is an auto parallel of the Myller configuration*

$$\mathcal{M}_t(\tilde{C}\,,\,\bar{\xi}_1\,,\,\bar{T}^m)\Leftrightarrow\chi_1^g=0\ \text{along}\ \tilde{C}\ \Leftrightarrow\bar{\alpha}_2\ \text{is colinear with}\ \bar{\nu}_1,$$

where $\bar{\nu}_1$ *is the first normal* $\tilde{\pi}$-*versor field of* $\mathcal{M}_t(\tilde{C}\,,\,\bar{\alpha}\,,\,\bar{T}^m)$.

The next theorem is useful to the study of the lines of curvatures in a Myller configuration:

Theorem 4.34 (Meusnier-Bonnet). *Let* χ_i *and* $\bar{\alpha}_i$ *be the curvatures and the principal versors of the curve* $\tilde{C}$. *If*

$$\bar{\alpha}_i=\sum_{b=1}^{m}\cos\epsilon_{ib}\bar{\mu}_b+\sum_{\alpha=1}^{p}\cos\varepsilon_{i\alpha}\bar{\nu}_\alpha\,,$$

$$\cos\epsilon_{1b}=\cos\varepsilon_{1\alpha}=0\ \forall b\in\overline{2,m}\,,\ \alpha\in\overline{1,p}\,,$$

$$\cos\epsilon_{2b}=\cos\varepsilon_{2\alpha}=0\ \forall b\in\overline{3,m}\,,\ \alpha\in\overline{2,p}\,,$$

with $(\cos\epsilon_{ib}, \cos\varepsilon_{i\alpha})$ *orthogonal,* $\det(\cos\epsilon_{ib}, \cos\varepsilon_{i\alpha}) = 1$, *then the following relations are true:*

$$\begin{aligned} & -\chi_{i-1}\cos\epsilon_{i-1,b} + \chi_i \cos\epsilon_{i+1,b} \\ (69) \quad = \; & \frac{d\cos\epsilon_{i,b}}{ds} - \chi_b^g \cos\epsilon_{i,b+1} + \chi_{b-1}^g \cos\epsilon_{i,b-1} - \sum_{\alpha=1}^{p} \chi_{b\alpha}\cos\varepsilon_{i\alpha}, \\ & -\chi_{i-1}\cos\varepsilon_{i-1,\alpha} + \chi_i\cos\varepsilon_{i+1,\alpha} \\ (70) \quad = \; & \frac{d\cos\varepsilon_{i\alpha}}{ds} + \sum_{b=1}^{m}\chi_{b\alpha}\cos\epsilon_{i,b} + \sum_{\beta=1}^{p}\chi_{\beta\alpha}^n \cos\varepsilon_{i\beta}, \end{aligned}$$

$$i \in \overline{1,n},\, b \in \overline{1,m},\, \alpha \in \overline{1,p},\, \chi_0 = \chi_n = \chi_0^g = \chi_m^g = 0 .$$

Particularly

$$(71) \qquad \begin{aligned} \chi_1\cos\epsilon_{22} &= \chi_1^g, \\ \chi_1\cos\varepsilon_{21} &= \chi_{11}, \\ \cos^2\epsilon_{22} + \cos^2\varepsilon_{21} &= 1, \\ \cos\epsilon_{2b} &= 0,\; b \in \overline{3,m}, \\ \cos\varepsilon_{2\alpha} &= 0,\; \alpha \in \overline{2,p}. \end{aligned}$$

A direct consequence of this result is :

Proposition 4.35. *The following statements are equivalent:*
1) $\tilde{C}$ *is an asymptote of rank* k *in* $\mathcal{M}_t(\tilde{C}, \bar{\xi}_1, \bar{T}^m)$, $k \in \overline{1, m-1}$;
2) $\chi_{i\beta} = 0$, $\forall i \in \overline{1,k}$, $\forall \beta \in \overline{1,p}$;
3) $\chi_i = \chi_i^g$, $\forall i \in \overline{1,k}$;
4) $\bar{\alpha}_i = \bar{\mu}_i$, $\forall i \in \overline{1,k+1}$.
Also, the formulae (66) *help us to characterize the lines of curvature:*
$\tilde{C}$ *is a line of curvature in* $\mathcal{M}_t(\tilde{C}, \bar{\xi}_1, \bar{T}^m) \Leftrightarrow$

$$\chi_{b\alpha} = 0 \;\; \forall b \in \overline{2,m},\; \forall \alpha \in \overline{1,p}.$$

Proof. Let $\bar{n}$ be an arbitrary normal $\tilde{\pi}$-vector field in $\bar{T}^p$,

$$\bar{n} = \sum_{\alpha=1}^{p}\cos\vartheta_\alpha \bar{\nu}_\alpha,\; \sum_{\alpha=1}^{p}\cos^2\vartheta_\alpha = 1 \Rightarrow$$

$$\frac{\nabla\bar{n}}{ds} = -\sum_{b=1}^{m}(\sum_{\beta=1}^{p}\cos\vartheta_\beta\chi_{b\beta})\bar{\mu}_b + \sum_{\alpha=1}^{p}(\frac{d\cos\vartheta_\alpha}{ds} + \sum_{\beta=1}^{p}\cos\vartheta_\beta\chi_{\beta\alpha}^n)\bar{\nu}_\alpha .$$

So, $\tilde{C}$ is a line of curvature in

$$\mathcal{M}_t(\tilde{C}\,,\,\bar{\xi}_1\,,\,\bar{T}^m) \Leftrightarrow$$

$$\bar{g}(\bar{\mu}_a, \frac{\nabla \bar{n}}{ds}) = 0\,,\ \forall a \in \overline{2,m}\,,\ \forall \bar{n} \in \bar{T}^p \Leftrightarrow$$

$$\sum_{\beta=1}^{p} \cos\vartheta_\beta \chi_{b\beta} = 0 \ \Leftrightarrow \quad \chi_{b\alpha} = 0\, \forall b \in \overline{2,m}\,,\ \forall \alpha \in \overline{1,p}\,.$$

□

The next result is a particularization of the theorem (4.6):

Proposition 4.36. *If*

$$\bar{\eta}_a = \sum_{b=1}^{m} \cos\zeta_{ab}\bar{\mu}_b\,, \quad a \in \overline{1,m}\,,$$

$$\bar{n}_\alpha = \sum_{\beta=1}^{p} \cos\varpi_{\alpha\beta}\bar{\nu}_\beta\,, \quad \alpha \in \overline{1,p}\,,$$

with the matrices $(\cos\zeta_{ab})$, $(\cos\varpi_{\alpha\beta})$ *orthogonal, then*

$$-G_{a-1}\cos\zeta_{a-1,b} + G_a \cos\zeta_{a+1,b} = \frac{d\cos\zeta_{ab}}{ds} - \chi_b^g \cos\zeta_{a,b+1} + \chi_{b-1}^g \cos\zeta_{a,b-1}\,, \tag{72}$$

$$\sum_{\beta=1}^{p} N_{\alpha\beta}\cos\varpi_{\beta\alpha} = \sum_{b=1}^{m} \chi_{b\alpha}\cos\zeta_{ab}\,, \tag{73}$$

$$\sum_{\beta=1}^{p} M_{\alpha\beta}\cos\varpi_{\beta\gamma} = \frac{d\cos\varpi_{\alpha\gamma}}{ds} + \sum_{\beta=1}^{p} \chi_{\beta\gamma}^n \cos\varpi_{\alpha\beta}\,. \tag{74}$$

Theorem 4.37. *If* $\tilde{C}$ *is a line of curvature in* $\mathcal{M}_t(\tilde{C}\,,\,\bar{\xi}_1\,,\,\bar{T}^m)$, *then* $\bar{\xi}_1$ *is conjugate of rank 1 with* $\bar{\alpha} \Leftrightarrow \bar{g}(\bar{\xi}_1, \bar{\alpha}) = 0$.

Proof. We take $a = 1$ and $\chi_{b\alpha} = 0$, $\forall b \in \overline{2,m}$, $\forall \alpha \in \overline{1,p}$ in (73) and obtain

$$N_{11}\cos\varpi_{11} = \chi_{11}\cos\zeta_{11} \Rightarrow N_{11} = 0 \ \Leftrightarrow \ \bar{g}(\bar{\xi}_1, \bar{\alpha}) = 0\,.$$

□

Using the same method we get:

Theorem 4.38. *If* $\tilde{C}$ *is a line of curvature in* $\mathcal{M}_t(\tilde{C}\,,\,\bar{\xi}_1\,,\,\bar{T}^m)$, *then* $\bar{\xi}_1$ *is conjugate of rank* k *with* $\bar{\alpha} \Leftrightarrow$ *the geodesic space of rank* k *of* $\bar{\xi}_1$ *is orthogonal to* $\bar{\alpha}$.

5. Applications of Myller configurations to the study of Finsler subspaces

Initially, the theory of Finsler subspaces $\check{F}^m$ in a Finsler space F^n was frustrating because of the enormous calculus involved. In the present, new methods appeared and we inspired our work from the books and articles of R. Miron [14], D. Bao [1], A. Bejancu [2], B. T. Hassan [7] and A. Tamim [20, 21].

The idea was to consider the geometric objects specific to a Finsler subspace like notions associated to a Myller configuration, naturally related to the subspace. For example, the normal curvature defined by A. Tamim and B. T. Hassan is exactly the normal curvature of the fundamental section $\bar{\alpha}$ restricted to the curve $\tilde{C}$, in the Myller configuration associated to the Finsler subspace. We studied the auto parallel curves, the asymptotes and the lines of curvatures of the Finsler subspace in the same way.

5.1. Some preliminaries about Finsler subspaces

Let $F : \widetilde{TM} \to \mathbb{R}$ be the fundamental function of a Finsler space F^n, $\bar{g} = g_{ij} d\tilde{x}^i \otimes d\tilde{x}^j \in \Gamma(\otimes^2 \pi^* T^* M)$ its fundamental tensor.

We consider an immersion $j : \check{M} \to M$ of a m dimensional submanifold $\check{M}$ in M. Locally, j is an embedding, and every point $x \in \check{M}$ will be identify with its images $j(x) \in M$. The same for all the geometric objects on $\check{M}$. We suppose that j is differentiable of class C^∞.

It is known [1, 2, 7] that the function

$$\check{F} : \widetilde{T\check{M}} \to \mathbb{R}, \quad \check{F} = F/_{\widetilde{T\check{M}}} \tag{75}$$

defines a Finsler structure on $\check{M}$, named the induced Finsler structure.

We take

$$\check{g} = \bar{g}/_{\pi^* T\check{M}} \,.$$

For any $\tilde{x} \in \widetilde{T\check{M}}$, we consider $\mathcal{N}_{\tilde{x}}$, the orthogonal complement of $(\pi^* T\check{M})_{\tilde{x}}$ in $(\pi^* TM)_{\tilde{x}}$ with respect to $\bar{g}_{\tilde{x}}$.

It results that

$$\mathcal{N} = \bigcup_{\tilde{x} \in \widetilde{T\check{M}}} \mathcal{N}_{\tilde{x}} \to \widetilde{T\check{M}} \tag{76}$$

is a vector bundle named the normal vector bundle induced by the given immersion. Its sections will be called $\tilde{\pi}$-normal vector fields and will be denoted by $\bar{n}$, $\bar{N}$, etc. We also denote $\dim(\mathcal{N}_{\tilde{x}}) = n - m := p$.

Let $C\Gamma = (\nabla, HTM)$ be the Cartan Finsler connection of F^n. We remember that HTM is the Cartan nonlinear connection on TM. We denote by $C\check{F} = (\check{\nabla}, HT\check{M})$ the induced Finsler connection on the given submanifold, and by $CF^\perp = (\nabla^\perp, HT\check{M})$ the Finsler connection induced on the normal bundle.

$H T\check{M}$ is the nonlinear connection induced on $T\check{M}$ by the Cartan nonlinear connection HTM.

Theorem 5.1 ([7, 2]). *The next Gauss-Weingarten formulae hold good:*

$$
\begin{aligned}
\nabla_{\tilde{X}}\bar{Y} &= \check{\nabla}_{\tilde{X}}\bar{Y} + \tilde{H}(\tilde{X},\bar{Y})\,, \quad \tilde{X} \in \chi(\widetilde{T\check{M}})\,, \ \bar{Y} \in \Gamma(\pi^* T\check{M})\,, \\
\nabla_{\tilde{n}}\bar{\xi} &= -\tilde{B}_{\bar{n}}\tilde{X} + \nabla^{\perp}_{\tilde{X}}\bar{n}\,, \quad \tilde{X} \in \chi(\widetilde{T\check{M}})\,, \ \bar{n} \in \Gamma(\mathcal{N})\,.
\end{aligned}
\tag{77}
$$

$\check{\nabla}$ is the induced connection on $\pi^* T\check{M}$ by the connection ∇ on $\pi^* TM$,
$\tilde{H}$-the second fundamental form of the Finsler subspace $\check{M}$,
$\tilde{B}_{\bar{n}}$-the Weingarten operator associated to the normal $\bar{\pi}$-vector field $\bar{n}$,
$\nabla^{\perp}$-the normal connection induced on the normal bundle $\mathcal{N}$.

Proposition 5.2 ([7, 2]). (1) *The induced connection $\check{\nabla}$ is $\check{g}$-metrical but it is not the Cartan connection of the subspace $\check{F}^m = (\check{M}, \check{F}(u,v))$ (it is not symmetrical);*

(2) *The second fundamental form $\tilde{H}$ is $\mathcal{F}(\widetilde{T\check{M}})$ bilinear;*

(3) *The Weingarten map $\tilde{B}$ is $\mathcal{F}(\widetilde{T\check{M}})$ bilinear and satisfies*

$$\check{g}(\tilde{B}_{\bar{n}}\tilde{X}\,,\,\bar{Y}) = \bar{g}(\tilde{H}(\tilde{X},\bar{Y})\,,\,\bar{n})\,;$$

(4) *The induced normal connection $\nabla^{\perp}$ is $g^{\perp}$-metrical, where $g^{\perp} = \bar{g}/_{\mathcal{N}}$.*

We denote $\beta/_{\pi^* T\check{M}} := \check{\beta}$ and define

$$H(\bar{X},\bar{Y}) = \tilde{H}(\check{\beta}\bar{X}\,,\,\bar{Y})\,, \quad Q(\bar{X},\bar{Y}) = \tilde{H}(\gamma\bar{X}\,,\,\bar{Y})\,, \quad \bar{X}\,,\,\bar{Y} \in \Gamma(\pi^* T\check{M})\,. \tag{78}$$

These are called the horizontal/vertical second fundamental form.

Let

$$N(\bar{X}) = H(\bar{X},\bar{\alpha}) \tag{79}$$

be the vector of normal curvature and

$$N_0 = N(\bar{\alpha}) \tag{80}$$

the normal curvature.

We also define the horizontal and vertical Weingarten operators:

$$B_{\bar{n}} = \tilde{B}_{\bar{n}} \circ \check{\beta}\,, \quad W_{\bar{n}} = \tilde{B}_{\bar{n}} \circ \gamma\,. \tag{81}$$

Theorem 5.3 ([10, 21]). *The induced Finsler connection $C\check{F}$ on $\check{F}^m = (\check{M}, \check{F})$ by the Cartan connection CF is the intrinsic Cartan connection of the subspace $\check{F}^m$ if and only if $N = 0 \Leftrightarrow N_0 = 0$.*

5.2. Tangent Myller configurations $\mathcal{M}_t$ associated to a Finsler subspace

Let C be a regular curve of class C^∞ on $\check{M}$, locally given by

$$C\,:\, s \rightarrow u^a(s)\,,\ \ s \in I\,,$$

with s the arc-length parameter:

$$\check{g}(\overline{\frac{dC}{ds}}, \overline{\frac{dC}{ds}}) = 1 \ \Leftrightarrow g_{ab}\frac{du^a}{ds}\frac{du^b}{ds} = 1 \ \Leftrightarrow g_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} = 1\,. \tag{82}$$

We consider the next family of linear spaces along C:

$$C(s) \rightarrow T_{C(s)}\check{M} \,:= T^m(C(s))\,.$$

Let ξ_1 a vector field along C, tangent to $\check{M}$:

$$\xi_1(s) \in T^m(C(s))\,, \forall s \in I\,.$$

The canonic lift $\tilde{C}$ of the curve C to $\widetilde{T\check{M}}$ is:

$$\tilde{C}\,:\, s \rightarrow (x^i(s), y^i(s) = \frac{dx^i}{ds}(s))\,,\ \ s \in I\,,$$

and $\bar{\xi}_1$, the lift of ξ_1 to a section of $\pi^*T\check{M}$ is:

$$\bar{\xi}_1(\tilde{C}(s)) = (\tilde{C}(s)\,,\,\xi_1(C(s)))\,.$$

We consider the case when $\bar{\xi}_1$ is a $\check{\pi}$-versor field along $\tilde{C}$ $\Leftrightarrow \check{g}(\bar{\xi}_1, \bar{\xi}_1) = 1$.

We also consider

$$\bar{T}^m\,:\,\tilde{C}(s) \rightarrow \{\tilde{C}(s)\} \times T_{C(s)}\check{M} \,= (\pi^*T\check{M})_{\tilde{C}(s)}\,,\ \ s \in I\,. \tag{83}$$

So, we defined a Myller configuration $\mathcal{M}_t(\tilde{C}\,,\,\bar{\xi}_1\,,\,\bar{T}^m)$ geometrically associated to the Finsler subspaces $\check{F}^m = (\check{M}, \check{F})$. Its invariants will be called *the invariants of the vector field ξ_1 along C in the Finsler subspace $\check{F}^m$.*

We notice that

$$\bar{T}^m(\tilde{C}(s)) = (\pi^*T\check{M})_{\tilde{C}(s)} \ \text{ and } \ \bar{T}^p(\tilde{C}(s)) = \mathcal{N}_{\tilde{C}(s)}\,.$$

For the Finsler space F^n we consider the Cartan Finsler connection $FC = (\nabla, HTM)$. It induces on $\check{F}^m$ the Finsler connection $\check{F}C = (\check{\nabla}, HT\check{M})$. We denote by $\frac{\nabla}{ds}$ and respectively $\frac{\check{\nabla}}{ds}$ the operators of covariant differentiation defined by the two Finsler connections.

Theorem 5.4. *Let C be a regular curve of class C^∞ on the submanifold $\check{M}$, locally represented by $C\,:\, s \rightarrow u^a(s)\,,\ s \in I$, with s the arc length parameter. We associate a Myller configuration $\mathcal{M}_t(\tilde{C}\,,\,\bar{\xi}_1\,,\,\bar{T}^m)$, with $\tilde{C}$ the canonic lift of C to $\widetilde{T\check{M}}$, ξ_1 a vector field along C, tangent to $\check{M}$, $\bar{\xi}_1$ the lift of ξ_1 to a section in $\pi^*T\check{M}$ and the distribution $\bar{T}^m$ tangent to the submanifold $\check{M}$ along $\tilde{C}$. Let*

$\bar{\eta}_a$, $a \in \overline{1,m}$ and $\bar{n}_\alpha$, $\alpha \in \overline{1,p}$ be the invariants of $\bar{\xi}_1$ in $\mathcal{M}_t(\tilde{C}, \bar{\xi}_1, \bar{T}^m)$. Then, the next formulae hold good:

$$
\begin{cases}
\frac{\check{\nabla}\bar{\eta}_a}{ds} & = -G_{a-1}\bar{\eta}_{a-1} + G_a\bar{\eta}_{a+1}\,, \\
\tilde{H}(\frac{d\tilde{C}}{ds}, \bar{\eta}_a) & = \sum_{\alpha=1}^{p} N_{a\alpha}\bar{n}_\alpha\,,\ a \in \overline{1,m}\,,\ G_0 = G_m = 0\,,
\end{cases} \tag{84}
$$

$$
\begin{cases}
\tilde{B}_{\bar{n}_\alpha}(\frac{d\tilde{C}}{ds}) & = \sum_{a=1}^{m} N_{a\alpha}\bar{\eta}_a\,, \\
\frac{\nabla^{\perp}\bar{n}_\alpha}{ds} & = \sum_{\beta=1}^{p} M_{\alpha\beta}\bar{n}_\beta\,,\ \alpha \in \overline{1,p}\,.
\end{cases} \tag{85}
$$

Proof. We use the Gauss-Weingarten formulae (77) and the fundamental equations (38). □

An immediate consequences is:

Proposition 5.5. *The invariants of ξ_1 on C in $\check{M}$ verify the next relations:*

$$G_a \;=\; \check{g}(\bar{\eta}_{a+1}\,, \frac{\check{\nabla}\bar{\eta}_a}{ds} - \check{g}(\bar{\eta}_a\,, \frac{\check{\nabla}\bar{\eta}_{a-1}}{ds})\bar{\eta}_{a-1})\,,\ a \in \overline{1,m-1}\,, \tag{86}$$

$$N_{a\alpha} \;=\; g^{\perp}(\bar{n}_\alpha\,, \tilde{H}(\frac{d\tilde{C}}{ds}, \bar{\eta}_a)) = \check{g}(\bar{\eta}_a\,, \tilde{B}_{\bar{n}_\alpha}(\frac{d\tilde{C}}{ds}))\,, \tag{87}$$

$$M_{\alpha\beta} \;=\; g^{\perp}(\bar{n}_\beta\,, \frac{\nabla^{\perp}\bar{n}_\alpha}{ds})\,,\ \alpha, \beta \in \overline{1,p}\,. \tag{88}$$

Corollary 5.6. *The geodesic curvatures G_a, $a \in \overline{1,m-1}$ of $\bar{\xi}_1$ in $\mathcal{M}_t(\tilde{C}, \bar{\xi}_1, \bar{T}^m)$ are the same with the invariants of $\bar{\xi}_1$ on $\tilde{C}$, with respect to the induced connection $\check{\nabla}$.*

Proof. In Section 3 we obtained the fundamental equations (16) of a $\tilde{\pi}$-versor field along a curve in $\widetilde{TM}$. When $\bar{\xi}_1$ is tangent to $\check{M}$ along a curve on $\check{M}$, we see that its invariants with respect to $\check{\nabla}$ are exactly those from the formulae (86). □

5.3. The study of some special curves on $\check{M}$

In the Section 4.3 we introduced some special curves in a Myller configuration: the auto parallels, the asymptotes, the lines of curvature. Now we'll define the asymptotes and the lines of curvature of a Finsler subspace and we'll establish relations between the auto parallels of the associated Myller configuration and the auto parallels of the induced nonlinear connection.

We already saw how we can associate to any Finsler submanifold $\check{M}$ a tangent Myller configuration. In this section we'll consider the particular case $\bar{\xi}_1 = \bar{\alpha}/\tilde{C} = \rho(\frac{d\tilde{C}}{ds})$. The fundamental equations of $\rho(\frac{d\tilde{C}}{ds})$ in the associated

Myller configuration will be the fundamental equations of the curve C on $\check{M}$. The notations and definitions are those of the Section 4.3.

A first result is the next one:

Theorem 5.7. *The normal curvature of the Finsler subspace $(\check{M}, \check{F})$ restricted to C is exactly the normal curvature of the fundamental section $\bar{\alpha}$ restricted to $\tilde{C}$, in the Myller configuration $\mathcal{M}_t(\tilde{C}, \bar{\alpha}, \bar{T}^m)$, with $\bar{T}^m$ given by* (83).

Proof. From (84) we have $\tilde{H}(\frac{d\tilde{C}}{ds}, \bar{\alpha}) = \chi_{11}\bar{\nu}_1$, so $N_0 = \chi_{11}$ along $\tilde{C}$. □

We start with the study of the auto parallels in $\mathcal{M}_t(\tilde{C}, \bar{\alpha}, \bar{T}^m)$.

Theorem 5.8. *Let $C : s \to u^a(s)$, $s \in I$ be a curve of class C^∞ on $\check{M}$. Then C is an auto parallel in $\mathcal{M}_t(\tilde{C}, \bar{\alpha}, \bar{T}^m)$ if and only if it is a geodesic of the induced Finsler connection $\check{F}C = (\check{\nabla}, HT\check{M})$.*

Proof. C is an auto parallel in $\mathcal{M}_t(\tilde{C}, \bar{\alpha}, \bar{T}^m)$ if and only if $\chi_1^g = 0$ along C. But from (84) we have

$$\frac{\check{\nabla}\bar{\alpha}}{ds} = \chi_1^g \bar{\nu}_2.$$

So $\chi_1^g = 0 \Leftrightarrow \frac{\check{\nabla}\bar{\alpha}}{ds} = 0 \Leftrightarrow C$ is a geodesic of the induced Finsler connection $\check{F}C = (\check{\nabla}, HT\check{M})$. □

Theorem 5.9. *If $\check{M}$ is an auto parallel submanifold of F^n, then any auto parallel in $\mathcal{M}_t(\tilde{C}, \bar{\alpha}, \bar{T}^m)$ is an auto parallel of the induced nonlinear connection $HT\check{M}$.*

Proof. In ([20]) a submanifold $\check{M}$ of a Finsler space is called auto parallel if $\forall X \in T_x\check{M}$ and for any curve σ on $\check{M}$, starting from x, the parallel transport of X along σ with respect to ∇ gets to a vector tangent to $\check{M}$:

$$\nabla_{\check{X}}\check{Y} \in \pi^*T\check{M}, \ \forall \check{X} \in \chi(T\check{M}), \ \forall \check{Y} \in \Gamma(\pi^*T\check{M}).$$

A necessary and sufficient condition for $\check{M}$ to be an auto parallel submanifold of M is $N_0 = 0$.

Then, the induced Finsler connection on $\check{F}^m$ is the intrinsic Cartan Finsler connection of the subspace and its geodesics are the auto parallels of the induced nonlinear connection $HT\check{M}$. □

Definition 5.10. A curve C on $\check{M}$ is an asymptote of rank k of the submanifold $\check{M}$ if it is an asymptote of rank k of the associated Myller configuration $\check{M}$.

Theorem 5.11. *A curve C on $\check{M}$ is an asymptote of rank k of the submanifold $\check{M}$ if and only if*

$$\tilde{H}(\frac{d\tilde{C}}{ds}, \bar{\mu}_a) = 0, \ \forall a \in \overline{1,k},$$

where $\bar{\mu}_a$, $\bar{\nu}_\alpha$ are the principal versors of $\bar{\alpha}/_{\tilde{C}} = \rho(\frac{d\tilde{C}}{ds})$ in the Myller configuration associated to C on $\check{M}$.

Particularly, C is an asymptote of rank 1 on $\check{M}$ if and only if $N_0 = 0$ along the lift of the curve C to $\widetilde{TM}$.

Proof. C is an asymptote of rank k of the submanifold $\check{M}$ if and only if

$$\bar{g}(\frac{\nabla \bar{n}}{ds}, \bar{\mu}_a) = 0\,,\ \forall a \in \overline{1,k}\,,\ \forall \bar{n} \in \Gamma(\mathcal{N}) \ \Leftrightarrow$$

$$g^{\perp}(\tilde{H}(\frac{d\tilde{C}}{ds}, \bar{\mu}_a), \bar{n}) = 0 \ \forall a \in \overline{1,k}\,,\ \forall \bar{n} \in \Gamma(\mathcal{N}) \ \Leftrightarrow \tilde{H}(\frac{d\tilde{C}}{ds}, \bar{\mu}_a) = 0\,,\ \forall a \in \overline{1,k}\,.$$

For $k = 1$ the condition is equivalent with $N_0 = 0$. □

A direct consequence is:

Corollary 5.12. *On an auto parallel submanifold of a Finsler space, any curve is an asymptote of rank 1.*

Definition 5.13. A curve C on $\check{M}$ is a line of curvature of the Finsler subspace $\check{F}^m$ if is a line of curvature in the associated Myller configuration.

Naturally, we obtain the next expected result:

Theorem 5.14. *The curve C on $\check{M}$ is a line of curvature of the subspace $\check{F}^m$ if and only if*

$$\tilde{B}_{\bar{\nu}_\alpha}(\frac{d\tilde{C}}{ds}) \text{ is colinear with } \bar{\alpha}$$

along $\tilde{C}$, the canonic lift of C to $\widetilde{T\check{M}}$.

Proof. C is a line of curvature of the subspace $\check{F}^m \Leftrightarrow$

$$g^{\perp}(\tilde{H}(\frac{d\tilde{C}}{ds}, \bar{\mu}_a), \bar{\nu}_\alpha) = 0\,,\ \forall \alpha \in \overline{1,p}\,,\ \forall a \in \overline{2,m} \ \Leftrightarrow$$

$$\check{g}(\tilde{B}_{\bar{\nu}_\alpha}(\frac{d\tilde{C}}{ds}), \bar{\mu}_a) = 0\,,\ \forall a \in \overline{2,m} \ \Leftrightarrow \tilde{B}_{\bar{\nu}_\alpha}(\frac{d\tilde{C}}{ds}) \text{ is colinear with } \bar{\alpha}\,.$$

□

We remember that a Finsler subspace is totally umbilical if it is umbilical with respect to any vector field normal to the subspace, it means

$$\tilde{B}_{\bar{\nu}} = \lambda\rho\,,\ \forall \bar{\nu} \in \Gamma(\mathcal{N})\,,\ \lambda : \widetilde{T\check{M}} \to \mathbb{R}\,.$$

Corollary 5.15. *On a totally umbilical subspace of a Finsler space, any curve is a line of curvature.*

5.4. Torse-forming vector fields tangent or normal to a Finsler subspace

Theorem 5.16. *A $\tilde{\pi}$-vector field $\bar{X}$ along a curve on $\widetilde{T\check{M}}$, tangent to $\check{M}$, is a torse forming (concurrent/parallel) $\tilde{\pi}$-vector field in the sense of Myller in $\mathcal{M}_t(\tilde{C}, \bar{\xi}_1, \bar{T}^m)$, with respect to the Cartan connection FC if and only if it is a torse forming (concurrent/parallel) $\tilde{\pi}$-vector field with respect to the induced Finsler connection $\check{F}C = (\check{\nabla}, HT\check{M})$.*

Proof. A $\tilde{\pi}$-vector field $\bar{X}$ along a curve on $\widetilde{T\check{M}}$ is tangent to $\check{M}$ if and only if $\bar{X} \in \bar{T}^m$ and $\bar{X}$ is torse forming in the sense of Myller $\Leftrightarrow \exists \alpha$, β two differentiable functions restricted to the given curve, such that

$$\frac{\nabla \bar{X}}{ds} - \alpha\rho(\frac{d\tilde{C}}{ds}) - \beta\bar{X} \in \bar{T}^p \Leftrightarrow \frac{\nabla \bar{X}}{ds}(s) - \alpha\rho(\frac{d\tilde{C}}{ds}) - \beta\bar{X} \in \mathcal{N}_{\tilde{C}(s)}\,, \forall s \in I\,.$$

From the Gauss Weingarten relations this formula is equivalent to

$$\frac{\check{\nabla}\bar{X}}{ds} = \alpha\rho(\frac{d\tilde{C}}{ds}) + \beta\bar{X}\,.$$

□

A similar result is obtained for vector fields normal to the submanifold.

Theorem 5.17. *A $\tilde{\pi}$-vector field $\bar{X}$ along a curve on $\widetilde{T\check{M}}$, normal to $\check{M}$, is a torse forming vector field in the sense of Myller in $\mathcal{M}_t(\tilde{C}, \bar{\xi}_1, \bar{T}^m)$, with respect to the Cartan connection FC, if and only if it is a recurrent $\tilde{\pi}$-vector field with respect to the induced normal connection $F^{\perp}C = (\nabla^{\perp}, HT\check{M})$.*

Proof. A $\tilde{\pi}$-vector field $\bar{X}$ along a curve on $\widetilde{T\check{M}}$ is normal to the submanifold if and only if $\bar{X} \in \bar{T}^p$, and $\bar{X}$ is torse forming in the sense of Myller $\Leftrightarrow \exists \epsilon$, δ two differentiable functions restricted to the given curve, such that

$$\frac{\nabla \bar{X}}{ds} - \epsilon\rho(\frac{d\tilde{C}}{ds}) - \delta\bar{X} \in \bar{T}^m \Leftrightarrow \frac{\nabla \bar{X}}{ds}(s) - \epsilon\rho(\frac{d\tilde{C}}{ds}) - \delta\bar{X} \in (\pi^* T\check{M})_{\tilde{C}(s)}\,, \forall s \in I.$$

From the Gauss Weingarten relations this formula is equivalent to $\frac{\nabla^{\perp}\bar{X}}{ds} = \delta\bar{X}$. □

Corollary 5.18. *A $\tilde{\pi}$-vector field $\bar{X}$ along a curve on $\widetilde{T\check{M}}$, normal to $\check{M}$, is a recurrent (parallel) vector field in the sense of Myller in $\mathcal{M}_t(\tilde{C}, \bar{\xi}_1, \bar{T}^m)$, with respect to the Cartan connection FC, if and only if it is a recurrent (parallel) $\tilde{\pi}$-vector field with respect to the induced normal connection $F^{\perp}C = (\nabla^{\perp}, HT\check{M})$.*

In the Section 3 we presented theorems of existence and uniqueness for $\tilde{\pi}$-torse forming vector fields in F^n, when initial conditions are given. We can reobtain these results for the $\tilde{\pi}$-vector fields tangent or normal to a Finsler subspace, like a consequence of the theorems of existence and uniqueness for $\tilde{\pi}$-torse forming vector fields in the sense of Myller in a Myller configuration.

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